Limits of geodesic rays and non-visible points of Teichmüller space

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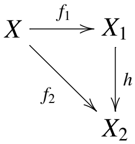
Notation

Let X be a Riemann surface of type (g, n) with $2g - 2 + n > 0$.
Let $T(X)$ be the Teichmüller space of X i.e.

$$T(X) = \{(Y, f) \mid f : X \to Y \text{ q.c.}\}/\sim$$

where $(Y_1, f_1) \sim (Y_2, f_2)$ if there is a conformal mapping $h : Y_1 \to Y_2$ such that $h \circ f_1$ is homotopic to f_2.



Let $\mathcal{S}$ be the set of non-trivial and non-peripheal s.c.c's on X.

$T(X)$ is topologized with the Teichmüller distance which is defined to be

$$d_T(y_1, y_2) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{S}} \frac{\mathrm{Ext}_{y_1}(\alpha)}{\mathrm{Ext}_{y_2}(\alpha)}$$

for $y_1, y_2 \in T(X)$ (known as Kerckhoff's formula), where $\mathrm{Ext}_y(\alpha)$ is the extremal length of α on $y = (Y, f)$:

$$\mathrm{Ext}_y(\alpha) = 1/\sup_A\{\mathrm{Mod}(A) \mid A \subset Y \text{ is an annulus with core} \sim f(\alpha)\}.$$

It is known that $(T(X), d_T)$ is complete and uniquely geodesic.

The space of measured foliations $\mathcal{MF}$ is the closure of the image of the embedding

$$\mathbb{R}_+ \otimes \mathcal{S} \ni t\alpha \mapsto [\mathcal{S} \ni \beta \mapsto t\, u(\beta, \alpha)] \in \mathbb{R}_+^{\mathcal{S}}.$$

The space of projective measured foliations $\mathcal{PMF}$ is the quotient

$$\mathcal{PMF} = (\mathcal{MF} - \{0\})/\mathbb{R}_{>0}.$$

It is known that $\mathcal{MF}$ and $\mathcal{PMF}$ are homeomorphic to the Euclidean space and the sphere respectively.

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Kerckhoff has shown that the extremal length function $\mathrm{Ext}_y(\cdot)$ on $\mathcal{S}$ extends as a continuous function

$$\mathrm{Ext}_y(\cdot) : \mathcal{MF} \to \mathbb{R}$$

with $\mathrm{Ext}_y(tF) = t^2\mathrm{Ext}_y(F)$.

Aim of this talk - Introduction

There are important ‘rays’ or ‘lines’ in the Teichmüller space and many investigations on behaviors and relations among them, For instance

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In this talk, I would like to review the recent progress on the behaviors of ‘rays’ or ‘lines’ in the other compactification, called Gardiner-Masur compactification.

Gardiner-Masur compactification

We consider a mapping

$$\Phi_{GM} : T(X) \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathrm{Ext}_y(\alpha)^{1/2}] \in \mathrm{P}\mathbb{R}_+^{\mathcal{S}}.$$

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The closure of the image is called the Gardiner-Masur compactification of $T(X)$. We call the complement $\partial_{GM} T(X)$ of the image from the closure the Gardiner-Masur boundary.

Define a continuous function on $\mathcal{MF}$ by

$$\mathcal{E}_y(F) = \left(\frac{\mathrm{Ext}_y(F)}{K_y}\right)^{1/2} \qquad K_y = \exp(2d_T(x_0, y)).$$

Notice that the Gardiner-Masur embeding above is equal to

$$\Phi_{GM} : T(X) \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathcal{E}_y(\alpha)] \in \mathrm{P}\mathbb{R}_+^{\mathcal{S}}.$$

Properties

- (Gardiner-Masur) $\mathcal{PMF} \subset \partial_{GM}T(X)$. $\mathcal{PMF} \neq \partial_{GM}T(X)$ if $\dim_{\mathbb{C}} T(X) \geq 2$.

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- (Kerckhoff) More precisely, any geodesic ray associated to rational foliation has a limit in the GM-compatification, and the limit is not contained in $\mathcal{PMF}$.
- (M) For any $p \in \partial_{GM} T(X)$, there is a continuous function $\mathcal{E}_p$ on $\mathcal{MF}$ such that
  - $\mathcal{S} \ni \alpha \mapsto \mathcal{E}_p(\alpha)$ represent p.
  - When $\{y_n\}_n \subset T(X)$ converges to p, there is a subsequence $\{y_{n_j}\}_j$ and $t_0 > 0$ such that $\mathcal{E}_{y_{n_j}}$ converges to $t_0 \mathcal{E}_p$ uniformly on any compact set of $\mathcal{MF}$.

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- (Liu-Su) The Gardiner-Masur compactification canonically coincides with the horofunction boundary with respect to the Teichmüller distance.

Naturality for the Teichmüller distance?

Recently, we also have the following evidence for the "naturality".

Proposition (M)

Let $x_0 \in T(X)$ be the base point. Then, the Gromov product

$$\langle y, z\rangle_{x_0} = \frac{1}{2}(d_T(x_0, y) + d_T(x_0, z) - d_T(y, z))$$

extends continuously on the GM-compactification (with value in $[0, \infty]$) such that

$$\exp(-2\langle y, z\rangle_{x_0}) = \frac{i(G, H)}{\mathrm{Ext}_{x_0}(G)^{1/2} \cdot \mathrm{Ext}_{x_0}(H)^{1/2}}$$

for $[G], [H] \in \mathcal{PMF} \subset \partial_{GM} T(X)$.

