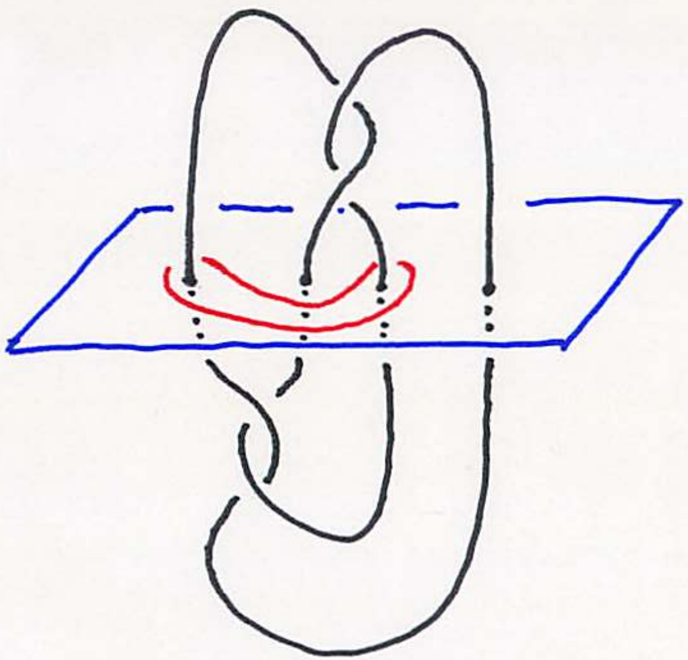
Essential Simple Loops on 2-bridge Spheres in 2-bridge Link Complements

– Dedicated to Professor Caroline Series on the occasion of her 60th birthday –



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作間 誠 (広島)

My personal encounter with Prof. Caroline Series

- Iain Aitchison's talk in Kobe (1992)

  The geometries of Markoff Numbers by C. Series

- Study of punctured torus groups (1996 ~)

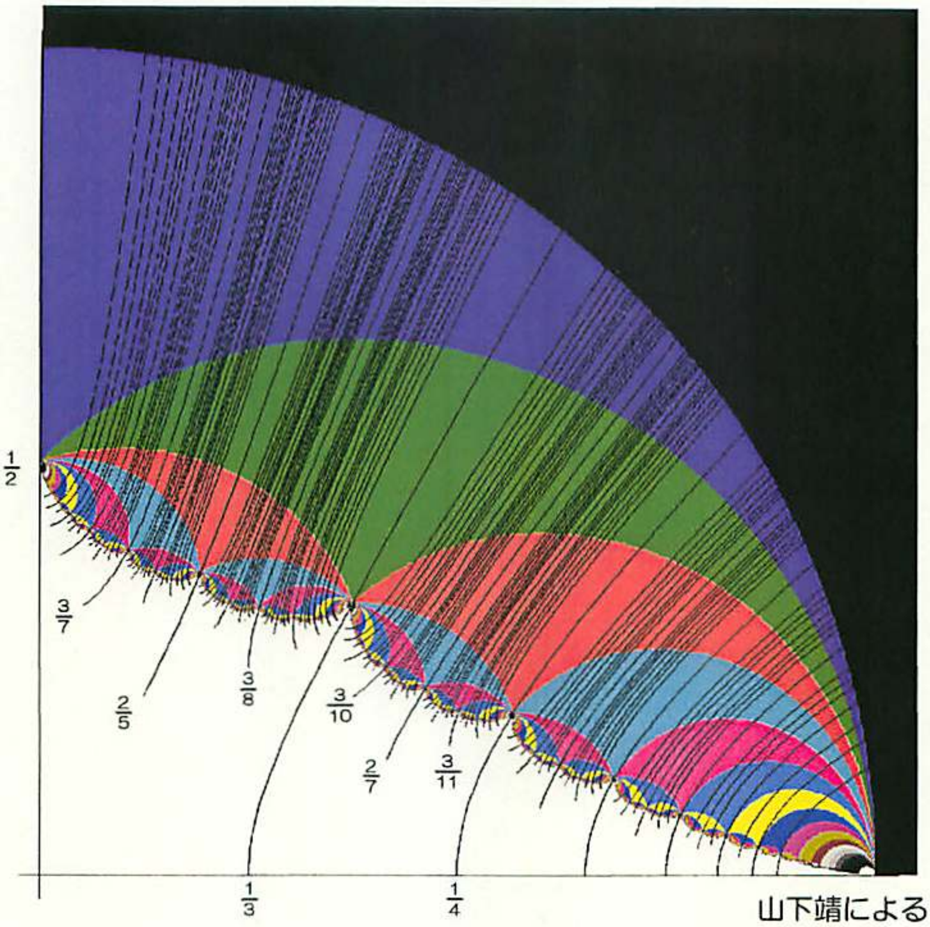
  Keen - Series theory of pleating coordinates

  [Akiyoshi - S - Wada - Yamashita] Jorgensen's theory and its extension

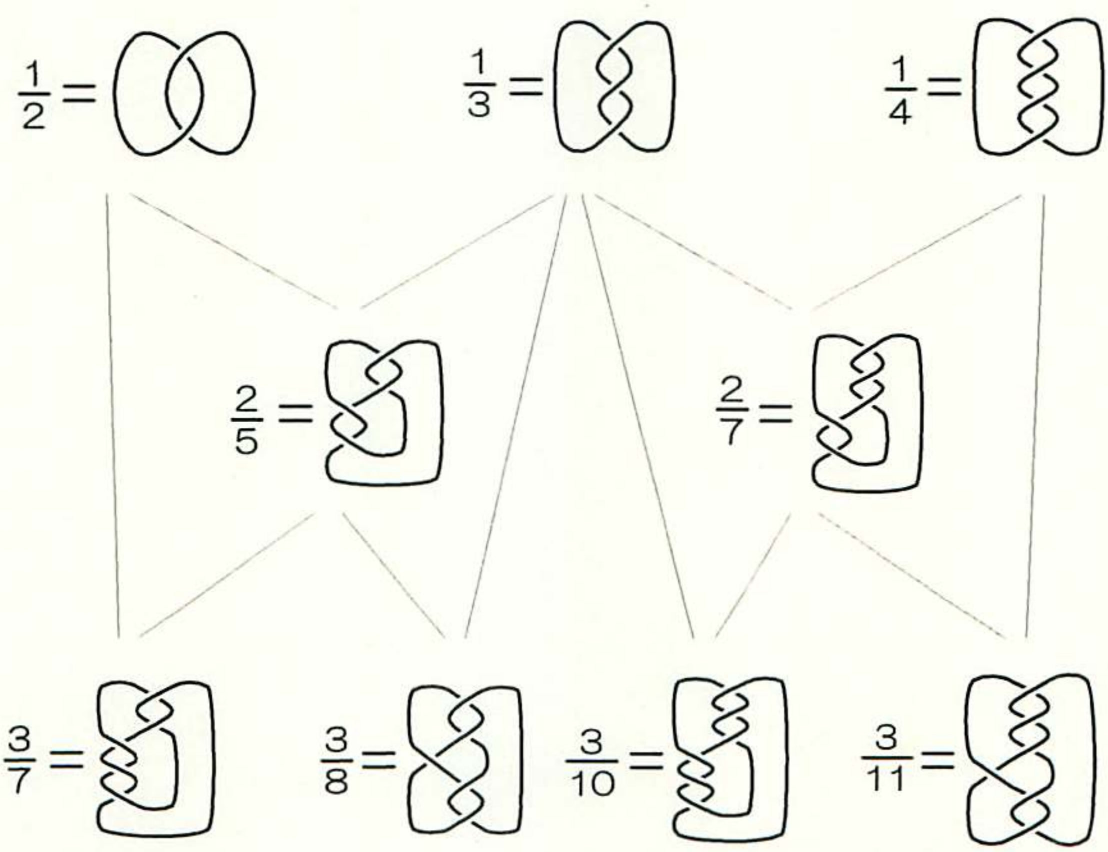
  Gueritaud proved the EPH-conjecture relating the two theories.

- Cone manifold conference in Tokyo organized by Kojima (1998)

ライリー切片



山下靖による



K : knot or link in S^3

S : punctured sphere in $S^3 - K$ obtained from a bridge sphere

Question

(1) For an essential simple loop in S,

when is it null-homotopic in $S^3 - K$?

: : peripheral : :

: : imprimitive : :

(2) For two essential simple loops in S,

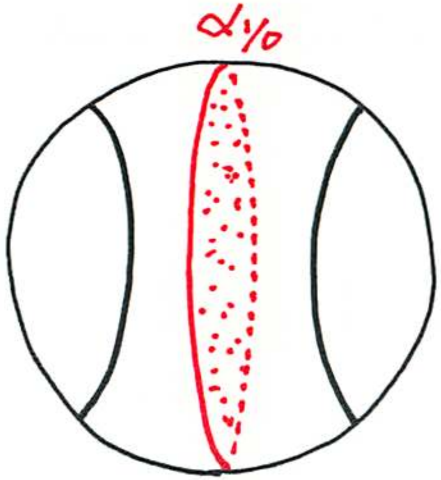
when are they homotopic in $S^3 - K$?

