CONNECTIVITY OF THE SPACE OF ENDING LAMINATIONS

CHRISTOPHER J. LEININGER and SAUL SCHLEIMER

Abstract
We prove that for any closed surface of genus at least four, and any punctured surface of genus at least two, the space of ending laminations is connected. A theorem of E. Klarreich [28, Theorem 1.3] implies that this space is homeomorphic to the Gromov boundary of the complex of curves. It follows that the boundary of the complex of curves is connected in these cases, answering the conjecture of P. Storm. Other applications include the rigidity of the complex of curves and connectivity of spaces of degenerate Kleinian groups.

Contents

1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface with genus g and with n marked points. The space of measured foliations on Σ is denoted $\mathcal{MF}(\Sigma)$. A measured foliation is *arational* if there are no leaf cycles (see Section 2.5; such foliations are necessarily minimal). We denote the space of arational measured foliations on Σ by $\mathcal{AF}(\Sigma) \subset \mathcal{MF}(\Sigma)$ and give it the subspace topology. The space $\mathcal{AF}(\Sigma)$ is related to the complex of curves and spaces of hyperbolic 3-manifolds as described below.

THEOREM 1.1
If $g \geq 4$ or $g \geq 2$ and $n \geq 1$, then $\mathcal{AF}(\Sigma_{g,n})$ is connected.

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We note that for $\Sigma = \Sigma_{1,1}$ and $\Sigma_{0,4}$, the space $\mathscr{AF}(\Sigma)$ is naturally homeomorphic to $\mathbb{R} - \mathbb{Q}$ and so is totally disconnected.

To describe some of the applications of this theorem, we recall that the work of Masur and Minsky [32], together with that of Klarreich [28] (see also [9], [21]), implies that Harvey's complex of curves $\mathscr{C}(\Sigma)$ is δ-hyperbolic, and the Gromov boundary is homeomorphic to the quotient

$$\partial\mathscr{C}(\Sigma) \cong \mathscr{AF}(\Sigma)/\sim.$$

Here $\sim$ denotes the equivalence relation obtained by forgetting the transverse measures. We thus have the following corollary of Theorem 1.1, affirmatively answering the Storm conjecture for most surfaces (see [25, Question 10], [34, page 8]).

COROLLARY 1.2
If $g \geq 4$ or $g \geq 2$ and $n \geq 1$, then $\partial\mathscr{C}(\Sigma_{g,n})$ is connected.

Connectivity of $\partial\mathscr{C}(\Sigma)$ is a useful property when trying to understand the quasi-isometric geometry of $\mathscr{C}(\Sigma)$, as was shown by Rafi and Schleimer [39]. Recall the measure of complexity $\xi(\Sigma_{g,n}) = 3g - 3 + n$.

THEOREM 1.3 (Rafi and Schleimer [39, Theorem 1.2])
Suppose that $\partial\mathscr{C}(\Sigma)$ is connected, and suppose that $\xi(\Sigma) \geq 4$. If Σ' is any surface for which $\mathscr{C}(\Sigma)$ and $\mathscr{C}(\Sigma')$ are quasi-isometric, then $\Sigma \cong \Sigma'$.

A version of this theorem holds for $\xi(\Sigma) \geq 2$: a quasi isometry from $\mathscr{C}(\Sigma)$ to $\mathscr{C}(\Sigma')$ is a bounded distance from a simplicial isomorphism. For $\xi(\Sigma) = 2, 3$ there are nonhomeomorphic surfaces with isomorphic curve complexes.

Now, view $\Sigma = \Sigma_{g,n}$ as a surface of genus g with n boundary components. The natural homeomorphism between $\mathscr{MF}(\Sigma)$ and $\mathscr{ML}(\Sigma)$, the space-measured laminations on Σ, sends $\mathscr{AF}(\Sigma)$ to the subspace of those laminations that are *filling*. The quotient of this space by forgetting the transverse measures is called the space of *ending laminations* and is denoted $\mathscr{EL}(\Sigma)$. Via the natural homeomorphism we see that the following are all homeomorphic:

$$\mathscr{EL}(\Sigma) \cong \mathscr{AF}(\Sigma)/\sim \cong \partial\mathscr{C}(\Sigma).$$

The space $\mathscr{EL}(\Sigma)$ is precisely the set of ending laminations of geometrically infinite Kleinian surface groups isomorphic to $\pi_1(\Sigma)$ without accidental parabolics.

The following is an easy consequence of Theorem 1.1 and its proof (see Theorems 4.8, 5.5).

THEOREM 1.4
The space $\mathscr{EL}(\Sigma_{g,n})$ *is connected and has no cut points when* $g \geq 4$ *or* $g \geq 2$ *and* $n \geq 1$.

The ending lamination theorem of Brock, Canary, and Minsky (see [35], [12]) asserts that the end invariants $\mathscr{E}(M) = (\mathscr{E}^-(M), \mathscr{E}^+(M))$ are complete invariants of the hyperbolic manifold $M \in \mathrm{AH}(\Sigma, \partial\Sigma)$. Moreover, on the space of doubly degenerate Kleinian surface groups

$$\mathrm{DD}(\Sigma, \partial\Sigma) = \left\{ M \in \mathrm{AH}(\Sigma, \partial\Sigma) \,\middle|\, \mathscr{E}(M) \in \mathscr{EL}(\Sigma) \times \mathscr{EL}(\Sigma) \right\},$$

the map $\mathscr{E}$ is a homeomorphism (see Theorem 6.5). Although this seems to be well known, we provide a proof in Section 6 for completeness. Thus, one has the following.