Hence, we may play and enjoy the Teichmüller geometry on the GM-compactification.... I think

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Horofunction boundary

The horofunction closure of a pointed metric space $((M, x_0), \rho)$ is a closure $\overline{M}^h$ of the image of embedding

$$M \ni y \mapsto \rho(y, x_0) \in C_*(M) = C(M)/\mathbb{R}$$

where $C(M)$ is the space of continuous functions on M equipped with topology of uniform convergence on any bounded set, and $\mathbb{R}$ is the subspace of constant function. The horofunction boundary is the complement $\overline{M}^h - M$.

A mapping $\gamma : T \to M$ ($T \subset [0, \infty)$ is an unbounded set with $0 \in T$) is an almost geodesic ray (with base point x_0) if

- $\gamma(0) = x_0$, and
- for all $\epsilon > 0$ there is an $N > 0$ such that for all $t, s \in T$ with $t \geq s \geq N$,

$$|\rho(\gamma(t), \gamma(s)) + \rho(\gamma(s), \gamma(0)) - t| < \epsilon.$$

Proposition (Rieffel)

Let (M, ρ) be a locally compact metric space. Then, any almost geodesic ray has a limit in the horofunction boundary.

Definition (Rieffel)

Let (M, ρ) be a locally compact metric space. A bounary point in the horofunction boundary is said to be a Busemann point if it is the limit of an almost geodesic ray.

We fix a base point $x_0 = (X, id) \in T(X)$.
From Liu-Su's result above and a property of horofunction compactifications (M. Rieffel), we can see the following.

Proposition (Liu-Su)

Any almost geodesic ray in the Teichmüller space has a limit in the Gardiner-Masur compactification.

Recently, C.Walsh defines the horofunction boundaries for asymmetric metric spaces and study the horofunction boundary of Thurston's (asymmetric) Lipschitz metric

$$d_L(x,y) = \log \sup_{\alpha \in \mathcal{S}} \frac{\ell_x(\alpha)}{\ell_y(\alpha)}$$

for $x, y \in T(X)$, where $\ell_x(\alpha)$ is the hyperbolic length of the geodesic representative of α on a marked Riemann surface x:

Theorem (Walsh)

The horofunction boundary of $(T(X), d_L)$ is canonically identified with the Thurston boundary. Moreover, any horofunction boundary point is a Busemann point. Namely, any boundary point is the limit of an almost geodesic ray.

The aim of this talk - Statements

Theorem 1.

For $G \in \mathcal{MF}$. Let $R_G : [0,\infty) \to T(X)$ be the Teichmüller geodesic ray associated with Hubbard-Masur differential with respect to G on x_0. Then, the mapping

$$\mathcal{PMF} \ni [G] \mapsto \lim_{t\to\infty} \Phi_{GM} \circ R_G(t)$$

is injective.

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is injective.

Notice

Proposition (Masur)

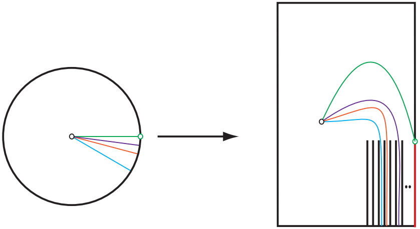
When $G = \sum_{k=1}^{m} w_k \alpha_k$ ($w_k > 0$, $\alpha_k \in \mathcal{S}$), the limit of $R_G(t)$ in the Thurston compactification exists and is equal to the 'barycenter' $[\sum_{k=1}^{m} \alpha_k]$.

Hence, in the case of Thurston compactification, even if we restrict the "limit map" to the set of measured foliations G with the property that R_G has a limit, the limit map cannot be injective.

Theorem 2 (Non-visibility via almost geodesic rays).

When $\dim_{\mathbb{C}} T(X) \geq 2$, the horofunction boundary of $(T(X), d_T)$ contains a non-Busemann point. Namely, there is a boundary point where cannot be a limit of any almost geodesic ray.

It is known that the horofunction boundary of any $\mathrm{CAT}(0)$-space consists of Busemann points. Hence, we obtain



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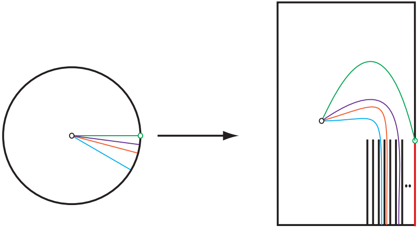
Corollary (Masur)

When $\dim_{\mathbb{C}} T(X) \geq 2$, a metric space $(T(X), d_T)$ is not a $\mathrm{CAT}(0)$-space.



Proof of Theorem 1

Proposition (Gardiner's differential formula)

Let $y = (Y, f) \in T(X)$ and $F \in \mathcal{MF}$. Let μ be a Beltrami differential on Y and denote by y_t be the marked surface obtained by the quasiconformal deformation with respect to $t\mu$ with $t \in \mathbb{R}$. Then, we have

$$\mathrm{Ext}_{y_t}(F) = \mathrm{Ext}_y(F) - 2t\,\mathrm{Re}\int_Y \mu\, J_{F,y} + o(t) \qquad (1)$$

as $t \to 0$, where $J_{F,y}$ is the holomorphic quadratic differential on Y whose vertical foliation is equal to $f(F)$.