[Lee - S] Complete answers to the above questions for 2-bridge knots and links

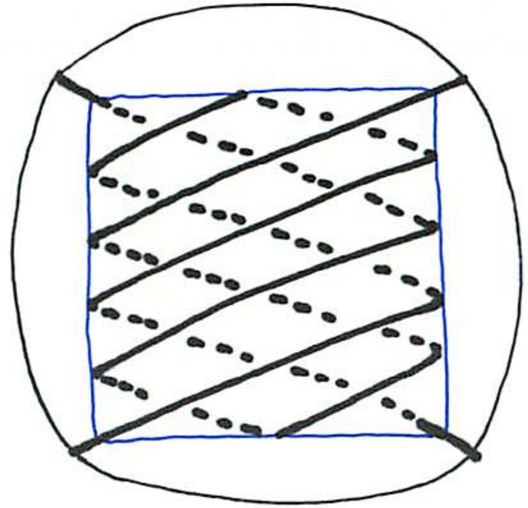
See arXiv 1004.2571, 1010.2232, 1103.0856

+ preliminary note, for exposition 1104.3462

Rational tangle $(B^3, t(r))$ of slope r :



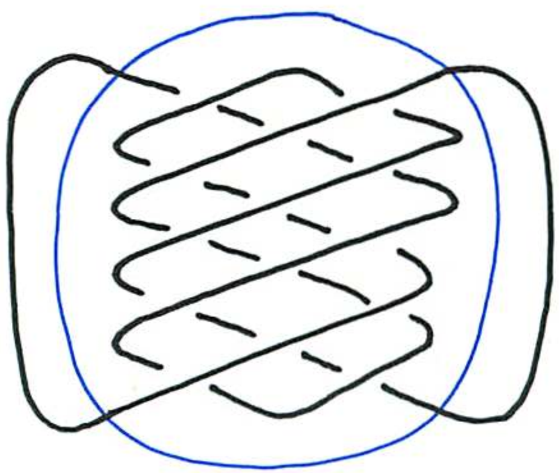
$(B^3, t(1/0))$



$(B^3, t(2/5))$

$$\pi_1(B^3 - t(r)) \cong \pi_1(S) / \langle\langle \alpha_r \rangle\rangle$$

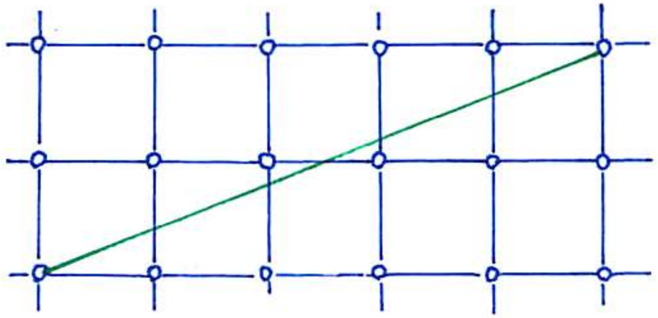
$(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$: 2-bridge link of slope r



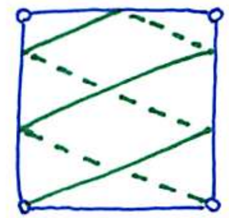
$$G(K(r)) := \pi_1(S^3 - K(r)) \cong \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle$$

$S := \mathbb{R}^2 - \mathbb{Z}^2 / \langle \pi\text{-rotations around punctures} \rangle$: 4-punctured sphere

(Conway sphere)

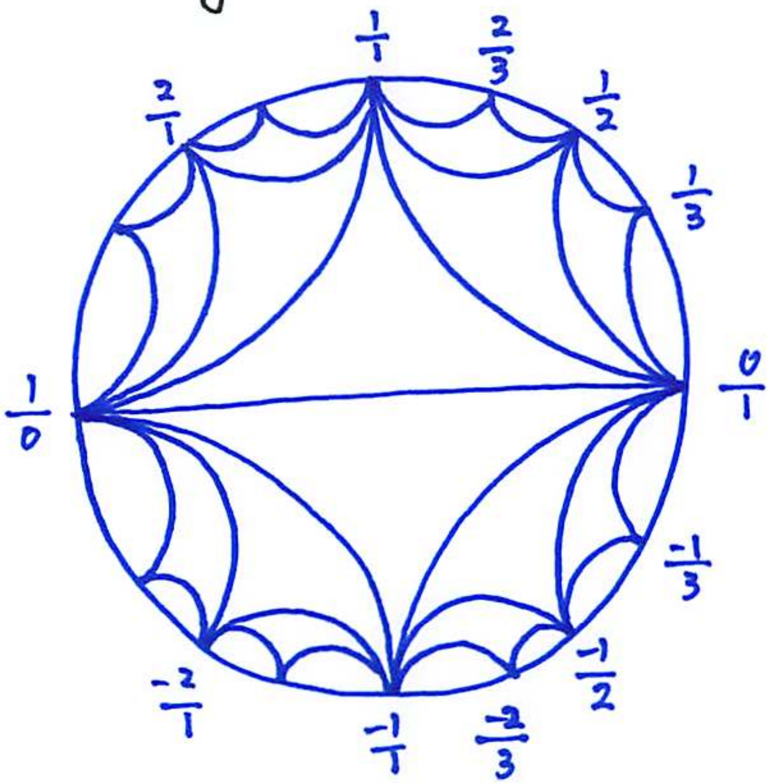


$\rightarrow$



$\delta_{2/5}$

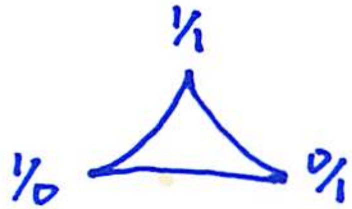
D : Farey tessellation



Vertex set of D $= \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\frac{1}{0}\} \ni r$

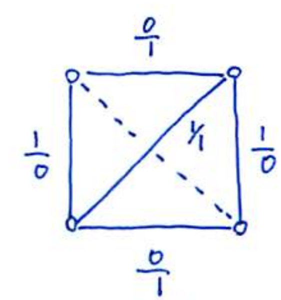
$\underset{1-1}{\leftrightarrow}$ {essential simple loops on S} $\ni \alpha_r$

$\underset{1-2}{\leftrightarrow}$ {essential simple arcs on S} $\ni \delta_r$



Farey triangle

$\leftrightarrow$

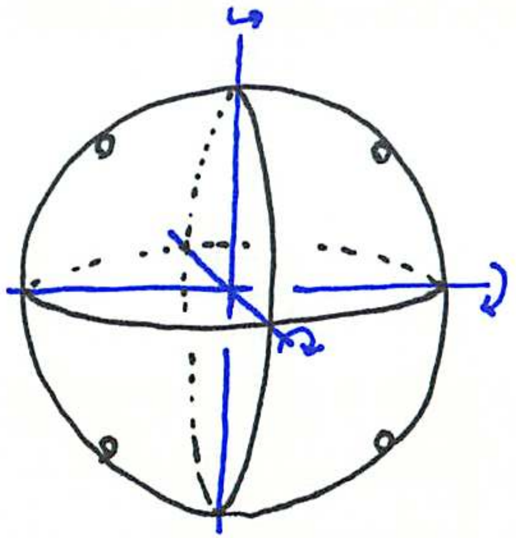


ideal triangulation of S

Mapping class group $\mathcal{M}(S) := \pi_0 \mathrm{Diff}(S)$

$$1 \to (\mathbb{Z}/2\mathbb{Z})^2 \to \mathcal{M}(S) \xrightarrow{\Phi} \underset{\substack{\cong \\ PGL(2,\mathbb{Z})}}{\mathrm{Aut}(\mathcal{D})} \to 1$$

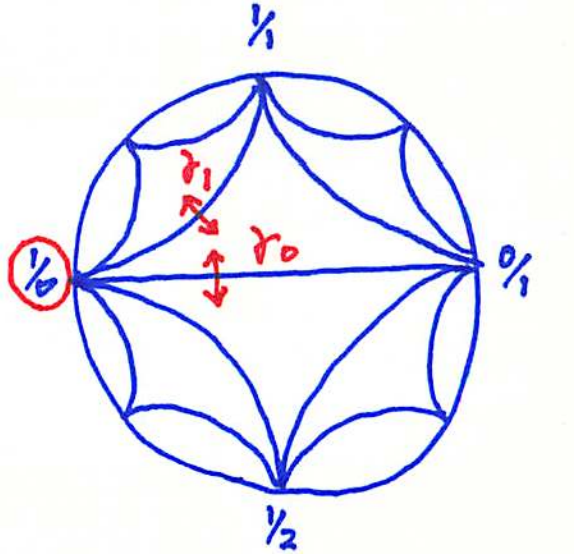
$(\mathbb{Z}/2\mathbb{Z})^2$-action on S acts trivially on $\mathcal{D}$.