PROPOSITION 1.5
The space $\mathrm{DD}(\Sigma_{g,n})$ *is connected if* $g \geq 4$ *or* $g \geq 2$ *and* $n \geq 1$.

Remark 1.6
On all of $\mathrm{AH}(\Sigma, \partial\Sigma)$, the map $\mathscr{E}$ fails to be continuous with respect to any of the usual topologies on the target (see [10]). However, discontinuity occurs in the presence of accidental parabolics, which is not an issue for the situation being discussed here.

A similar result holds if one considers the subset of the boundary of a Bers slice $\partial_0 B_Y \subset \partial B_Y$ consisting of those Kleinian groups without accidental parabolics. Then $\partial_0 B_Y \cong \mathscr{EL}(\Sigma)$ (see Theorem 6.6). So, as a corollary of Theorems 1.4 or 1.1, we have the following.

COROLLARY 1.7
The space $\partial_0 B_Y$ *is connected if* $g \geq 4$ *or* $g \geq 2$ *and* $n \geq 1$.

Finally, we remark that Theorem 1.1 has some bearing on problems regarding negatively curved 4-manifolds. More precisely, an open question is whether or not there exists a closed, negatively curved 4-manifold M that fibers over a surface (see [5, Question 12.3(b)]). The fiber is also a surface Σ, and a consequence of the work of Farb and Mosher [17] is that if such a manifold exists, then there is an embedding of a circle to $\mathscr{AF}(\Sigma)$. One approach to proving that there are no such 4-manifolds would thus be to prove that $\mathscr{AF}(\Sigma)$ is totally disconnected or, at least, to prove that $\mathscr{AF}(\Sigma)$ contains no circles. Theorem 1.1 thus removes this obstruction—indeed, during the course of the proof of Theorem 1.1, we construct many circles embedded in $\mathscr{AF}(\Sigma)$ (see Section 4.5).

Plan of the article

The proof of Theorem 1.1 for surfaces with a single marked point is considerably simpler and follows from a suggestion made to us by Ken Bromberg. Proceed as follows. Fix an arational measured foliation on $\Sigma_{g,0}$. Introducing a marked point yields an arational measured foliation on $\Sigma_{g,1}$, and moving the point produces paths of such foliations. Indeed, the set of foliations so obtained forms a dense, path-connected subset of $\mathscr{AF}(\Sigma_{g,1})$. Thus, $\mathscr{AF}(\Sigma_{g,1})$ is connected. The fact that the closure of a connected set is connected is exploited several times.

Remark 1.8

In joint work with Mahan Mj [30], we develop further tools, combining ideas from [26] and [36] to provide a more precise picture for surfaces with one marked point. As a consequence we prove that $\mathscr{EL}(\Sigma_{g,1})$ is path connected and locally path connected. We also note that using different techniques, David Gabai has now proven this for all surfaces Σ with $\xi(\Sigma) \geq 2$ (see [19]).

For surfaces $\Sigma_{g,n}$ with $n \geq 2$, a similar strategy can be employed. However, the position of the marked points is much more delicate. Specifically, starting with an arational measured foliation on $\Sigma_{g,0}$, arbitrary placement of the n marked points does not result in an arational measured foliation on $\Sigma_{g,n}$. Consequently, we must first devise an effective criteria that guarantees that the position of n marked points determines an arational measured foliation on $\Sigma_{g,n}$. With such a criteria at our disposal, we can begin producing paths in $\mathscr{AF}(\Sigma_{g,n})$.

Second, we must come to terms with the fact that the reduced flexibility in the placement of the marked points means that we have fewer paths to work with. In particular, there is no obvious dense path-connected set. We use the dynamics of pseudo-Anosov mapping classes to produce paths that connect to pseudo-Anosov fixed points. Concatenating such paths, we are able to produce a dense path-connected set and thus prove that $\mathscr{AF}(\Sigma_{g,n})$ is connected.

Our criteria for the positions of the marked points requires that we choose orientable foliations in $\mathscr{AF}(\Sigma_{g,0})$ when constructing paths. Moreover, to construct our dense path-connected subset of $\mathscr{AF}(\Sigma_{g,n})$, we need some connected space of (not necessarily arational) orientable foliations to get started. The space of complex-valued 1-forms (holomorphic with respect to varying complex structures on $\Sigma_{g,0}$) provides such a space.

To prove that $\mathscr{AF}(\Sigma_{g,0})$ is connected, we apply a branched cover construction to find a family of connected subsets so that

(a) the union of the connected sets is dense, and

(b) for any two subsets in the family, there is a finite chain of such subsets so that consecutive subsets in the chain nontrivially intersect.

It follows that the union is a dense connected set, and hence $\mathscr{AF}(\Sigma_{g,0})$ is connected.

We end the article by explaining the applications to hyperbolic 3-manifolds mentioned above in more detail.

2. Preliminaries

In this section, we briefly describe the background material we need, make some notational conventions (most of which are standard), and make some preliminary observations.