In comparing the formula (1) with the original Gardiner's formula, we should notice from the definition that $-J_{F,y}$ is the holomorphic quadratic differential with horizontal foliation F.

For $G \in \mathcal{MF} - \{0\}$ let $y_t = R_G(t)$ for $t \geq 0$.
From the Gardiner's differential formula, we can see the following.

Lemma

For any $F \in \mathcal{MF}$, a function

$$[0, \infty) \ni t \mapsto \mathcal{E}_{y_t}(F) = e^{-t}\mathrm{Ext}_{y_t}(F)^{1/2}$$

is a positive non-increasing function. Furthermore, this function is strictly decreasing if and only if F is not projectively equivalent to the horizontal foliation of J_{G,x_0}.

Proof of Lemma

Notice that the infinitesimal Beltrami differential along R_{G,x_0} at y_t is the Teichmüller differential

$$\mu_t = \frac{|J_{G,y_t}|}{J_{G,y_t}}.$$

By the Gardiner's differential formula, we have

$$\frac{d}{dt}e^{-2t}\mathrm{Ext}_{y_t}(F) = -2e^{-2t}\left\{\mathrm{Ext}_{y_t}(F) + \mathrm{Re}\int_{Y_t}\mu_t\,J_{F,y_t}\right\} \leq 0. \tag{2}$$

From (2), the derivative vanishes at $t \geq 0$ if and only if

$$\mathrm{Re}\int_{Y_t}\left(1+\frac{|J_{G,y_t}|}{J_{G,y_t}}\frac{J_{F,y_t}}{|J_{F,y_t}|}\right)|J_{F,y_t}| = \mathrm{Ext}_{y_t}(F) + \mathrm{Re}\int_{Y_t}\mu_t\,J_{F,y_t} = 0.$$

Hence, $J_{F,y_t} = -J_{G,y_t}$ almost everywhere. Therefore, F is projectively equivalent to the horizontal foliation of J_{G,x_0}.

Proof of Theorem 1.

We first give a simple proof of the existence of the limit of any Teichmüller ray. From Lemma, for any $\alpha \in \mathcal{S}$, the limit

$$e_\alpha = \lim_{t\to\infty} e^{-t}\mathrm{Ext}_{R_G(t)}(\alpha)^{1/2}$$

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exists.
Let $\alpha \in \mathcal{S}$ with $i(G,\alpha) \neq 0$. By Minsky's inequality

$$\begin{aligned} 0 < i(G,\alpha) &\leq \mathrm{Ext}_{R_G(t)}(G)^{1/2}\mathrm{Ext}_{R_G(t)}(\alpha)^{1/2} \\ &= \mathrm{Ext}_{x_0}(G)^{1/2} \cdot e^{-t}\mathrm{Ext}_{R_G(t)}(\alpha)^{1/2} \to \mathrm{Ext}_{x_0}(G)^{1/2} e_\alpha \end{aligned}$$

Hence $e_\alpha \neq 0$ when $i(G,\alpha) \neq 0$. Thus,

$$\Phi_{GM} \circ R_G(t) = [\mathcal{S} \ni \alpha \mapsto \mathrm{Ext}_{R_G(t)}(\alpha)^{1/2}] \to p_G := [\mathcal{S} \ni \alpha \mapsto e_\alpha]$$

as $t \to \infty$ in $\mathrm{P}\mathbb{R}_+^{\mathcal{S}}$.

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as $t\to\infty$ in $\mathrm{P}\mathbb{R}_+^{\mathcal{S}}$.

For $F \in \mathcal{MF}$, we re-define

$$\mathcal{E}_{p_G}(F) = \lim_{t\to\infty}\left(\frac{\mathrm{Ext}_{R_G(t)}(F)}{K_{R_G(t)}}\right)^{1/2}$$

Let us prove the injectivity of the limit map.

Let $[G_1], [G_2] \in \mathcal{PMF}$ with $[G_1] \neq [G_2]$. Let $p_1 = p_{[G_1]}$ and $p_2 = p_{[G_2]}$.

Let H_i be the horizontal foliation of of J_{G_i,x_0}. We normalize H_i with $\mathrm{Ext}_{x_0}(H_i) = 1$ for $i = 1, 2$. By Hubbard-Masur theorem, H_1 is not projectively equivalent to H_2.

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From Lemma,

$$\mathcal{E}_{p_i}(H_i) = \mathrm{Ext}_{x_0}(H_i) = 1$$
$$\mathcal{E}_{p_i}(H_{3-i}) < \mathrm{Ext}_{x_0}(H_i) = 1.$$

for $i = 1, 2$. Hence

$$\frac{\mathcal{E}_{p_1}(H_1)}{\mathcal{E}_{p_2}(H_1)} > 1 > \frac{\mathcal{E}_{p_1}(H_2)}{\mathcal{E}_{p_2}(H_2)}.$$

This means that $p_1 \neq p_2$.

Comments on Theorem 1

We can also see the following “expected” result.

Proposition (M)

Let $G \in \mathcal{MF} - \{0\}$ be a unquely ergodic measured foliation. Let $p \in \partial_{GM} T(X)$. If $\mathcal{E}_p(G) = 0$, there is a $t_0 > 0$ such that

$$\mathcal{E}_p(F) = t_0\, i(F, G)$$

for all $F \in \mathcal{MF}$. Namely, $p = [G]$ as points in $\mathrm{P}\mathbb{R}_+^{\mathcal{S}}$.