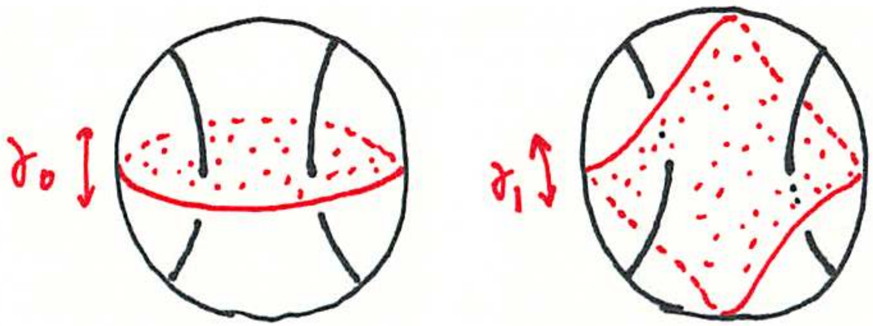


$$\begin{array}{l} \mathcal{M}(S) \overset{\Phi}{\twoheadrightarrow} \mathrm{Aut}(D) \\ \cup \\ \mathcal{M}(B^3, t(\infty)) := \pi_0 \mathrm{Diff}(B^3, t(\infty)) \\ \cup \\ \mathcal{M}_0(B^3, t(\infty)) := \{ f \in \mathcal{M}(B^3, t(\infty)) \mid f_* = \mathrm{id} \in \mathrm{Out}(\pi_1(B^3 - t(\infty))) \} \end{array}$$

Observation

$$\begin{array}{l} \mathrm{Aut}(D) \\ \cup \\ \Gamma_\infty := \Phi(\mathcal{M}_0(B^3, t(\infty))) = \left\langle \begin{array}{l} \text{reflections in the edges of } D \\ \text{with endpoint } \infty \end{array} \right\rangle \end{array}$$



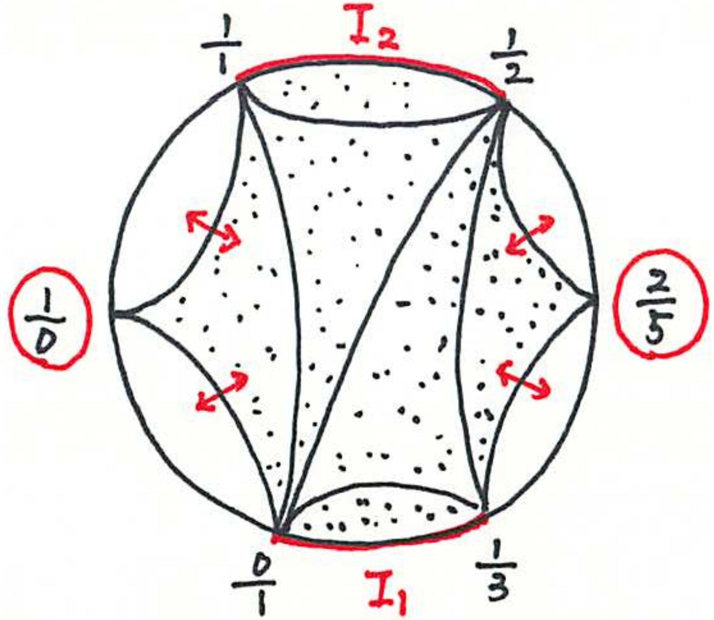


Similarly

$$\begin{array}{c} \mathrm{Aut}(D) \\ \cup \end{array}$$

$$\Gamma_r := \Phi(\pi_0(B^3, t(r))) = \left\langle \begin{array}{l} \text{reflections in the edges of } D \\ \text{with an endpoint } r \end{array} \right\rangle$$

Consider $\hat{\Gamma}_r := \langle \Gamma_\infty, \Gamma_r \rangle < \mathrm{Aut}(D)$



- The limit set $\Lambda(\hat{\Gamma}_r) =$ closure of $\hat{\Gamma}_r\{\infty, r\}$
- $I_1 \cup I_2$ is a fundamental domain of the action of $\hat{\Gamma}_r$ on the domain of discontinuity $\Omega(\hat{\Gamma}_r) := \partial \mathbb{H}^2 - \Lambda(\hat{\Gamma}_r)$

Observation [Ohtsuki - Riley - S]

(1) For any $s \in \hat{\mathbb{Q}}$, there is a unique $s_0 \in I_1 \cup I_2 \cup \{\infty, r\}$

st $s = \gamma(s_0)$ for some $\gamma \in \hat{\Gamma}_r$

(2) $\alpha_s \sim \alpha_{s_0}$ in $S^3 - K(r)$

(3) If $s_0 = \infty$ or r, then $\alpha_s \sim 1$ in $S^3 - K(r)$

(Proof of (2))

- If $s = \gamma(s_0)$ with $\gamma \in \Gamma_\infty$,

  then $\alpha_s \sim \alpha_{s_0}$ in $B^3 - t(\infty)$ and so in $S^3 - K(r)$.

- If $s = \gamma(s_0)$ with $\gamma \in \Gamma_r$

  then $\alpha_s \sim \alpha_{s_0}$ in $B^3 - t(r)$ and so in $S^3 - K(r)$.

Question Is the converse true ?

[Lee - S : arXiv : 1004.2571]

$\alpha_s \sim 1$ in $S^3 - K(r)$ iff $s \in \hat{\Gamma}_r\{\infty, r\}$.

i.e, if $s \in I_1 \cup I_2$, then $\alpha_s \not\sim 1$ in $S^3 - K(r)$.

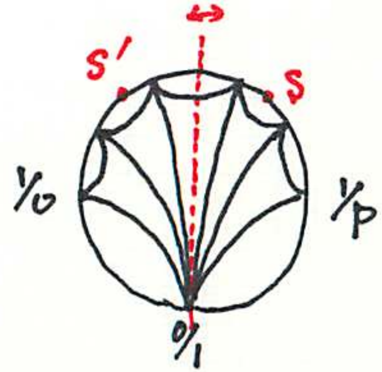
[Lee-S : arXiv : 1010.2232, 1103.0856, preliminary note]

$\alpha_s \sim \alpha_{s'}$ in $S^3 - K(r)$ for distinct $s, s' \in I_1 \cup I_2$,

iff one of the following holds :

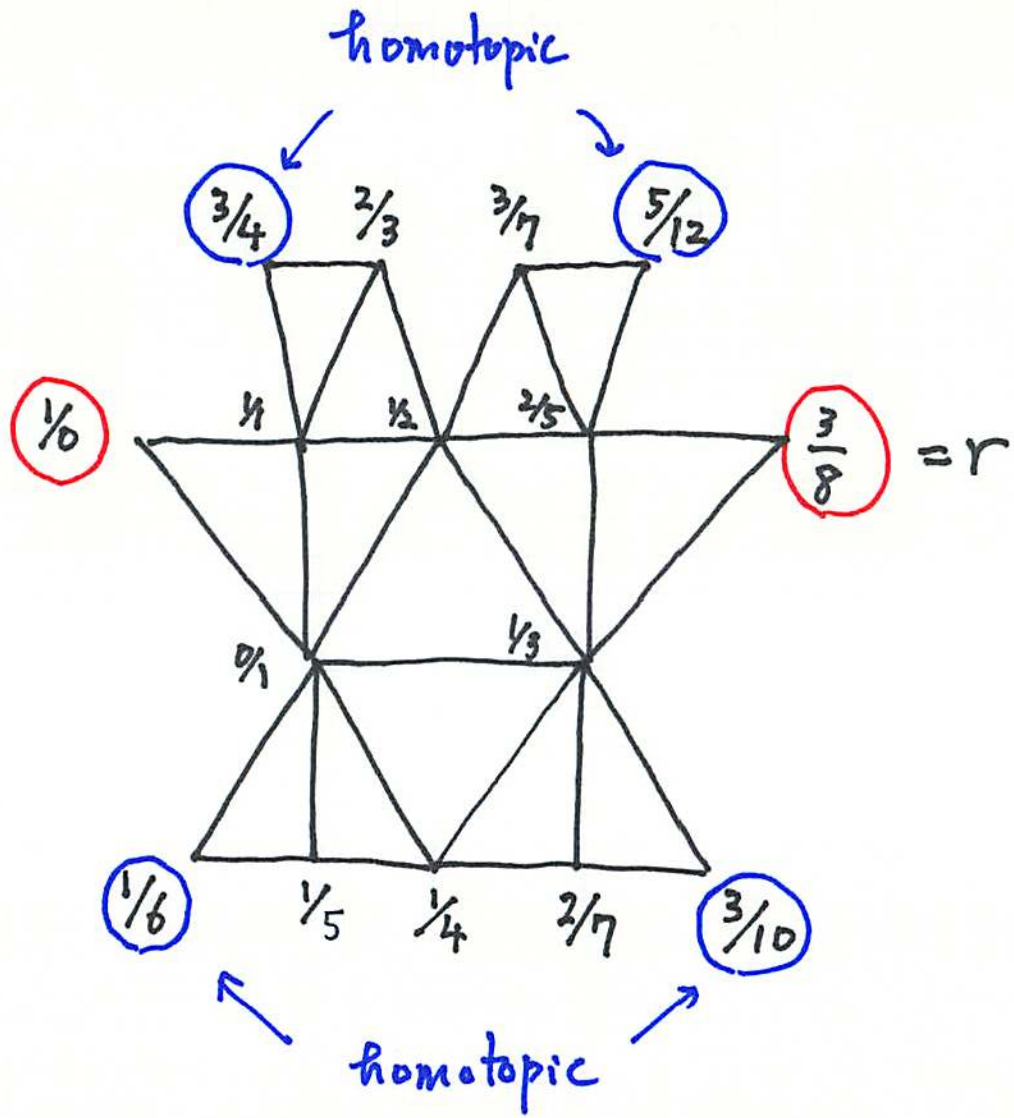
(1) $r = 1/p$ (2-bridge torus link) and $s = q_1/p_1$, $s' = q_2/p_2$

st $q_1 = q_2$ and $q_1/(p_1+p_2) = 1/p$.