We let S denote a closed surface of genus at least 2, and we let $\mathbf{z} = \{z_1, \ldots, z_n\} \subset S$ denote a set with $n \geq 0$ points. Because we wish to refer to the marked points by name, we write $(S, \mathbf{z})$ instead of $\Sigma_{g,n}$. We view $\mathbf{z} \subset S$ as an ordered set of distinct points. We sometimes refer to it as a point in the n-fold product $S \times \cdots \times S$.

We frequently make definitions for $(S, \mathbf{z})$ and consider them valid for S unless they clearly apply only when $|\mathbf{z}| \neq 0$.

2.1. Curves and paths

We let $\mathscr{C}^0(S, \mathbf{z})$ denote the set of isotopy classes of essential simple closed curves contained in $S - \mathbf{z}$. It is also convenient to define $\mathscr{A}^0(S, \mathbf{z}) \supset \mathscr{C}^0(S, \mathbf{z})$ by adding to $\mathscr{C}^0(S, \mathbf{z})$ the set of all isotopy classes of essential arcs meeting $\mathbf{z}$ precisely in the endpoints of the arcs. Isotopies must fix $\mathbf{z}$. A curve or arc is essential if it cannot be isotoped into an arbitrarily small neighborhood of a point of $\mathbf{z}$. The geometric intersection number $\mathbf{i}(\cdot, \cdot)$ is defined for pairs of points in $\mathscr{A}^0(S, \mathbf{z})$ as the minimal number of points of intersection between representatives of the curves/arcs.

We let $\Gamma(S)$ denote the set of homotopy classes of oriented closed curves on S, and we let $\Gamma(S, z_i, z_j)$ denote the set of homotopy classes of oriented paths from z_i to z_j. Note that $\Gamma(S, z_i, z_j)$ and $\Gamma(S, z_j, z_i)$ differ simply by reversing the orientations on all homotopy classes. This latter homotopy is relative to the endpoints, but for example, the path may be homotoped through other marked points.

2.2. Diffeomorphisms and mapping classes

The orientation-preserving diffeomorphism group of S is denoted $\mathrm{Diff}^+(S)$. There are several subgroups in which we are interested: $\mathrm{Diff}^+(S, \mathbf{z})$, the subgroup consisting of those diffeomorphisms fixing each $z_i \in \mathbf{z}$, $\mathrm{Diff}_0(S)$ and $\mathrm{Diff}_0(S, \mathbf{z})$, the respective components containing the identity, as well as the intersection

$$\mathrm{Diff}_{0,\mathbf{z}}(S, \mathbf{z}) = \mathrm{Diff}_0(S) \cap \mathrm{Diff}^+(S, \mathbf{z}).$$

The *mapping class groups* we are interested in are

$$\mathrm{Mod}(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S),$$

$$\mathrm{Mod}(S, \mathbf{z}) = \mathrm{Diff}^+(S, \mathbf{z})/\mathrm{Diff}_0(S, \mathbf{z}).$$

Given a diffeomorphism $\phi \in \mathrm{Diff}^+(S)$, we denote its image in $\mathrm{Mod}(S)$ by $\bar{\phi}$. If $\phi \in \mathrm{Diff}^+(S, \mathbf{z})$, then we denote its image in $\mathrm{Mod}(S, \mathbf{z})$ by $\hat{\phi}$.

We also have need to consider the $(S, \mathbf{z})$-*braid group*

$$\mathscr{B}(S, \mathbf{z}) = \mathrm{Diff}_{0,\mathbf{z}}(S, \mathbf{z})/\mathrm{Diff}_0(S, \mathbf{z}) < \mathrm{Mod}(S, \mathbf{z}).$$

See Section 2.3 for the discussion of the Birman exact sequence and the connection to the usual definition of the surface braid group.

The mapping class groups act on the sets of curves and paths. More precisely, $\mathrm{Mod}(S, \mathbf{z})$ acts on $\mathscr{C}^0(S, \mathbf{z})$, $\mathscr{A}^0(S, \mathbf{z})$, $\Gamma(S)$, $\Gamma(S, z_i, z_j)$ in the usual way by pushing forward homotopy/isotopy classes. We denote the result of the mapping class $\hat{\phi}$ acting on the homotopy/isotopy class of curve/path α by $\hat{\phi}(\alpha)$.

2.3. Configuration spaces

The configuration space of n ordered points on S ($n \geq 1$) is the subspace of the n-fold product $S \times \cdots \times S$ obtained by removing the locus where two coordinates are equal:

$$\mathrm{Conf}_n(S) = \big\{(p_1, \cdots, p_n) \,\big|\, p_i \in S \text{ for all } i \text{ and } p_i \neq p_j \text{ for all } i \neq j\big\}.$$

Observe that $\mathrm{Conf}_1(S) \cong S$, and for $n \geq 2$, $\mathrm{Conf}_n(S)$ fibers over $\mathrm{Conf}_{n-1}(S)$ with fibers homeomorphic to S with $(n-1)$ points removed. Applying the long exact sequence of a fibration inductively, we see that all higher homotopy groups of $\mathrm{Conf}_n(S)$ vanish. It follows that the universal covering $\widetilde{\mathrm{Conf}}_n(S)$ is contractible.