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In particular, we have

Corollary

When G is uniquely ergodic,

$$\lim_{t\to\infty} \Phi_{GM} \circ R_G(t) = [G] \in \mathcal{PMF} \subset \partial_{GM} T(X)$$

Furthermore, combining Masur's result, we can conclude

Proposition (M)

For a uniquely ergodic measured foliation $G \in \mathcal{MF}$, the following are equivalent for a sequence $\{y_n\}_n$ in $T(X)$.

- $\{y_n\}_n$ converges to $[G]$ in the Thurston compactification.
- $\{y_n\}_n$ converges to $[G]$ in the Gardiner-Masur compactification.

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- $\{y_n\}_n$ converges to $[G]$ in the Gardiner-Masur compactification.

In particular, from R.Diaz and S.Series' result, when G is as above, for $F \in \mathcal{MF}$ such that F fills up X with G, the line of minima associated to (F, G) has the limit (at the "G-direction") in the Gardiner-Masur compactification and converges to $[G]$.

Thus, the line of minima for (F, G) has the same limit (at the G-direction) as the Teichmüller ray associated to G under the Gardiner-Masur embedding.

Proof of Theorem 2

We recall

Theorem 2 (Non-visibility via almost geodesic rays).

When $\dim_{\mathbb{C}} T(X) \geq 2$, the horofunction boundary of $(T(X), d_T)$ contains a non-Busemann point. Namely, there is a boundary point where cannot be arrived by any almost geodesic ray.

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To show Theorem 2, we shall show the following

Theorem 3 (Maximal rational foliations are non-visibile).

When $\dim_{\mathbb{C}} T(X) \geq 2$, any maximal rational foliation $[G] \in \mathcal{PMF} \subset \partial_{GM} T(X)$ cannot be the limit of any almost geodesic ray.

Key of the proof of Theorem 2 : Non-twisting property

Let $[G]$ be the projective class of a maximal rational foliation. Suppose that $[G]$ is the limit of an almost geodesic ray

$$\gamma : T \to T(X)$$

where $T \subset [0, \infty)$ with $0 \in T$ and $\gamma(0) = x_0$.

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Let $G = \sum_{i=1}^{k} w_i\alpha_i$ ($k = \dim_{\mathbb{C}} T(X) \geq 2$). Let $\gamma(t) = (Y_t, f_t)$ and J_t the Jenkins-Strebel differential of G on $\gamma(t)$. Let $A_{i,t}$ the characteristic annulus of J_t.

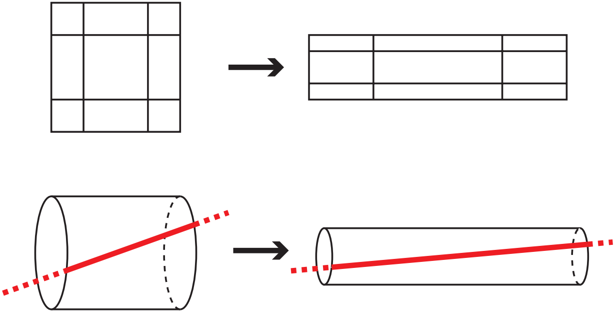
Key obserbation

Any simple closed curve is not so “twisted” on any characteristic annulus $A_{i,t}$ along an almost geodesic ray $\gamma : T \to T(X)$.

Idea of the proof of Theorem 2 : Geodesic rays

We first recall Kerckhoff's calculation for the case where γ is the Teichmüller geodesic ray associated to the Jenkins-Strebel differential of G.

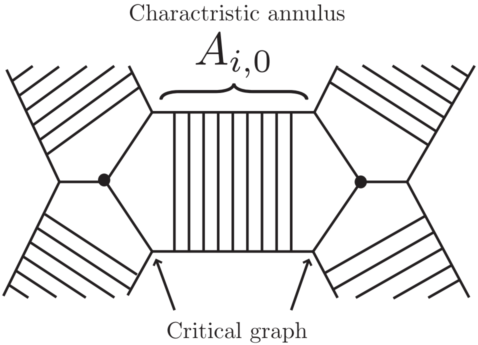
The deformation along the Teichmüller geodesic ray is given by "stretching".



Not twisted

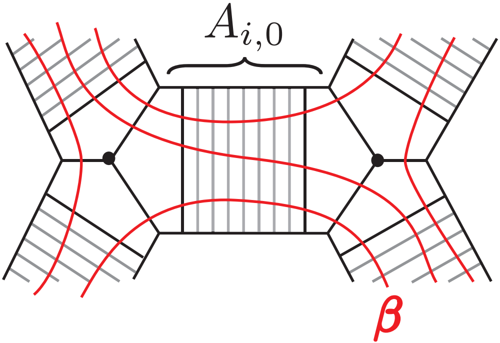
Idea of the proof of Theorem 2 : Geodesic rays

The characteristic annulus of the Hubbard-Masur differential for G on the initial point x_0.