(2) $r = 3/8$ (Whitehead link) and

$\{s, s'\} = \{1/6, 3/10\}$ or $\{3/4, 5/12\}$.

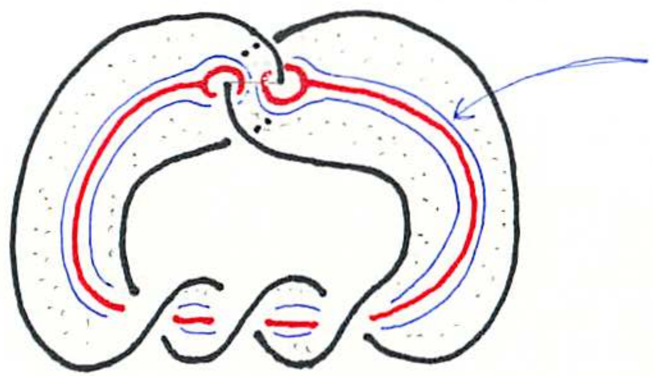

homotopic
3/4
2/3
3/7
5/12
1/0
1/1
1/2
2/5
3/8
= r
0/1
1/3
1/6
1/5
1/4
2/7
3/10
homotopic

[Peripheral Problem]

For a hyperbolic 2-bridge link $K(r)$,
the loop α_s $(s \in I_1 \cup I_2)$ is peripheral,
iff one of the following holds.

(1) $r = 2/5$ and $s = 1/5$ or $3/5$.

(2) $r = n/(2n+1)$ and $s = n+1/2n+1$



This peripheral loop
in $S^3 - K(n/2n+1)$
is isotopic to $\alpha_{n+1/2n+1}$
in $S^3 - K(n/2n+1)$.

[Primitiveness Problem]

For a hyperbolic 2-bridge link $K(r)$,
the loop α_s $(s \in I_1 \cup I_2)$ is not primitive
iff one of the following holds.

(1) $r = 2/5$ and $s = 2/7$ or $3/4$.
In this case, $\alpha_s = \beta^3$ for some primitive $\beta \in G(K(r))$.

(2) $r = 3/7$ and $s = 2/7$.
In this case, $\alpha_s = \beta^2$ for some $\beta \in G(K(r))$.

Question

For a hyperbolic link $K(r)$,
when the loop α_s is isotopic to a simple closed geodesic ?

Speculation following [Minsky, Geom. Top. Monograph 12]

- $(S^3, K) = (B_1^3, t_1) \underset{S}{\cup} (B_2^3, t_2)$ n-bridge decomposition
- $\mathcal{M}(S) = \pi_0 \mathrm{Diff}(S)$ where $S = \partial B_i^3 - t_i$

$\cup$

$\mathcal{M}_0(B_i^3, t_i) := \{ f \in \pi_0 \mathrm{Diff}(B_i^3, t_i) \mid f_* = id \in \mathrm{Out}(\pi_1(B_i^3 - t_i)) \}$

$\Gamma := \langle \mathcal{M}_0(B_1^3, t_1), \mathcal{M}_0(B_2^3, t_2) \rangle \subset \mathcal{M}(S)$

- $\mathcal{C}^{(0)}(S) = \{$essential simple loops on $S\}/$isotopy

$\cup$

$\Delta_i := \{$the boundaries of essential disks in $B_i^3 - t_i\}$

$\Delta := \Delta_1 \cup \Delta_2$

Observation If $\alpha \in \Gamma \cdot \Delta$, then $\alpha \sim 1$ in $S^3 - K$

Question Is the converse true ?

[Masur] $\mathcal{M}_0(B_i^3, t_i) \curvearrowright \mathcal{PML}(S)$ has a non-empty domain of discontinuity.

Question Suppose the bridge decomposition is "sufficiently complicated".

(1) Does $\Gamma = \langle \mathcal{M}_0(B_1^3, t_1), \mathcal{M}_0(B_2^3, t_2) \rangle \curvearrowright \mathcal{PML}(S)$ have a non-empty domain of discontinuity?

(2) $\Gamma \cong \mathcal{M}_0(B_1^3, t_1) * \mathcal{M}_0(B_2^3, t_2)$?

(3) Suppose $\alpha \in \mathcal{C}^{(0)}(S)$ is contained in the domain of discontinuity $\Omega(\Gamma) \subset \mathcal{PML}(S)$. Then can $\alpha \sim 1$ in $S^3 - K$?

(4) Does $cl\{\alpha \in \mathcal{C}^{(0)}(S) \mid \alpha \sim 1 \text{ in } S^3 - K(r)\} \subset \mathcal{PML}(S)$ have measure 0 ?

(Idea of Proof)

- Starting point is :

  [Keen - Series], [Komori - Series]

  (0) $\alpha_s \sim 1$ in $B^3 - t(\infty)$ iff $s = \infty$

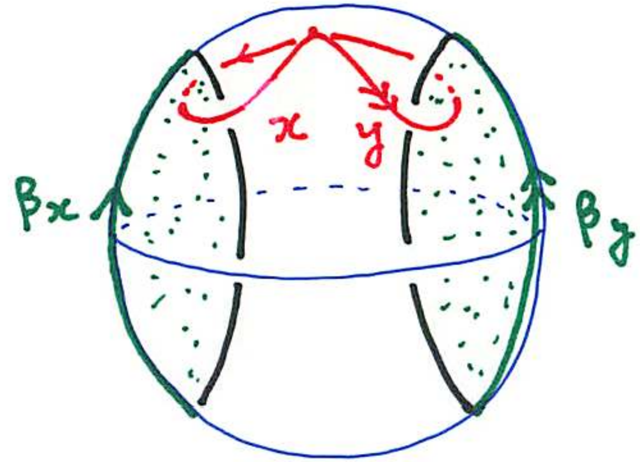
  (1) $\alpha_s \sim \alpha_{s'}$ in $B^3 - t(\infty)$ iff $s' \in \Gamma_\infty \cdot s$

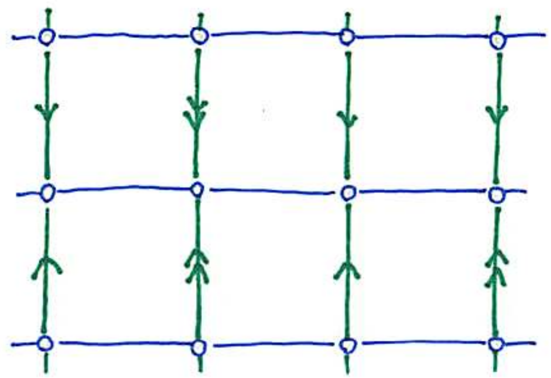
- Key tool is the Small Cancellation Theory, where analysis of the "cutting sequence" of a straight line in $\mathbb{R}^2$ plays a crutial role.

  cf. [Series : The geometry of Markoff numbers]

Upper presentation of $G(K(r)) = \pi_1(S)/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle = \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r \rangle\rangle$