We think of $\mathbf{z}$ as a basepoint for $\mathrm{Conf}_n(S)$. This determines an evaluation map

$$\mathrm{ev}_{\mathbf{z}} : \mathrm{Diff}_0(S) \to \mathrm{Conf}_n(S)$$

given by $\mathrm{ev}_{\mathbf{z}}(\phi) = \phi(\mathbf{z})$. As in Birman's work [6], [7], the group $\mathrm{Diff}_{0,\mathbf{z}}(S, \mathbf{z})$ acts on the fibers and makes $\mathrm{Diff}_0(S)$ into a principal $\mathrm{Diff}_{0,\mathbf{z}}(S, \mathbf{z})$-bundle. We use local trivializations for this fibration, which we discuss in more detail in Section 2.4.

The long exact sequence of homotopy groups of a fibration, together with the contractibility of $\mathrm{Diff}_0(S)$—due to Earle and Eells [15]—gives isomorphisms

$$\mathscr{B}(S, \mathbf{z}) = \pi_0\big(\mathrm{Diff}_{0,\mathbf{z}}(S, \mathbf{z})\big) \cong \pi_1\big(\mathrm{Conf}_n(S)\big). \tag{1}$$

This justifies our referring to $\mathscr{B}(S, \mathbf{z})$ as the braid group since the last group $\pi_1(\mathrm{Conf}_n(S))$ is the usual definition for the (pure) n-strand braid group on S. This isomorphism and the short exact sequence below were obtained by Birman [6], [7] for the homeomorphism group.

It follows that the quotient of $\mathrm{Diff}_0(S)$ by the smaller group $\mathrm{Diff}_0(S, \mathbf{z})$ is the universal cover $\widetilde{\mathrm{Conf}}_n(S)$. We thus obtain $\mathrm{Diff}_0(S)$ as a principal $\mathrm{Diff}_0(S, \mathbf{z})$–bundle over $\widetilde{\mathrm{Conf}}_n(S)$. Contractibility of $\widetilde{\mathrm{Conf}}_n(S)$ implies

$$\mathrm{Diff}_0(S) \cong \widetilde{\mathrm{Conf}}_n(S) \times \mathrm{Diff}_0(S, \mathbf{z}). \tag{2}$$

The basepoint $\mathbf{z}$ for $\mathrm{Conf}_n(S)$ has a canonical lift $\tilde{\mathbf{z}}$ to $\widetilde{\mathrm{Conf}}_n(S)$, namely, the image of the identity in $\mathrm{Diff}_0(S)$.

The inclusion $\mathrm{Diff}^+(S, \mathbf{z}) < \mathrm{Diff}^+(S)$ induces a homomorphism

$$\mathrm{Mod}(S, \mathbf{z}) \to \mathrm{Mod}(S).$$

The discussion above, together with the isomorphism theorems from group theory situates this homomorphism into the *Birman exact sequence*

$$1 \to \mathscr{B}(S, \mathbf{z}) \to \mathrm{Mod}(S, \mathbf{z}) \to \mathrm{Mod}(S) \to 1.$$

We use this to view $\mathscr{B}(S, \mathbf{z})$ as a subgroup of $\mathrm{Mod}(S, \mathbf{z})$.

2.4. Local trivializations

We now describe the local trivializations (i.e., local sections) for the principal bundles $\mathrm{Diff}_0(S) \to \widetilde{\mathrm{Conf}}_n(S)$ and $\mathrm{Diff}_0(S) \to \mathrm{Conf}_n(S)$ which we use. We describe these only near the point $\mathbf{z}$ as this is our primary case of interest.

Let $B_1, \ldots, B_n$ be open disk neighborhoods of $z_1, \ldots, z_n$, respectively, in S with pairwise disjoint disk closures $\overline{B}_1, \ldots, \overline{B}_n$. We let $U_1, \ldots, U_n$ be pairwise disjoint open disks with $\overline{B}_i \subset U_i$. Write $\mathbf{B} = B_1 \times \cdots \times B_n$ and $\mathbf{U} = U_1 \times \cdots \times U_n$ with points denoted $\mathbf{b} = (b_1, \ldots, b_n)$.

Consider a smooth map

$$f : S \times \mathbf{B} \to S.$$

For $\mathbf{b} \in \mathbf{B}$, let $f_{\mathbf{b}} = f(\cdot, \mathbf{b}) : S \to S$. We suppose that f has the following properties:

- $f_{\mathbf{z}} = \mathrm{Id}$,
- $f_{\mathbf{b}}$ is a diffeomorphism for every $\mathbf{b} \in \mathbf{B}$,
- $f_{\mathbf{b}}$ is the identity outside $\bigcup_i U_i$,
- $f(z_i, \mathbf{b}) = b_i$ for every $\mathbf{b} \in \mathbf{B}$ and $i = 1, \ldots, n$.

Note that $\mathbf{B} \subset \mathrm{Conf}_n(S)$ is a neighborhood of $\mathbf{z}$, and note that the map

$$f_{\mathbf{B}} : \mathbf{B} \to \mathrm{Diff}_0(S)$$

given by $f_{\mathbf{B}}(\mathbf{b}) = f_{\mathbf{b}}$ is a local trivialization for $\mathrm{Diff}_0(S) \to \mathrm{Conf}_n(S)$ over this neighborhood. Similarly, this determines a local trivialization for $\mathrm{Diff}_0(S) \to \widetilde{\mathrm{Conf}}_n(S)$ over the neighborhood of $\tilde{\mathbf{z}}$ obtained by lifting $\mathbf{B}$ to $\widetilde{\mathrm{Conf}}_n(S)$. We call either f or $f_{\mathbf{B}}$ a $\mathbf{B}$-trivialization.