Idea of the proof of Theorem 2 : Geodesic rays

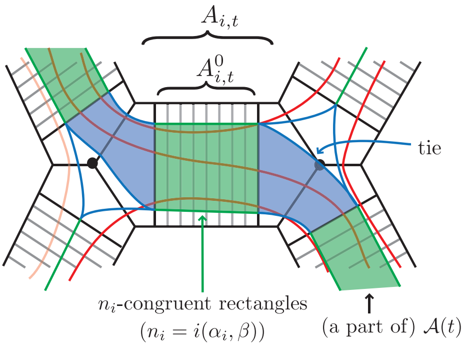
Let $\beta \in \mathcal{S}$. We shall recall briefly the calculation of the asymptotic behaviour of the extremal length $\mathrm{Ext}_{\gamma(t)}(\beta)$ along the Teichmüller ray. The method here is due to S.Kerckhoff.



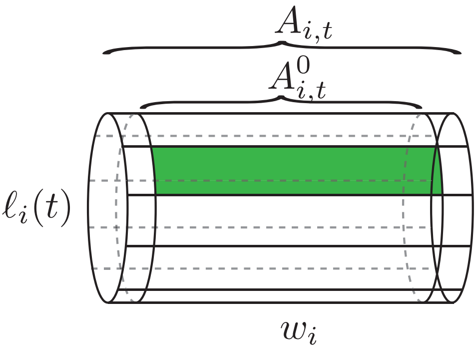
Idea of the proof of Theorem 2 : Geodesic rays

Let $n_i = i(\alpha_i, \beta)$, where $G = \sum_{i=1}^{k} w_i \alpha_i$.

- Let $A_{i,t}^0$ be the subannulus of $A_{i,t}$ which is a component of the "regular neighborhood" of the critical graph.
- Divide each characteristic annulus $A_{i,t}$ into n_i-congruent rectanges.
- Connecting rectangles via "ties" to obtain an annulus $\mathcal{A}(t)$ whose core is homotopic to β.



Idea of the proof of Theorem 2 : Geodesic rays



Ext(hori. paths in a cong. rectangle) $= w_i/(\ell_i(t)/n_i) + O(1) = n_i \mathrm{Mod}(A_{i,t}) + O(1)$

Hence,

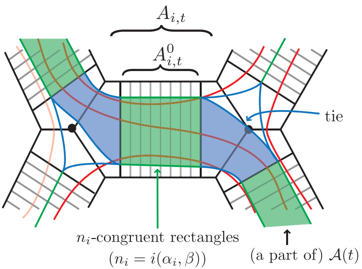
$$\text{(Ext. leng. of all congruent rectangles)} = \sum_{i=1}^{k} n_i^2 \mathrm{Mod}(A_{i,t}) + O(1)$$

Idea of the proof of Theorem 2 : Geodesic rays

By applying some technical thing (including the length area method), we get

$$\mathrm{Ext}(\mathcal{A}(t)) \le \text{(Ext. leng. of all congruent rectangles)} + \text{(Ext. leng. of ties)}$$
$$\le \sum_{i=1}^{k} n_i^2 \mathrm{Mod}(A_{i,t}) + o(K_t).$$

as $t \to \infty$, that is, the major part comes from the congruent rectangles
Notice that $\mathrm{Mod}(A_{i,t}) \asymp K_t := e^{2d_T(x_0,\gamma(t))}$.

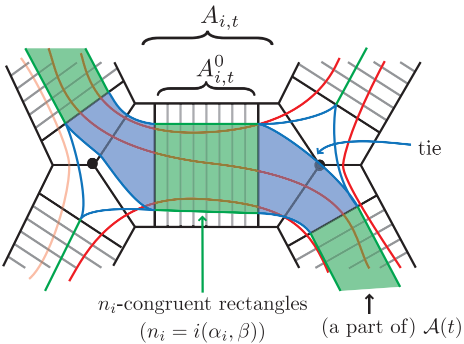


Idea of the proof of Theorem 2 : Geodesic rays

The "non-twisting property" implies that rectangles in $A^0_{i,0}$ are mapped to rectangles in $A^0_{i,t}$. Hence, the core of $\mathcal{A}(t)$ is homotopic to β. we can see that

$$\mathrm{Ext}_{\gamma(t)}(\beta) \leq \mathrm{Ext}(\mathcal{A}(t)) \leq \sum_{i=1}^{k} n_i^2 \mathrm{Mod}(A_{i,t})^2 + o(K_t)$$

as $t \to \infty$.

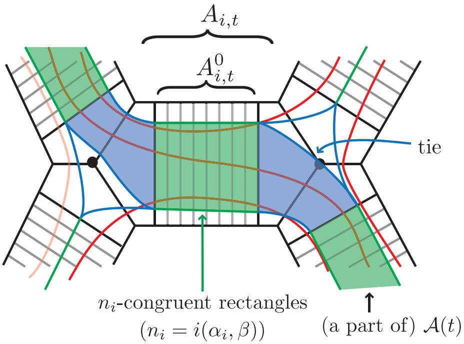


Idea of the proof of Theorem 2 : Geodesic rays

By the standard (but technical) argument, we have the upper bound of modulus of the the characteristic annulus of JS-differential of β, and we get

$$\mathrm{Ext}_{\gamma(t)}(\beta) \geq \sum_{i=1}^{k} n_i^2 \mathrm{Mod}(A_{i,t})^2 + O(1)$$

as $t \to \infty$.