$\pi_1(B^3 - t(\infty)) = \langle x, y \rangle$

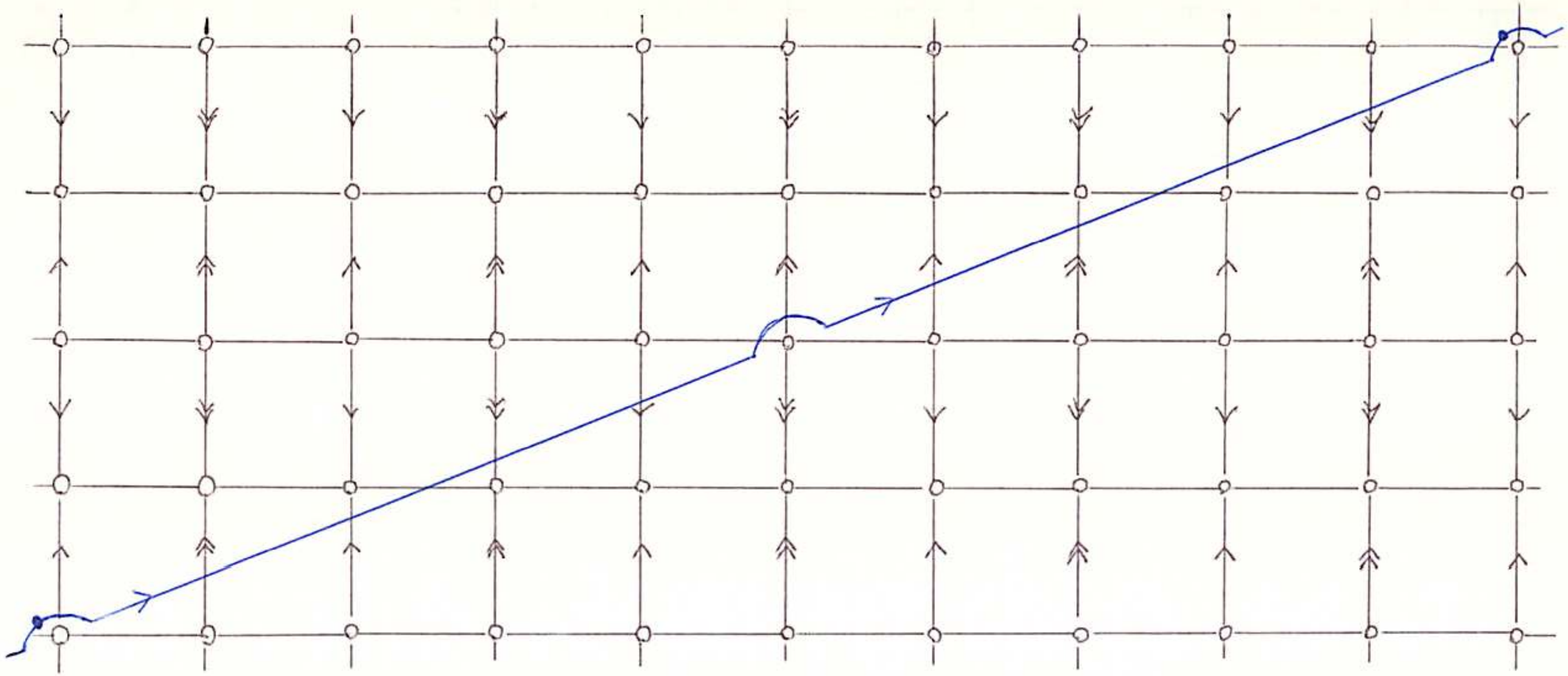




For a loop $\alpha \subset S$,

$[\alpha] \in \pi_1(B^3 - t(\infty)) = \langle x, y \rangle$

is obtained by "reading" the intersections of α with β_x and β_y.



$$[\alpha_{2/5}] := u_{2/5} = x \cdot y x \bar{y} \bar{x} \cdot y \cdot x y \bar{x} \bar{y}$$

$$= x y x \cdot \bar{y} \bar{x} \cdot y x y \cdot \bar{x} \bar{y}$$

$S(2/5) := S(u_{2/5}) := (3, 2, 3, 2)$ S-sequence

$CS(2/5) := ((3, 2, 3, 2))$ Cyclic S-sequence

Observation

- $u_{2/5} = x\, y\, x\, \bar{y}\, \bar{x}\, y\, x\, y\, \bar{x}\, \bar{y}$ is alternating,
  i.e. x and y apper alternatively.
- $u_{2/5}$ is determined by its S-sequence $(3, 2, 3, 2)$
  and the initial letter x.
- Any alternating word w with $S(w) = S(2/5)$ is
  conjugate to $u_{2/5}$ or $\bar{u}_{2/5}$.

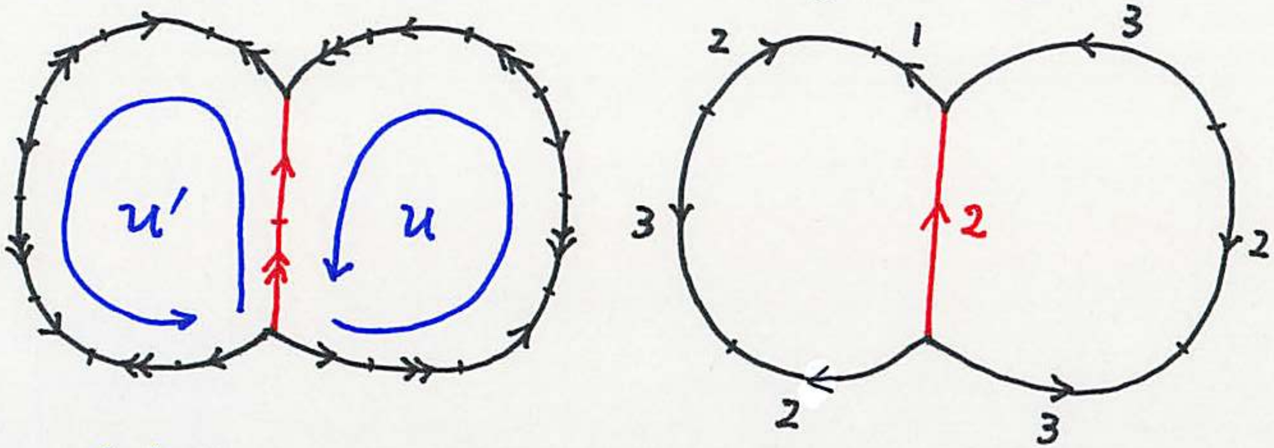
$$x\, y\, x\, \bar{y}\, \bar{x}\, y\, x\, y\, \bar{x}\, \bar{y} = u_{2/5}$$
$$y\, x\, y\, \bar{x}\, \bar{y}\, x\, y\, x\, \bar{y}\, \bar{x} \sim u_{2/5}$$
$$\bar{x}\, \bar{y}\, \bar{x}\, y\, x\, \bar{y}\, \bar{x}\, \bar{y}\, y\, x \sim \bar{u}_{2/5}$$
$$\bar{y}\, \bar{x}\, \bar{y}\, x\, y\, \bar{x}\, \bar{y}\, \bar{x}\, y\, x \sim \bar{u}_{2/5}$$

- Conjugacy class of $\{u_{2/5}, \bar{u}_{2/5}\}$ is determined by $CS(2/5)$.

- $G(K(2/5)) = \langle x, y \mid u_{2/5} \rangle$, $u_{2/5} = x y x \bar{y} \bar{x} y x y \bar{x} \bar{y}$

- van Kampen diagram over $\{u_r\}$

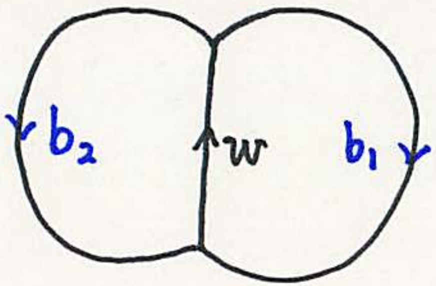
  = simply connected 2-dim cell complex in $\mathbb{R}^2$, where each oriented edge is labeled with an element in $F[x, y]$.
  st. the boundary label of a 2-cell is a cyclically reduced word representing the cyclic word $u_r^{\pm 1}$.



boundary label $= u \cdot u' = x y x \bar{y} \bar{x} y x y \underbrace{\bar{x} \bar{y}} \cdot \underbrace{y x} y \bar{x} \bar{y} x y x \bar{y} \bar{x}$

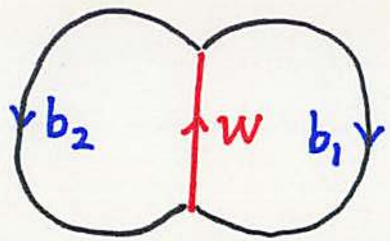
piece

- Reducible pair in a van-Kampen diagram



where $b_1 = b_2$

- A van-Kampen diagram is reduced if it has no reduced pairs.



- $b_1 \neq b_2$ though wb_1 and wb_2 are cyclic conjugates of $u_r^{\pm 1}$.
- w is called a piece.

Key Lemma : a complete characterization of pieces for $\{u_r\}$.

Cor $\{u_r\}$ satisfies the condition $C(4)$.
i.e. the cyclic word u_r is not a product of 3 $(= 4-1)$ pieces.

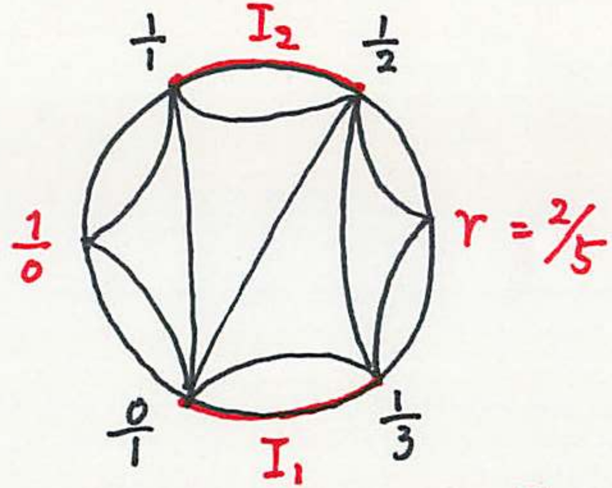
(Proof of "$\alpha_s \neq 1$ in $S^3 - K(r)$ if $s \in I_1 \cup I_2$")

Show that there is no reduced van-Kampen diagram with boundary label α_s $(s \in I_1 \cup I_2)$.