Using local coordinates, one can construct a **B**-trivialization for any **U** and **B** as above.

2.5. Measured foliations

We refer the reader to [18] and [37] for a more detailed discussion of measured foliations on surfaces. We note that the definition of measured foliations for surfaces with marked points (or punctures) is treated in [18] by replacing the puncture with a boundary component and making all definitions on compact surfaces with boundary.

A *measured foliation* on $(S, \mathbf{z})$ is a singular foliation $\mathscr{F}$ on S together with a transverse measure μ of full support. The singularities of $\mathscr{F}$ are all required to be p-prong singularities for $p \geq 1$, or for $p \geq 3$ if the singularity does not occur at a marked point. We denote the set of singularities by $\operatorname{sing}(\mathscr{F}) \subset S$.

Given a measured foliation $(\mathscr{F}, \mu)$, and $\alpha \in \mathscr{C}^0(S, \mathbf{z})$, the geometric intersection number $\mathbf{i}(\alpha, (\mathscr{F}, \mu))$ is defined as the infimum

$$\mathbf{i}\big(\alpha, (\mathscr{F}, \mu)\big) = \inf_{\alpha_0 \in \alpha} \int_{\alpha_0} \mu.$$

This is the infimum of the total variation of α_0 as α_0 ranges over all representatives of α.

Two measured foliations $(\mathscr{F}, \mu)$ and $(\mathscr{F}', \mu')$ are declared to be equivalent on $(S, \mathbf{z})$ if

$$\mathbf{i}\big(\alpha, (\mathscr{F}, \mu)\big) = \mathbf{i}\big(\alpha, (\mathscr{F}', \mu')\big)$$

for all $\alpha \in \mathscr{C}^0(S, \mathbf{z})$. Théorème 1 of exposé 11 in [18] states (in particular) that this is the same as the equivalence relation on measured foliations generated by Whitehead equivalence and isotopy on $(S, \mathbf{z})$. We denote the space of equivalence classes by $\mathscr{MF}(S, \mathbf{z})$, topologized as a subspace of $\mathbb{R}^{\mathscr{C}^0(S,\mathbf{z})}$ via the inclusion

$$[\mathscr{F}, \mu] \mapsto \big\{\mathbf{i}\big(\alpha, (\mathscr{F}, \mu)\big)\big\}_{\alpha \in \mathscr{C}^0(S,\mathbf{z})}.$$

It is convenient at times to denote the equivalence class of $(\mathscr{F}, \mu)$ by μ rather than $[\mathscr{F}, \mu]$. Thus, when we write μ, we are referring to an equivalence class of measured foliation (even if we inappropriately call it a measured foliation), whereas the notation $(\mathscr{F}, \mu)$ means the actual measured foliation, not just the equivalence class it determines. We also write $\mathbf{i}(\alpha, \mu)$ to denote $\mathbf{i}(\alpha, (\mathscr{F}, \mu))$ when convenient.

A measured foliation $\mu \in \mathscr{MF}(S, \mathbf{z})$ is *orientable* if it has a representative that is transversely orientable. We note that any transversely orientable foliation is Whitehead equivalent to one that is not transversely orientable. The reason is that a Whitehead move can turn an even prong singularity into a pair of odd prong singularities.

We define a *saddle connection* of a foliation $\mathscr{F}$ to be the image of a path $\epsilon : I \to S$, defined on a compact interval I, with the following properties:

- ϵ is injective and tangent to $\mathscr{F}$ on the interior of I,
- ϵ maps the interior disjoint from $\mathrm{sing}(\mathscr{F}) \cup \mathbf{z}$, and
- ϵ maps the endpoints of I into $\mathrm{sing}(\mathscr{F}) \cup \mathbf{z}$.

A *leaf cycle* is an embedded loop or an embedded path connecting points of $\mathbf{z}$ which is a concatenation of saddle connections. A leaf cycle that is a loop is called a *closed leaf cycle*.

A measured foliation $(\mathscr{F}, \mu)$ is *arational* if $\mathscr{F}$ has no leaf cycle. The existence of a leaf cycle is not changed by Whitehead moves, and so we may say that μ is arational if any representative is. An equivalent formulation is that a measured foliation $(\mathscr{F}, \mu)$ is arational if

$$\mathbf{i}\big(\alpha, (\mathscr{F}, \mu)\big) > 0 \quad \text{for every } \alpha \in \mathscr{C}^0(S, \mathbf{z}).$$

If a measured foliation $(\mathscr{F}, \mu)$ on $(S, \mathbf{z})$ has no 1-prong singularities, then it determines points in both $\mathscr{MF}(S, \mathbf{z})$ and $\mathscr{MF}(S) = \mathscr{MF}(S, \emptyset)$. We write μ and $\pi(\mu)$ for these respective points.

LEMMA 2.1

If $(\mathscr{F}, \mu)$ has no 1-prong singularities and $\pi(\mu)$ is arational, then μ is arational if and only if it has no leaf cycle connecting two distinct points $z_i, z_j \in \mathbf{z}$.