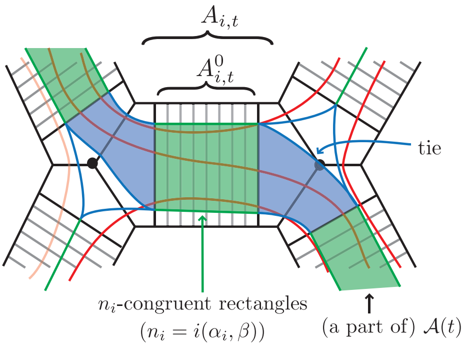


Idea of the proof of Theorem 2 : Geodesic rays

Thus we get

$$\sum_{i=1}^{k} n_i^2 \mathrm{Mod}(A_{i,t})^2 + O(1) \leq \mathrm{Ext}_{\gamma(t)}(\beta) \leq \sum_{i=1}^{k} n_i^2 \mathrm{Mod}(A_{i,t})^2 + o(K_t)$$

as $t \to \infty$.



“Non”-twisting property from almost geodesic rays

Recall that an almost geodesic ray

$$\gamma : T \to T(X)$$

converges to the projective class $[G]$ of a maximal rational foliation G. We assume that $\mathrm{Ext}_{x_0}(G) = 1$ and there is $t_0 > 0$ such that

$$\mathcal{E}_{\gamma(t)}(\cdot) \to t_0\, i(\cdot, G)$$

unformly on any compact set of $\mathcal{MF}$.

Lemma

Under the notation above, we have $t_0 = 1$.

Proof.

Indeed,

$$1 = \max_{\mathrm{Ext}_{x_0}(F)=1} \mathcal{E}_{\gamma(t)}(F) \to t_0 \max_{\mathrm{Ext}_{x_0}(F)=1} i(F, G) = t_0.$$

□

Lemma

Under the asumption as above, we have

$$K_{\gamma(t)} \cdot \mathrm{Ext}_{\gamma(t)}(G) \to 1 \quad (t \to \infty).$$

[Proof] Recall that an almost geodesic $\gamma : T \to T(X)$ satisfies that for any $\epsilon > 0$ there is an $N > 0$ such that

$$|d_T(\gamma(t), \gamma(s)) + d_T(\gamma(s), x_0) - t| < \epsilon$$

for $t \geq s \geq N$. By Kerchhoff's formula, this inequality is re-stated as

$$e^{t-\epsilon} \leq \max_{\mathrm{Ext}_{x_0}(F)=1} \frac{\mathrm{Ext}_{\gamma(t)}(F)^{1/2}}{\mathrm{Ext}_{\gamma(s)}(F)^{1/2}} \cdot \max_{\mathrm{Ext}_{x_0}(F)=1} \frac{\mathrm{Ext}_{\gamma(t)}(F)^{1/2}}{\mathrm{Ext}_{x_0}(F)^{1/2}} \leq e^{t+\epsilon},$$

equivalently,

$$e^{t-\epsilon} \leq \max_{\mathrm{Ext}_{x_0}(F)=1} \frac{\mathrm{Ext}_{\gamma(t)}(F)^{1/2}}{\mathrm{Ext}_{\gamma(s)}(F)^{1/2}} \cdot K_{\gamma(s)}^{1/2} \leq e^{t+\epsilon},$$

for $t \geq s \geq N$.

We have

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for $t \geq s \geq N$. In particular, when $t = s$,

$$e^{s-\epsilon} \leq K_{\gamma(s)}^{1/2} \leq e^{s+\epsilon},$$

for $s \geq N$.

We have

$$e^{t-\epsilon} \leq \max_{\mathrm{Ext}_{x_0}(F)=1} \frac{\mathrm{Ext}_{\gamma(t)}(F)^{1/2}}{\mathrm{Ext}_{\gamma(s)}(F)^{1/2}} \cdot K_{\gamma(s)}^{1/2} \leq e^{t+\epsilon}, \quad (3)$$

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for $s \geq N$. Dividing each sides of (3) by $K_{\gamma(t)}^{1/2}$, we obtain

$$e^{-2\epsilon} \leq \max_{\mathrm{Ext}_{x_0}(F)=1} \frac{\mathcal{E}_{\gamma(t)}(F)}{\mathrm{Ext}_{\gamma(s)}(F)^{1/2}} \cdot K_{\gamma(s)}^{1/2} \leq e^{2\epsilon}$$

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for $t \geq s \geq N$. Letting $t \to \infty$, we get

$$e^{-2\epsilon} \leq \max_{\mathrm{Ext}_{x_0}(F)=1} \frac{i(F,G)}{\mathrm{Ext}_{\gamma(s)}(F)^{1/2}} \cdot K_{\gamma(s)}^{1/2} \leq e^{2\epsilon},$$

equivalently,

$$e^{-2\epsilon} \leq \mathrm{Ext}_{\gamma(s)}(G)^{1/2} \cdot K_{\gamma(s)}^{1/2} \leq e^{2\epsilon}$$

when $s \geq N$.