Intuition behind the proof

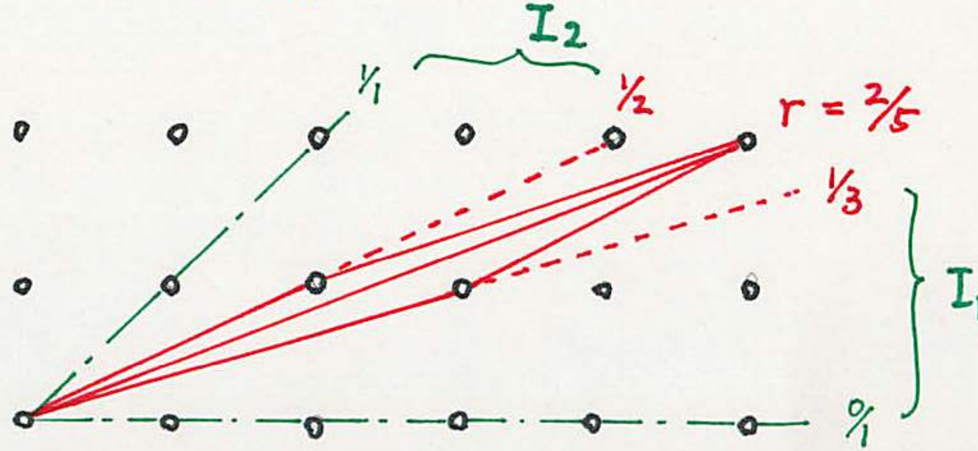
$s \in I_1 \cup I_2$

$\Leftrightarrow$ The slope s is far from ∞ and r



$\Leftrightarrow$ α_s and $u_r = \alpha_r$ cannot share a long subword.

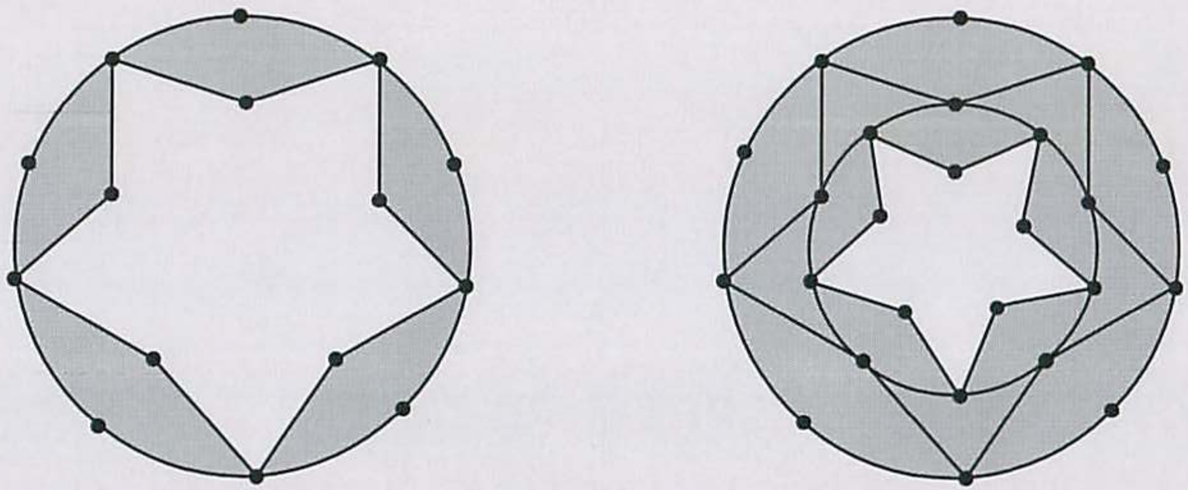
ie α_s admits only small cancellations.



(Proof of Conjugacy Theorem)

Structure Theorem

The following illustrates the only possible annular diagrams between α_s and $\alpha_{s'}$ with $s, s' \in I_1 \cup I_2$.



The proof of Conjugacy Thm is divided into the following 3 cases.

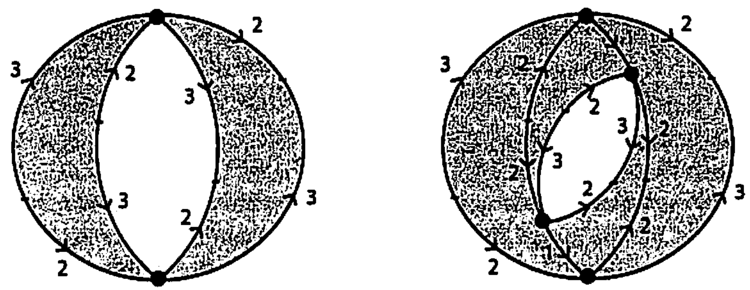
(I) $r = 1/p$

(II) $r = [m, n]$ or $[m, 2, n]$ (various exceptional homotopies)

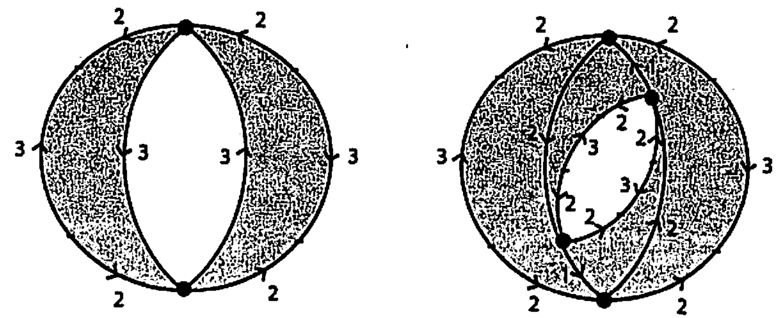
(III) Inductive argument for general cases.

Remark The figure-eight knot case is the most complicated!

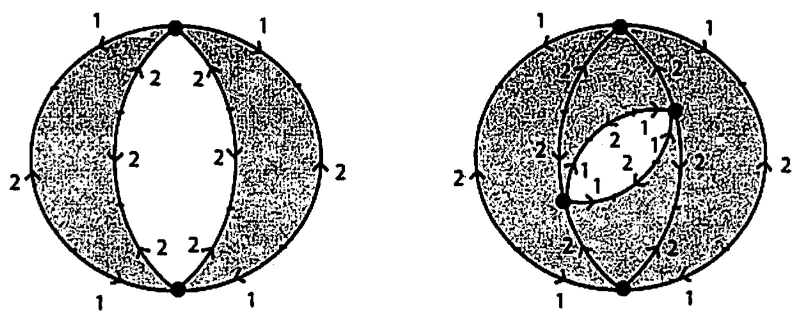
Annular diagrams for $G(K(2/5))$



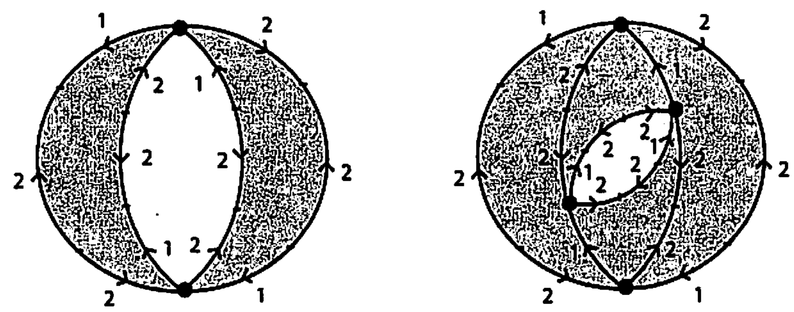
$u_{1/6}$ is commutative with a meridian, and so peripheral



$u_{2/7} \sim (yx)^3$

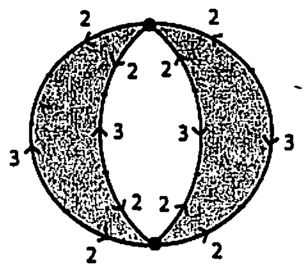


$u_{3/4} \sim (\bar{y}\,\bar{x}\,y\,x)^3$



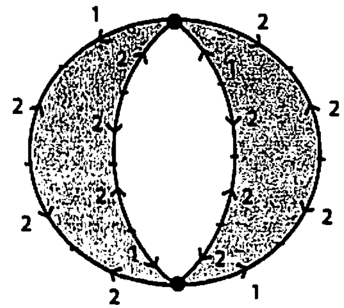
$u_{3/5}$ is peripheral, because it is commutative with a meridian

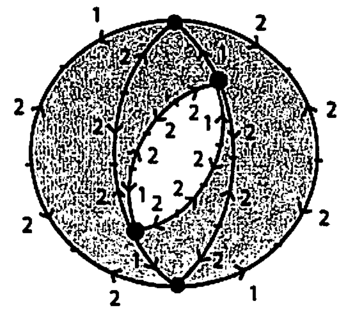
Annular diagram for $G(K(\frac{n}{2n+1}))$ $(n \geqq 3)$



$u_{3/7} = (x y^2 x \bar{y} \bar{x} \bar{y})^2$

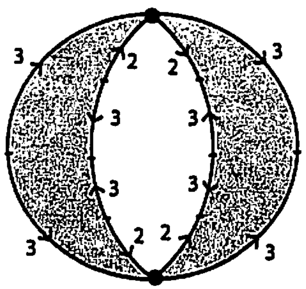
in $G(K(3/7))$





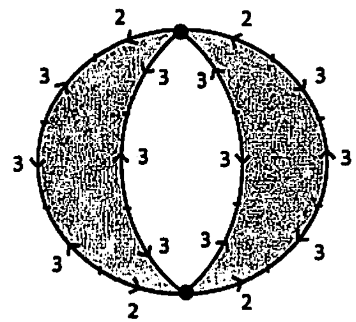
$u_{\frac{n+1}{2n+1}}$ is peripheral, because it is commutative with a meridian.