Proof

To distinguish whether we are viewing the foliation $\mathscr{F}$ on S or $(S, \mathbf{z})$, we write $(\mathscr{F}, \pi(\mu))$ and $(\mathscr{F}, \mu)$, respectively. The only minor subtlety involved in the proof is that a saddle connection for $(\mathscr{F}, \mu)$ which has at least one endpoint in $\mathbf{z}$ is not necessarily a saddle connection for $(\mathscr{F}, \pi(\mu))$.

If μ is arational, then there are no leaf cycles by definition. In particular, there are no leaf cycles connecting z_i to z_j for any $i \neq j$.

We prove the other implication by proving the contrapositive. Suppose that μ is not arational, so that there exists a leaf cycle γ. Because γ is embedded, one can check that one of the following must happen:

(1) γ is a closed leaf cycle for $(\mathscr{F}, \pi(\mu))$,
(2) γ is a closed leaf of $(\mathscr{F}, \pi(\mu))$, or
(3) γ is a nonclosed leaf cycle.

Case (1) cannot happen since we are assuming that $(\mathscr{F}, \pi(\mu))$ is arational. Likewise, case (2) implies that there is a cylinder, the boundary of which contains a closed leaf cycle, again ruled out by arationality of $(\mathscr{F}, \pi(\mu))$. It follows that case (3) must occur, which is the desired conclusion. □

The group $\mathrm{Mod}(S, \mathbf{z})$ acts on $\mathscr{MF}(S, \mathbf{z})$, and this can be most easily described via the change in intersection numbers. Specifically, if α is the isotopy class of a closed curve or arc, $\mu \in \mathscr{MF}(S, \mathbf{z})$, and $\hat{\phi} \in \mathrm{Mod}(S, \mathbf{z})$, then

$$\mathbf{i}(\alpha, \hat{\phi} \cdot \mu) = \mathbf{i}\big(\hat{\phi}^{-1}(\alpha), \mu\big).$$

That is, the action should preserve geometric intersection number.

It is also convenient to have the space $\mathbb{P}\mathscr{MF}(S, \mathbf{z})$ of *projective measured foliations*. This is the quotient of $\mathscr{MF}(S, \mathbf{z})$ by the action of $\mathbb{R}_+$ by scaling the transverse measure. The action of $\mathrm{Mod}(S, \mathbf{z})$ on $\mathscr{MF}(S, \mathbf{z})$ descends to an action on $\mathbb{P}\mathscr{MF}(S, \mathbf{z})$. An element of $\mathbb{P}\mathscr{MF}(S, \mathbf{z})$ is arational if and only if any (equivalently, every) of its preimages is.

If $\hat{\phi} \in \mathrm{Mod}(S, \mathbf{z})$ is pseudo-Anosov, we let $\mathbb{P}(\mu_s), \mathbb{P}(\mu_u) \in \mathbb{P}\mathscr{MF}(S, \mathbf{z})$ denote the *stable and unstable projective measured foliations* of ϕ. These are attracting and repelling fixed points, respectively. That is, on $\mathbb{P}\mathscr{MF}(S, \mathbf{z}) - \{\mathbb{P}(\mu_u)\}$, iteration of $\hat{\phi}$ converges uniformly on compact sets to the constant map with value $\mathbb{P}(\mu_s)$. Inverting $\hat{\phi}$, we obtain the same dynamics after interchanging $\mathbb{P}(\mu_s)$ and $\mathbb{P}(\mu_u)$. We also call μ_s and μ_u the stable and unstable measured foliations of $\hat{\phi}$, though they are only well defined up the action of $\mathbb{R}_+$.

LEMMA 2.2
If $\hat{\phi} \in \mathscr{B}(S, \mathbf{z})$ is pseudo-Anosov and μ_s is its stable foliation, then any representative of μ_s has a 1-prong singularity.

Proof
Let $\phi_0 : (S, \mathbf{z}) \to (S, \mathbf{z})$ be a representative *pseudo-Anosov homeomorphism* of ϕ with stable and unstable measured foliations $(\mathscr{F}_s, \mu_s)$ and $(\mathscr{F}_u, \mu_u)$ (see [18]). Suppose that $\mathscr{F}_s$ has no 1-prong singularity. By transversality, $\mathscr{F}_u$ does not have one either. It follows that we can forget $\mathbf{z}$, and $\phi_0 : S \to S$ is still a pseudo-Anosov homeomorphism. Therefore, the class in $\mathrm{Diff}^+(S)$ determined by ϕ_0 is pseudo-Anosov. However, $\hat{\phi} \in \mathscr{B}(S, \mathbf{z})$ means that any representative lies in $\mathrm{Diff}_0(S)$ and so is trivial in $\mathrm{Mod}(S)$ and cannot be pseudo-Anosov on S.

