

THREE-DIMENSIONAL MANIFOLDS  
MICHAELMAS TERM 1999

EXAMPLES SHEET 2

1. Show that any compact non-orientable 3-manifold  $M$  having no  $\mathbb{R}P^2$  boundary components has infinite  $H_1(M)$ , and hence has a 2-sided properly embedded non-separating incompressible surface.
2. Construct a Haken 3-manifold with the same homology as  $S^3$ .
3. Show that if  $M$  is a compact orientable irreducible 3-manifold with  $\pi_1(M)$  a free group, then  $M$  is a handlebody. [Hint: consider a map from  $M$  to a bouquet of circles.]
4. Let  $M$  be a compact orientable irreducible 3-manifold and let  $F$  be a compact surface in  $\partial M$ . Show that if  $\pi_1(F) \rightarrow \pi_1(M)$  is an isomorphism, then there is a homeomorphism from  $M$  to  $F \times [0, 1]$  taking  $F$  to  $F \times \{0\}$ .
5. Show that if  $S$  is an orientable incompressible surface properly embedded in a compact orientable 3-manifold  $M$ , then  $\pi_1(M_S) \rightarrow \pi_1(M)$  is injective.
6. Using a hierarchy argument, show that the fundamental group of a Haken 3-manifold is torsion-free.
7. What extra assumptions do we need to make about a properly embedded incompressible surface with non-empty boundary in a compact irreducible 3-manifold to guarantee that it can be ambient isotoped into normal form?
8. Show that any Haken 3-manifold  $M$  has a hierarchy

$$M = M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \dots \xrightarrow{S_{n-1}} M_n,$$

where  $n \leq 5$  (but where each surface  $S_i$  may be disconnected). Here,  $n$  is the *length* of this hierarchy. Which compact orientable 3-manifolds have a hierarchy of length one?

9. Suppose that a compact 3-manifold  $M$  contains  $k$  properly embedded 2-spheres, none of which bounds a 3-ball and no two of which are parallel. Then show that, for any triangulation of  $M$ , we may find such a collection of 2-spheres in normal form. Deduce that any compact 3-manifold can be

expressed as a connected sum of prime 3-manifolds.

10. Show that, in the statement of the Loop Theorem, we may remove the hypotheses that the 3-manifold is irreducible and compact.
11. Disprove the following conjecture. 'Let  $F$  be a surface properly embedded in a 3-manifold  $M$ , such that, with respect to any triangulation of  $M$ ,  $F$  may be ambient isotoped into normal form. Then  $F$  is incompressible.' [Hint: let  $M$  be the lens space obtained by 6/1 surgery on the unknot.]
12. Show that, for a fixed triangulation of a 3-manifold with  $t$  tetrahedra, its normal surfaces are in one-one correspondence with the integral lattice points in a subset  $C$  of  $\mathbb{R}^{7t}$ . If  $S_1$ ,  $S_2$  and  $S_3$  are normal surfaces corresponding to points  $[S_1]$ ,  $[S_2]$  and  $[S_3]$  in  $C$ , such that  $[S_1] + [S_2] = [S_3]$ , how are the Euler characteristics of  $S_1$ ,  $S_2$  and  $S_3$  related?