Asymptotic behavior of moduli of annuli

Lemma

Suppose G contains a foliated annulus A. Namely, $G = F + w\alpha$ for some $F \in \mathcal{MF}$ and $\alpha \in \mathcal{S}$. Let A_t be the characteristic annulus of the Hubbard-Masur differential J_t for G on $\gamma(t)$. Then,

$$\mathrm{Mod}(A_t) \asymp K_t \quad (t \to \infty).$$

[Proof] From the geometric definition of the extremal length,

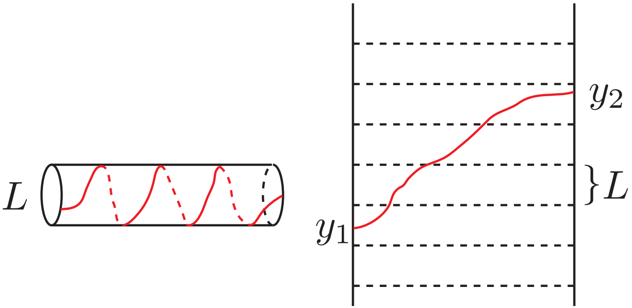
$$\mathrm{Mod}(A_t) \leq 1/\mathrm{Ext}_{\gamma(t)}(\alpha) \leq K_t/\mathrm{Ext}_{x_0}(\alpha).$$

On the other hand,

$$\begin{aligned} \frac{1}{\mathrm{Mod}(A_t)} &= \frac{\ell_{J_t}(\alpha)}{w} = w^2 \cdot (J_t\text{-area of } A) \\ &\leq w^2 \|J_t\| = w^2 \mathrm{Ext}_{\gamma(t)}(G) \asymp K_t \quad \square \end{aligned}$$

Twisting number on a flat annulus

Let A be a flat annulus and η a path connecting boundary components of A.



The twisting number $\mathrm{tw}_A(\eta)$ is defined to be

$$\mathrm{tw}_A(\eta) = \frac{|y_1 - y_2|}{L}.$$

“Non”-twisting on flat annuli

Suppose $G = F + w\alpha$. Let β^* be the geodesic representative of β with respect to J_t on $\gamma(t)$.

Since there are no critical points of J_t in the charactristic annulus A_t of α, the intersection $\beta^* \cap A_t$ consists of (atmost $i(\alpha, \beta)$) straight lines connecting boundary components.

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The following implies that any almost geodesic ray behaves like a geodesic ray in view of markings.

Lemma (“Non”-twisting)

For each component σ of $\beta^* \cap A_t$,

$$\mathrm{tw}_{A_t}(\sigma) = o(K_t).$$

as $t \to \infty$.

[Proof] Let $q_t = J_t/\|J_t\|$. Let $\sigma_1, \cdots, \sigma_{n_0}$ be components of $\beta^* \cap A_t$. Let $\{\eta_j\}_j$ the straight segments in $\beta^* \setminus \cup_i \sigma_i$. Then,

$$\begin{aligned}
\|J_t\|^{-1} i(\beta, G) = i(\beta, V_{q_t}) \leq \ell_{q_t}(\beta^*) \\
&= \sum_{i=1}^{n_0} \sqrt{i(\sigma_i, H_{q_t})^2 + i(\sigma_i, V_{q_t})^2} + \sum_j \sqrt{i(\eta_j, H_{q_t})^2 + i(\eta_j, V_{q_t})^2} \\
&= \sum_{i=1}^{n_0} \sqrt{i(\sigma_i, H_{q_t})^2 + \|J_t\|^{-1} w^2} + \sum_j \sqrt{i(\eta_j, H_{q_t})^2 + i(\eta_j, V_{q_t})^2} \\
&\leq \mathrm{Ext}_{\gamma(t)}(\beta)^{1/2}
\end{aligned}$$

Since

$$\|J_t\| \cdot \mathrm{Ext}_{\gamma(t)}(\beta)^{1/2} = (1+o(1)) \cdot K_t^{-1/2} \cdot \mathrm{Ext}_{\gamma(t)}(\beta)^{1/2} \to i(\beta, G) = n_0 w + \sum_j i(\eta_j, V_{J_t}),$$

$$\sum_{s=1}^{n_0} \left(\sqrt{i(\sigma_s, H_{J_t})^2 + w^2} - w \right) + \sum_j \left(\sqrt{i(\eta_j, H_{J_t})^2 + i(\eta_j, V_{J_t})^2} - i(\eta_j, V_{J_t}) \right) \to 0$$

Therefore, for any $s = 1, \cdots, n_0$,

$$i(\sigma_s, H_{J_t}) \to 0.$$

Therefore, for any $s = 1, \cdots, n_0$,

$$i(\sigma_s, H_{J_t}) \to 0.$$

Notice

$$K_t \asymp \mathrm{Mod}(A_t) = w/\ell_{J_t}(\alpha).$$

Fix s. Let $\tilde{A}_t$ be the universal covering of A_t and y_1, y_2 be endpoints of a lift of σ_s. Then,

$$i(\sigma_s, H_{J_t}) = |y_1 - y_2|_{J_t} \quad (J_t\text{-height})$$

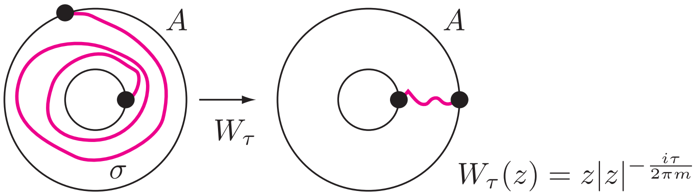
and

$$\mathrm{tw}_{A_t}(\sigma_s) = \frac{|y_1 - y_2|_{J_t}}{\ell_{J_t}(\alpha)} \asymp i(\sigma_s, H_{J_t})K_t = o(K_t).$$

as $t \to \infty$.