Note that if $(\mathscr{F}'_s, \mu'_s)$ is any other representative of the class μ_s, then $\mathscr{F}_s$ is obtained from $\mathscr{F}'_s$ by isotopy and *collapsing* Whitehead moves only. This is because $\mathscr{F}_s$ can have no saddle connections. If $\mathscr{F}'_s$ had a 1-prong singularity, then $\mathscr{F}_s$ must also have had a 1-prong singularity. □

2.6. Teichmüller space and holomorphic 1-forms
To discuss the space of holomorphic 1-forms, which is the primary space of interest for us, we first recall some facts about Teichmüller space. The space of complex structures on S, compatible with the smooth structure and orientation, is denoted $\mathscr{H}(S)$. The

group $\mathrm{Diff}^+(S)$ acts on $\mathscr{H}(S)$ *on the right* by pulling back complex structures, and the Teichmüller space of S is the quotient by the action of the subgroup $\mathrm{Diff}_0(S)$:

$$\mathscr{T}(S) = \mathscr{H}(S)/\mathrm{Diff}_0(S).$$

The action on the fibers of the map $\mathscr{H}(S) \to \mathscr{T}(S)$ is simply transitive, giving $\mathscr{H}(S)$ the structure of a principal $\mathrm{Diff}_0(S)$-bundle (see [15]). Contractibility of $\mathscr{T}(S)$ implies

$$\mathscr{H}(S) \cong \mathscr{T}(S) \times \mathrm{Diff}_0(S).$$

Keeping track of the marked points $\mathbf{z} \subset S$ amounts to taking the quotient by the smaller group $\mathrm{Diff}_0(S, \mathbf{z})$. That is, the Teichmüller space of $(S, \mathbf{z})$ is

$$\mathscr{T}(S, \mathbf{z}) = \mathscr{H}(S)/\mathrm{Diff}_0(S, \mathbf{z}).$$

Combining this discussion with (2), we obtain

$$\mathscr{H}(S) \cong \mathscr{T}(S) \times \widetilde{\mathrm{Conf}}_n(S) \times \mathrm{Diff}_0(S, \mathbf{z}) \cong \mathscr{T}(S, \mathbf{z}) \times \mathrm{Diff}_0(S, \mathbf{z}), \tag{3}$$

and so

$$\mathscr{T}(S, \mathbf{z}) \cong \mathscr{T}(S) \times \widetilde{\mathrm{Conf}}_n(S).$$

Remark 2.3
In the case where $\mathbf{z}$ is a single marked point, Bers [4] proved that the quotient map $\mathscr{T}(S, \mathbf{z}) \to \mathscr{T}(S)$ is a holomorphic fibration. Bers's theorem holds in the more general situation, where S has finite type. From this and an inductive argument, it follows that the fibration $\mathscr{T}(S, \mathbf{z}) \to \mathscr{T}(S)$ is holomorphic for any finite set $\mathbf{z}$ (not just a single point), though we do not use this fact here.

For each $X \in \mathscr{H}(S)$, we have the vector space of 1-forms that are holomorphic with respect to X. This determines a g-dimensional complex vector bundle over $\mathscr{H}(S)$, and we denote the bundle obtained from this by removing the zero section by

$$\widetilde{\Omega}(S) = \big\{(X, \omega) \,\big|\, X \in \mathscr{H}(S) \text{ and } \omega \text{ is a holomorphic 1-form on } (S, X)\big\}.$$

We refer to a point of $\widetilde{\Omega}(S)$ as (X, ω) or sometimes simply ω since the complex structure X is determined by the 1-form ω. We let $\mathrm{Zeros}(\omega)$ denote the set of zeros of ω.

We sometimes view $\omega \in \widetilde{\Omega}(S)$ as a *translation structure* on S (see, e.g., [16]). This is a singular flat metric on S with trivial holonomy and a preferred vertical direction in each tangent space. The singularities are isolated cone-type singularities occurring precisely at the points of $\mathrm{Zeros}(\omega)$ and having cone angles in $2\pi\mathbb{Z}$. The metric (and notion of vertical) are pulled back from $\mathbb{C}$ via *natural coordinates* obtained by integrating ω over a sufficiently small simply connected open neighborhood U of a point p_0 in $S - \text{Zeros}(\omega)$:

$$\zeta(p) = \int_{p_0}^{p} \omega.$$

We say that the natural coordinate ζ is based at p_0. In the natural coordinates, ω has the simple form $\omega = d\zeta$.

The metric on S associated to ω is locally CAT(0). Given $\alpha \in \mathscr{C}^0(S, \mathbf{z})$, there may not be a geodesic representative in $S - \{\mathbf{z}\}$ as this surface is incomplete. However, a sequence of representatives with lengths approaching the infimum has a limit in S (which may nontrivially intersect $\mathbf{z}$) by the Arzela-Ascoli theorem. This is a geodesic, except possibly at points of $\mathbf{z}$, where incoming and outgoing geodesic segments can make an angle less than π. We refer to such a curve as a geodesic representative for α.

The right action of $\text{Diff}^+(S)$ on $\mathscr{H}(S)$ naturally lifts to an action on $\widetilde{\Omega}(S)$. This action is equivalently the restriction of the action of $\text{Diff}^+(S)$ on all 1-forms

$$\omega \cdot \phi = \phi^*(\omega)$$

for $\phi \in \text{Diff}^+(S)$ and $\omega \in \widetilde{\Omega}(S)$. We consider two quotients

$$\Omega(S) = \widetilde{\Omega}(S)/\text{Diff}_0(S) \qquad \text{and} \qquad \Omega(S, \mathbf{z}) = \widetilde{\Omega}(S)/\text{Diff}_0(S, \mathbf{z}).$$

Equation (3) implies a product structure

$$\Omega(S, \mathbf{z}) \cong \Omega(S) \times \widetilde{\text{Conf}}_n(S). \tag{4}$$

Let

$$\pi : \Omega(S, \mathbf{z}) \to \Omega(S)$$

denote the projection.

