Hitting measures on $\mathcal{PMF}$

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Random walks on groups

Let G be a group with a finite generating set S. Let $C_S(G)$ be the Cayley graph of G w.r.t S. The *nearest neighbor random walk on* G is a random walk on $C_S(G)$.

General setup:

- μ: probability distribution on G.
- $w_n = g_1 g_2 .... g_n$ is a *sample path* of length n where each increment g_i is sampled by μ.
- Distribution of w_n is $\mu^{(n)}$.

$$\mu^{(2)}(g) = \sum_h \mu(h)\mu(h^{-1}g)$$

μ-boundaries for random walks

(Furstenberg)

- G acting on a topological space B
- After projection to B, a.e. sample path converges in B.

Examples:

- $S^1 = \partial\mathbb{H}$ is a μ-boundary for $SL(2, \mathbb{R})$.
- The space of full flags is a μ-boundary for $SL(d, \mathbb{R})$.
- $\mathcal{PMF} = \partial\mathcal{T}(S)$ is a μ-boundary for $Mod(S)$.

Teichmüller space and the mapping class group

Let S be an orientable surface with non-negative Euler characteristic.

- Mapping class group:

$$Mod(S) = \pi_0(Diffeo^+(S))$$

- Teichmüller space:

$$\mathcal{T}(S) = \text{marked conformal structures on } S \text{ modulo isotopy}$$

- $Mod(S)$ acts on $\mathcal{T}(S)$ by changing the marking. The quotient

$$M = \mathcal{T}(S)/Mod(S)$$

is the moduli space of curves.

- Thurston compactification:

$$\overline{\mathcal{T}(S)} = \mathcal{T}(S) \sqcup \mathcal{PMF}$$

Random walks on $Mod(S)$

Theorem (Maher, Rivin)

pseudo-Anosov mapping classes are generic with respect to random walks.

- Rivin: quantitative but applies to $< Supp(\mu) > \twoheadrightarrow Sp(2g, \mathbb{Z})$.
- Maher: applies to the Torelli group but is less quantitative.

Theorem (Kaimanovich-Masur)

Fix $X \in \mathcal{T}(S)$. If $< Supp(\mu) >$ is non-elementary then for a.e sample path the sequence $w_n X$ converges to $\mathcal{PMF} = \partial\mathcal{T}(S)$.

- This defines hitting measure h on $\mathcal{PMF}$.
- Furthermore, they show $h(\mathcal{PMF} \setminus UE) = 0$. By Klarreich's theorem, no information is lost if the random walk is projected to curve complex (or relative space) instead of $\mathcal{T}(S)$.

Applications of Kaimanovich-Masur

- *Farb-Masur rigidity*: A homomorphic image in $Mod(S)$ of a lattice of $\mathbb{R}$-rank $\geqslant 2$ is finite.

compare to

- *Furstenberg rigidity*: No lattice in $SL(d, \mathbb{R})$; $d \geqslant 2$ is isomorphic to a subgroup of $SL(2, \mathbb{R})$.

Hitting measures

Lebesgue measure class on $\mathcal{PMF}$:

- *MF* has piecewise linear structure by maximal train tracks.
- Projectivizing, get charts on $\mathcal{PMF}$ with Lebesgue measure.
- Transition functions are absolutely continuous.

The main theorem:

Theorem (G)

If μ finitely supported and $< Supp(\mu) >$ non-elementary then h is singular w.r.t Lebesgue.

Theorem (Guivarc'h-LeJan)

For a non-compact lattice $G < SL(2,\mathbb{R})$ ($\mathbb{H}/G$ finite volume), h is singular w.r.t Lebesgue on S^1.

Analogy really lies in the proof.

Hitting measures continued

- *Conjecture* (Guivarc'h-Kaimanovich-Ledrappier): true for any lattice in $SL(2,\mathbb{R})$.
- Kaimanovich-LePrince have examples of initial distributions on any Zariski dense subgroup of $SL(d,\mathbb{R})$ that are singular on the boundary.
- *Conjecture* (Kaimanovich-LePrince): true for any lattice in $SL(d,\mathbb{R})$.
- McMullen has an example of a non-discrete subgroup of $SL(2,\mathbb{R})$ for which experiments suggest that h is absolutely continuous on S^1. Also some examples by Peres-Simon-Solomyak.

$SL(2, \mathbb{Z})$

- $SL(2, \mathbb{Z})$ is quasi-isometric to the tree dual to the Farey tessellation.

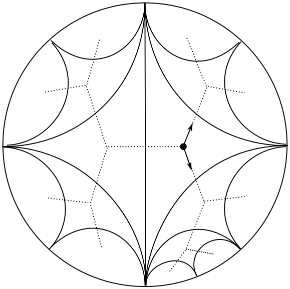

Figure: Farey graph and the dual tree

- With the base-point as shown, every $r \in (0, 1) \setminus \mathbb{Q}$ is encoded by an infinite path $R^{a_1} L^{a_2}$.....