Twisting deformation on an annulus

Let A be a round annulus of modulus M. Let σ a path connecting components of ∂A. $\tau = \mathrm{tw}_A(\sigma)$.



By calculation,

$$\frac{\overline{\partial} W_\tau}{\partial W_\tau} = \frac{-i(\tau/m)}{4\pi - i(\tau/m)} \frac{z}{\overline{z}} \frac{d\overline{z}}{dz}.$$

In particular

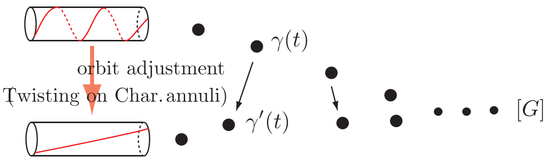
$$\left\| \frac{\overline{\partial} W_\tau}{\partial W_\tau} \right\|_\infty \to 0$$

when $\tau = o(M)$ as $M \to \infty$.

Orbit adjustment

By "Non"-twisting and Behavior of moduli, we can do twisting deformations on characteristic annuli such that the twisting number of β on each char. annulus is uniformly bounded (say 0 or 1) such that

$$d_T(\gamma(t), \gamma'(t)) \to 0 \quad (t \to 0).$$



Observation

Since β is really NON-twisted on the adjustment $\gamma'(t)$, we can apply the Kerckhoff's calculation of β on the adjustment $\gamma'(t)$!!

Summarize and Conclusion

We summarize the situation. Let γ be an almost geodesic ray converging to the projective class G of a maximal rational foliation

$$G = \sum_{i=1}^{k} w_i \alpha_i$$

with $k \geq 2$. Let $\beta \in \mathcal{S}$.

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...... After a lot of technical things ......

We can do an "orbit adjustment" to obtain new almost geodesic ray $\gamma'(t)$ such that β is not-twisted on the flat annulus of of the Hubbard-Masur differential J_t of G.

Recall that $A_{i,t}$ is the characteristic annulus of J_t for α_i.

We can apply Kerckhoff's calculation of the extremal length of β, and get (after taking subsequence)

$$\lim_{t\to\infty}\left(\frac{\mathrm{Ext}_{\gamma(t)}(\beta)}{K_t}\right)^{1/2} = \sqrt{\sum_{i=1}^{k} n_i^2 \frac{\mathrm{Mod}(A_{i,t})}{K_t}} = \sqrt{\sum_{i=1}^{k} n_i^2 M_i}$$

for some $M_i > 0$ where $n_i = i(\alpha, \beta)$.

Notice that M_i does NOT depend on $\beta \in \mathcal{S}$.

On the other hand, from the assumption, the limit above shoud be equal to $i(\beta, G)$. Hence

$$\sum_{i=1}^{k} M_i i(\alpha, \beta)^2 = i(\beta, G)^2 = \left(\sum_{i=1}^{k} w_i i(\alpha_i, \beta)\right)^2$$

for all $\beta \in \mathcal{S}$. Since the intersection number is continuous, the equality above holds for all $\beta \in \mathcal{MF}$.

We substitute $\beta = x\beta_1 + y\beta_2$ ($i(\beta_1, \beta_2) = 0$) to the equality and get

$$\sum_{i=1}^{k} M_i i(\alpha, x\beta_1 + y\beta_2)^2 = \left(\sum_{i=1}^{k} w_i i(\alpha_i, x\beta_1 + y\beta_2)\right)^2$$

and

$$\left(\sum_{i=1}^{k} M_i n_{1,i}^2\right)^2 x^2 + 2\left(\sum_{i=1}^{k} M_i n_{1,i} n_{2,i}\right) xy + \left(\sum_{i=1}^{k} M_i n_{2,i}^2\right)^2 y^2 = (\cdots)^2$$

where $n_{j,i} = \imath(\alpha_i, \beta_j)$. Hence the discriminant satisfies

$$\left(\sum_{i=1}^{k} M_i n_{1,i} n_{2,i}\right)^2 = \left(\sum_{i=1}^{k} M_i n_{1,i}^2\right)^2 \left(\sum_{i=1}^{k} M_i n_{2,i}^2\right)^2.$$

We substitute $\beta = x\beta_1 + y\beta_2$ ($i(\beta_1,\beta_2) = 0$) to the equality and get

$$\sum_{i=1}^{k} M_i i(\alpha, x\beta_1 + y\beta_2)^2 = \left(\sum_{i=1}^{k} w_i i(\alpha_i, x\beta_1 + y\beta_2)\right)^2$$

and

$$\left(\sum_{i=1}^{k} M_i n_{1,i}^2\right)^2 x^2 + 2\left(\sum_{i=1}^{k} M_i n_{1,i} n_{2,i}\right) xy + \left(\sum_{i=1}^{k} M_i n_{2,i}^2\right)^2 y^2 = (\cdots)^2$$

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This means that two vectors

$$(\sqrt{M_1} n_{1,1}, \cdots, \sqrt{M_k} n_{1,k}), \quad (\sqrt{M_1} n_{2,1}, \cdots, \sqrt{M_k} n_{2,k})$$

are always parallel for β_1 and β_2 with $i(\beta_1,\beta_2) = 0$. This is a contradiction.

Thank you for your attention.

and please do not forget to go outside for the workshop picture.