Annular diagram for $G(K(3/8))$



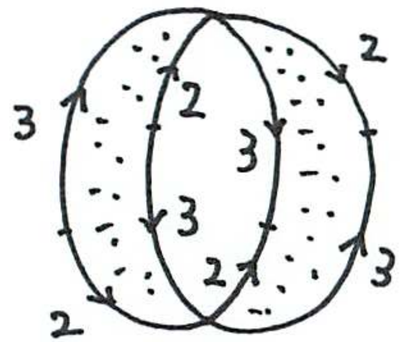
$u_{1/6} \sim u_{3/8}$

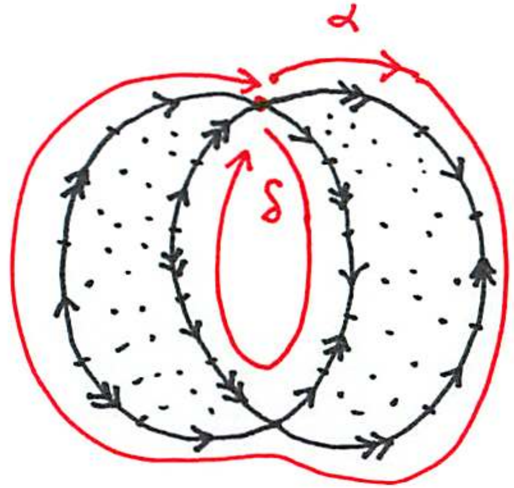
in $G(K(3/10))$



$u_{3/4} \sim u_{5/12}$

in $G(K(3/8))$





Outer boundary label $\alpha = y x \bar{y} \bar{x} \bar{y} \bar{x} \bar{y} x y x$

$= \bar{x} \bar{y} \bar{x} (x y x y x \bar{y} \bar{x} \bar{y} \bar{x} \bar{y}) x y x$

$= w_1 \, u_{1/5} \, \overline{w_1} \qquad (w_1 = \bar{x} \bar{y} \bar{x})$

Inner boundary label $\delta = x y x \bar{y} \bar{x} \bar{y} \bar{x} \bar{y} x y$

$= \bar{y} \bar{x} (x y x y x \bar{y} \bar{x} \bar{y} \bar{x} \bar{y}) x y$

$= w_2 \, u_{1/5} \, \overline{w_2} \qquad (w_2 = \bar{y} \bar{x})$

Hence $w_1 u_{1/5} \overline{w_1} = w_2 u_{1/5} \overline{w_2}$

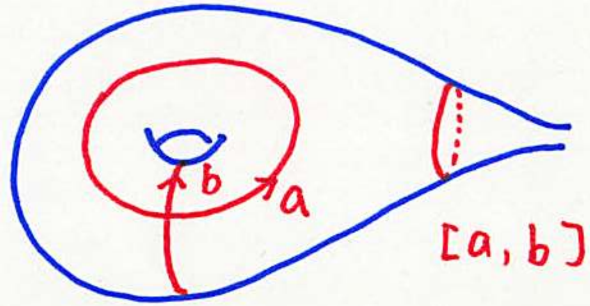
So $u_{1/5}$ is commutative with $\overline{w_1} w_2 = (x y) x (\bar{y} \bar{x})$ = a meridian

Hence $u_{1/5}$ is peripheral.

Applications

- A variation of McShane's identity for 2-bridge links
- Bowditch, Tan-Wong-Zhang's end invariants of $SL(2,\mathbb{C})$-representations of $\pi_1(T)$ where T is the once-punctured torus

T : once-punctured torus



$\pi_1(T) = \langle a, b \mid - \rangle$

Ψ

$[a, b]$: peripheral

Def $\rho : \pi_1(T) \to (P)SL(2, \mathbb{C})$ is type-preserving, if

(1) $\rho([a, b])$ is parabolic

(2) ρ is irreducible

$$\tilde{\mathcal{R}} := \{ \rho : \pi_1(T) \to SL(2, \mathbb{C}) \text{ type-pres} \} / \text{conj}$$

$$\downarrow \ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

$$\mathcal{R} := \{ \rho : \pi_1(T) \to PSL(2, \mathbb{C}) \text{ type-pres} \} / \text{conj}$$

Fact For ${}^{\forall}\rho \in \tilde{\mathcal{R}}$, $(x, y, z) := (\operatorname{tr}\rho(a), \operatorname{tr}\rho(ab), \operatorname{tr}\rho(b))$ is a nontrivial Markoff triple, i.e.

$$x^2 + y^2 + z^2 = xyz$$
$$(x, y, z) \neq (0, 0, 0)$$

$$\begin{array}{ccl} \tilde{\mathcal{R}} & \cong & \Phi := \{\text{nontrivial Markoff triples}\} \\ \downarrow & & \downarrow \\ \mathcal{R} & \cong & \Phi/\sim \end{array}$$

where $(x, y, z) \sim (x, -y, -z)$
$\qquad\qquad\qquad\quad\; (-x, y, -z)$
$\qquad\qquad\qquad\quad\; (-x, -y, z)$

D : Farey tessellation

$$D^{(0)} = \hat{\mathbb{Q}} = \{\text{essential simple loops on } T\} / \text{isotopy}$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ s & \longleftrightarrow & \beta_s \end{array}$$

Each $\rho \in \tilde{R}$ determines a Markoff map

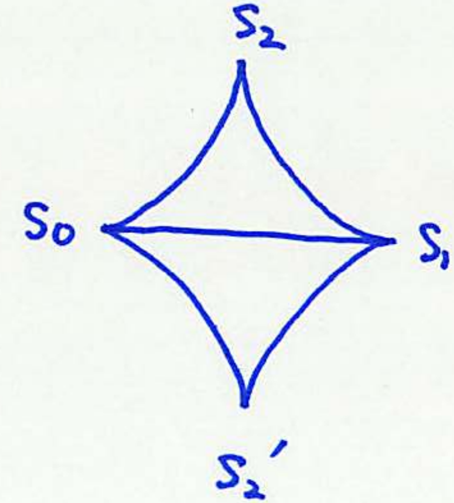
$$\phi : D^{(0)} \longrightarrow \mathbb{C}$$

$$s \longmapsto \mathrm{tr}(\rho(\beta_s))$$

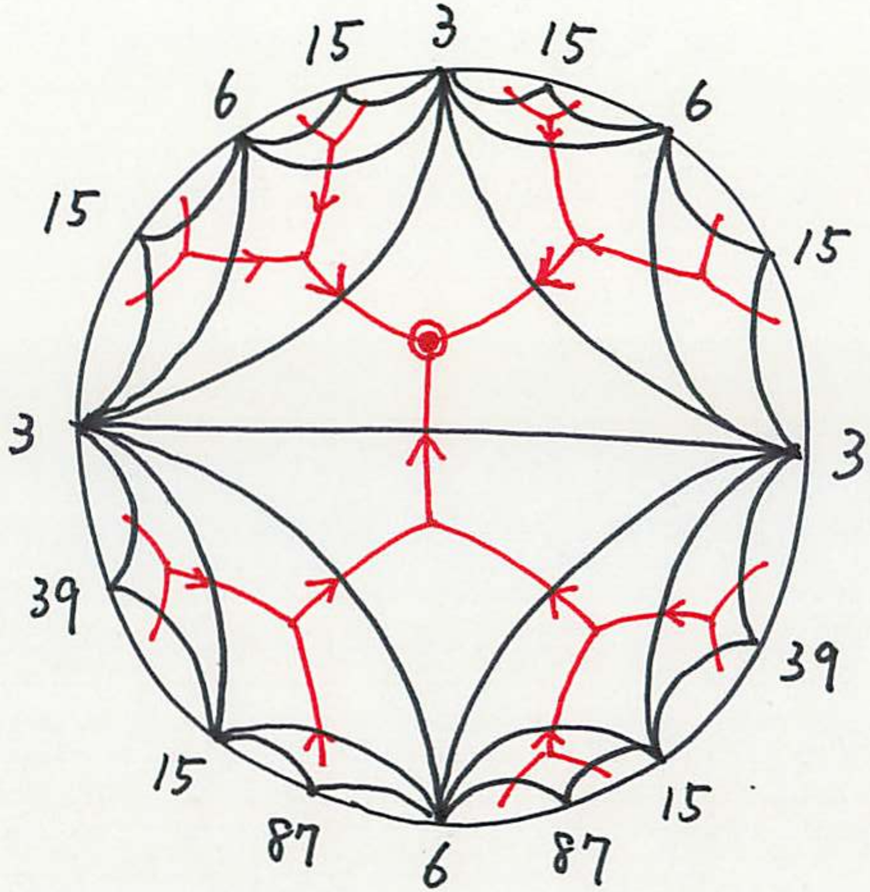
(i) $(x, y, z) := (\phi(s_0), \phi(s_1), \phi(s_2))$