- In fact,

$$r = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

  which is the classical connection to continued fractions.
- Distribution of a_n w.r.t Lebesgue:

$$\ell(a_n \geqslant m) \approx \frac{1}{m}$$

- Distribution of a_n w.r.t the measure h:

$$h(a_n \geqslant m) \approx \exp(-m)$$

- Borel-Cantelli to construct the singular set.
- Use Bowen-Series coding for $G < SL(2, \mathbb{R}); \mathbb{H}/G$ finite volume with cusps, to get Guivarc'h-LeJan.

$SL(2,\mathbb{Z})$ as mapping class group of the torus

- The expansion $R^{a_1}L^{a_2}.....$ or $L^{a_1}R^{a_2}....$ can be recognized as Rauzy-Veech expansion of an interval exchange with two subintervals with widths satisfying

$$r = \frac{\lambda_1}{\lambda_2}$$

- R and L correspond to Dehn twists in the curves $(1,0)$ and $(0,1)$ respectively, on the torus.

General setup for $Mod(S)$

- Encode measured foliations on S by Rauzy-Veech expansions of non-classical interval exchanges (maximal train tracks with a single switch).

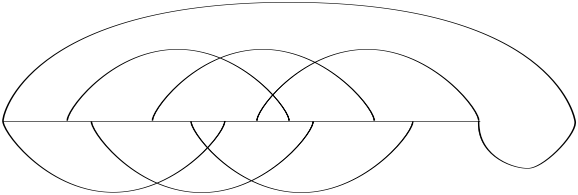

Figure: Genus 2

- Find combinatorics for a non-classical exchange such that there is a finite splitting sequence that returns to the same combinatorics and is a Dehn twist in a vertex cycle.
- Get the measure theory to work!

Rauzy-Veech renormalization

- Parameter space is the standard simplex Δ cut out by normalizing $\lambda_1 + \lambda_2 = 1$.
- Suppose band 1 splits band 2, then associated matrix is R.
- Denote initial widths: $\lambda = (\lambda_1, \lambda_2)$.
- Denote new widths: $\lambda^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)})$
- Notice $\lambda_1^{(1)} = \lambda_1, \lambda_2^{(1)} = \lambda_2 - \lambda_1$ so $\lambda = R\lambda^{(\mathbf{1})}$
- Projectivize to get $\Gamma R : \Delta \to \Delta$ i.e.

$$\Gamma R(\mathbf{x}) = \frac{R\mathbf{x}}{|R\mathbf{x}\ |}$$

where $|\mathbf{x}| = |x_1| + |x_2|$.

- Iterations produce a matrix Q and a projective linear map $\Gamma Q : \Delta \to \Delta$.
- Normalizing $vol(\Delta) = 1$,

$$\ell(\Gamma Q(\Delta)) \approx \text{probability calculated from continued fractions}$$

- Splitting is non-Markov.
- Distortion is uniform every time we switch from R to L and vice versa. Consequently, a_n as random variables are almost independent w.r.t Lebesgue.

Uniform distortion and estimating measures

After fixing combinatorics, the parameter space of a non-classical exchange is a codimension 1 subset of Δ.

Theorem (G)

For almost every non-classical exchange, the splitting sequence becomes uniformly distorted.

If a stage $\jmath$ with matrix $Q_\jmath$ is uniformly distorted i.e. the Jacobian $\mathcal{J}(\Gamma Q_\jmath)$ is roughly the same at all points then

$$\ell(\Gamma Q_\jmath(A)) \approx \ell(A)$$

Control: The probability that a finite permissible sequence κ follows a uniformly distorted stage $\jmath$ is roughly the same as the probability that an expansion begins with κ.

Dehn twist splitting

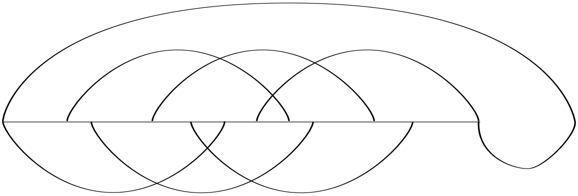

Figure: Genus 2

- Split down for all subintervals on top to return to the same combinatorics. This is a Dehn twist in a vertex cycle.
- Call this splitting sequence $\jmath$. Call the parameter space W.

$$\ell(\Gamma Q_{n\jmath}(W)) \approx \frac{1}{n^d}$$

Estimating the hitting measure and concluding singularity

- The Dehn twist splitting repeated n times increases subsurface projection to the annulus given by the vertex cycle.
- (Maher) The hitting measure h decays exponentially with increase in subsurface projections (more precisely, nesting distance w.r.t subsurface projection).
- Run the measure theory technology to conclude singularity.

Three cheers for Caroline!!!
Happy B'day