is a Markoff triple

(ii) $z + w = xy$ where $w = \phi(s_2')$

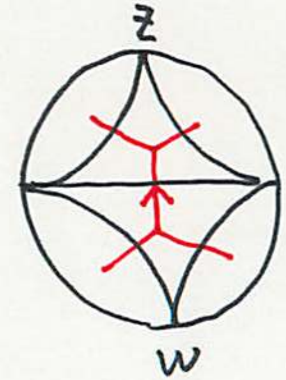


Integral Markoff map



$\exists$ unique sink

where



$|z| < |w|$

[Bowditch] [Tan - Wong - Zhang]

- $\lambda \in \partial \mathbb{H}^2 = \hat{\mathbb{R}}$ is an end invariant of ρ. if

  $\exists K > 0$, $\exists \{r_n\}$ distinct elements of $\hat{\mathbb{Q}}$, st

  (i) $r_n \to \lambda$

  (ii) $|\phi(r_n)| < K$ for $\forall n$ (Recall $\phi(r_n) = tr(\rho(\beta_n))$)

- $\mathcal{E}(\rho) := \{$ end invariants of $\rho \} \subset \hat{\mathbb{R}}$

Examples

(1) If $\rho \leftrightarrow (3, 3, 3)$, then $\mathcal{E}(\rho) = \emptyset$

(2) If ϕ is real (ie $\phi(\hat{\mathbb{Q}}) \subset \mathbb{R}$), then

ρ is Fuchsian and hence $\mathcal{E}(\rho) = \emptyset$.

(3) If ρ is quasi-fuchsian, then $\mathcal{E}(\rho) = \emptyset$

<u>Conjecture (Bowditch)</u>

ρ is quasi-fuchsian iff $\mathcal{E}_B(\rho) = \emptyset$,

where $\mathcal{E}_B(\rho) = \mathcal{E}(\rho) \cup \{ s \in \hat{\mathbb{Q}} \mid \phi(s) = \pm 2 \}$

ρ is $\underbrace{\text{geometrically}}_{\text{discrete, faithful and}}$ finite iff $\mathcal{E}(\rho) = \emptyset$.

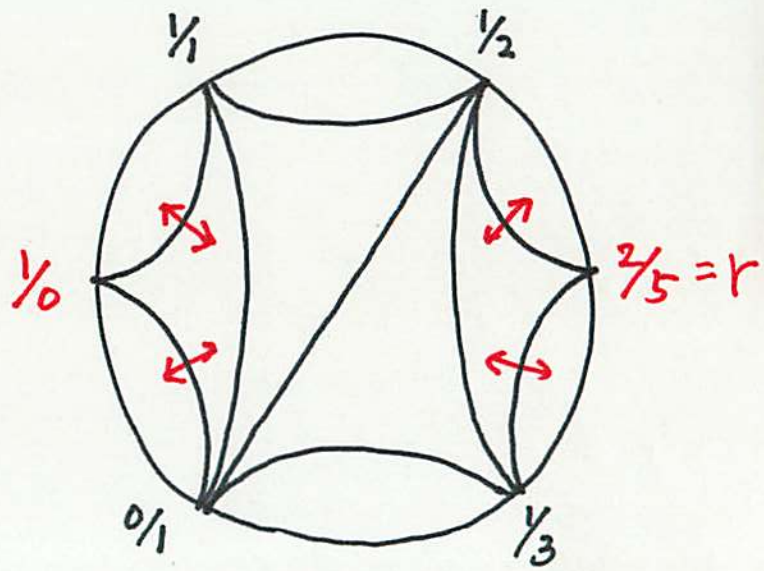
[Tan-Wong-Zhang]

(1) $\mathcal{E}(\rho)$ is a closed subset of $\hat{\mathbb{R}}$

(2) If ρ is discrete in the sense that $\phi(\hat{\mathbb{Q}})$ is a discrete subset of $\mathbb{C}$, and if $\mathcal{E}(\rho)$ has at least 3 elements, then $\mathcal{E}(\rho)$ is a Cantor set or $\hat{\mathbb{R}}$.

Main Theorem

If ρ corresponds to the discrete faithful representation of a hyperbolic 2-bridge knot/link $K(r)$, then $\mathcal{E}(\rho) = \Lambda(\hat{\Gamma}_r)$, the limitset of the reflection group $\hat{\Gamma}_r$ as in the figure.



$\hat{\Gamma}_{2/5}$

(Proof of the main theorem $\mathcal{E}(\rho) = \Lambda(\hat{P}_r)$)

- $\Lambda(\hat{P}_r) \subset \mathcal{E}(\rho)$ is obvious
- $\mathcal{E}(\rho) \subset \Lambda(\hat{P}_r) \iff \mathcal{E}(\rho) \cap \Omega(\hat{P}_r) = \emptyset$

$$\iff \mathcal{E}(\rho) \cap (I_1 \cup I_2) = \emptyset$$

The last property follows from the following facts.

(1) $(I_1 \cup I_2) \cap \mathbb{Q} \to \{\text{loops in } S^3 - K(r)\}/\text{homotopy}$

$$s \longmapsto \alpha_s$$

is almost 1-1

(2) $\#\{s \in (I_1 \cup I_2) \cap \mathbb{Q} \mid \alpha_s : \text{peripheral}\} < +\infty$

(3) Discreteness of the length spectrum for geometrically finite hyperbolic manifolds.

Conjecture

If $\rho \in R$ satisfies $\mathcal{E}(\rho) = \Lambda(\hat{\Gamma}_n)$,

then ρ corresponds to the holonomy representation

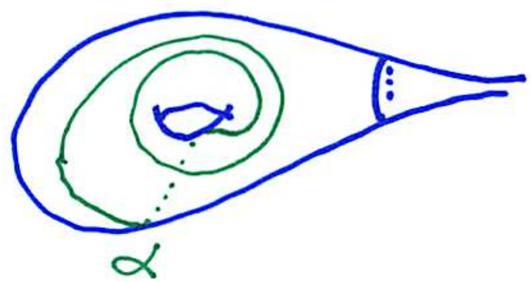
of the complete hyperbolic manifold $S^3 - K(r)$.

McShane's identity

For any hyperbolic once-punctured torus T

$$\sum_{\alpha} \frac{1}{1 + \exp(l(\alpha))} = \frac{1}{2}$$

where α runs over essential simple closed geodesics



- Generalization + Application : Mirzakhani, Tan-Wong-Zhang
- 3-dim variation : Bowditch, Akiyoshi-Miyachi-S.

- $K(r)$: hyperbolic, i.e. $S^3 - K(r)$ admits a complete hyperbolic structure of finite volume

$\Leftrightarrow$ $q \not\equiv \pm 1 \mod p$, where $r = q/p$ $\Leftrightarrow$ $d(\infty, r) \geqq 3$

- $\rho_r : \pi_1(S) \twoheadrightarrow \pi_1(S)/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle \cong G(K(r)) \hookrightarrow PSL(2, \mathbb{C})$

$\Updownarrow$

$\rho_r : \pi_1(T) \longrightarrow PSL(2, \mathbb{C})$

where $T := \mathbb{R}^2 - \mathbb{Z}^2 / \mathbb{Z}^2$ is a punctured torus

($\because$) T and S are "commensurable"

- ρ_r is a discrete non-faithful representation of $\pi_1(T)$ or $\pi_1(S)$

Main Theorem (Variation of McShane's identity) For $\rho = \rho_r$,

$$2 \sum_{s \in \mathring{I}_1} \frac{1}{1 + \exp(\lambda(\rho(\alpha_s)))} + \sum_{s \in \partial I_1} \frac{1}{1 + \exp(\lambda(\rho(\alpha_s)))}$$

$$= -1 - 2 \sum_{s \in \mathring{I}_2} \frac{1}{1 + \exp(\lambda(\rho(\alpha_s)))} - \sum_{s \in \partial I_2} \frac{1}{1 + \exp(\lambda(\rho(\alpha_s)))}$$

= Modulus of the cusp torus

with a suitable choice of (meridian and) longitude.