Perhaps the most important point of what follows is the distinction between points of $\Omega(S, \mathbf{z})$ and of $\Omega(S)$. Given $\omega \in \widetilde{\Omega}(S)$, we write $\hat{\omega} \in \Omega(S, \mathbf{z})$ and $\bar{\omega} = \pi(\hat{\omega}) \in \Omega(S)$ for the associated points in the quotient spaces.

The right action of $\text{Diff}^+(S)$ on $\widetilde{\Omega}(S)$ determines a left action in the usual way by defining

$$\phi \cdot \omega = \phi^{-1*}(\omega). \tag{5}$$

This descends to left actions of $\mathrm{Mod}(S, \mathbf{z})$ and $\mathrm{Mod}(S)$ on $\Omega(S, \mathbf{z})$ and $\Omega(S)$, respectively:

$$\hat{\phi} \cdot \hat{\omega} = \widehat{\phi^{-1*}(\omega)} \qquad \text{and} \qquad \bar{\phi} \cdot \bar{\omega} = \overline{\phi^{-1*}(\omega)}.$$

Let $\omega \in \widetilde{\Omega}(S)$ be any point. We denote the fiber of π over $\bar{\omega}$ by $F_{\bar{\omega}} = \pi^{-1}(\bar{\omega})$ and note that with respect to the product structure of (4), we have

$$F_{\bar{\omega}} \cong \{\hat{\omega}\} \times \widetilde{\mathrm{Conf}}_n(S).$$

From here we see the isomorphism (2) clearly; the action of $\mathscr{B}(S, \mathbf{z}) < \mathrm{Mod}(S, \mathbf{z})$ on $\mathscr{F}_{\bar{\omega}}$ is by covering transformations.

2.7. A neighborhood in the fiber

We frequently need to consider families of 1-forms and not just their isotopy classes, and we use the trivializations described in Section 2.4 to construct these. More precisely, consider any $\mathbf{B}$-trivialization $f : S \times \mathbf{B} \to S$. Given $\omega \in \widetilde{\Omega}(S)$, f determines a map we denote

$$f^{\omega} : \mathbf{B} \to \widetilde{\Omega}(S),$$

which is defined by

$$f^{\omega}(\mathbf{b}) = f_{\mathbf{b}}^{*}\omega.$$

We can compose f^{ω} with the projections to both $\Omega(S, \mathbf{z})$ and $\Omega(S)$. Since $f_{\mathbf{b}} \in \mathrm{Diff}_0(S)$ for all $\mathbf{b} \in \mathbf{B}$, the latter map is simply the constant map with value $\bar{\omega}$. We are primarily interested in the composition with the former projection, which we denote

$$\hat{f}^{\omega} : \mathbf{B} \to \Omega(S, \mathbf{z}).$$

The image of $\hat{f}^{\omega}$ lies in $F_{\bar{\omega}}$. Since this is a local trivialization of the bundle

$$\mathrm{Diff}_0(S) \to \widetilde{\mathrm{Conf}}_n(S),$$

$\hat{f}^{\omega}$ maps onto a neighborhood of $\hat{\omega} = \hat{f}^{\omega}(\mathbf{z})$ in $F_{\bar{\omega}}$.

2.8. From 1-forms to foliations

An element $\omega \in \widetilde{\Omega}(S)$ determines a harmonic 1-form, $\mathrm{Re}(\omega)$, on S. Let γ_0 denote any representative of a homotopy class γ in $\Gamma(S)$, and let $\omega \in \widetilde{\Omega}(S)$. Since $\mathrm{Re}(\omega)$ is harmonic, it is closed, and hence the integral

$$\int_{\gamma_0} \mathrm{Re}(\omega)$$

is independent of the choice of representative γ_0 of γ.

By definition of the left action of $\mathrm{Diff}^+(S)$ on $\widetilde{\Omega}(S)$, if $\phi \in \mathrm{Diff}^+(S)$ and $\omega \in \widetilde{\Omega}(S)$, then

$$\int_{\gamma_0} \mathrm{Re}(\phi \cdot \omega) = \int_{\gamma_0} \phi^{-1*}\big(\mathrm{Re}(\omega)\big) = \int_{\phi^{-1}(\gamma_0)} \mathrm{Re}(\omega). \tag{6}$$

If $\phi \in \mathrm{Diff}_0(S)$, then $\phi^{-1} \in \mathrm{Diff}_0(S)$, and so $\phi^{-1}(\gamma_0)$ also represents γ. Therefore,

$$\int_{\gamma_0} \mathrm{Re}(\phi \cdot \omega) = \int_{\phi^{-1}(\gamma_0)} \mathrm{Re}(\omega) = \int_{\gamma_0} \mathrm{Re}(\omega).$$

It follows that we can well define

$$\int_{\gamma} \mathrm{Re}(\bar{\omega}) = \int_{\gamma_0} \mathrm{Re}(\omega)$$

for any $\gamma \in \Gamma(S)$ and $\bar{\omega} \in \Omega(S)$.

By the same reasoning, we can well define

$$\int_{\gamma} \mathrm{Re}(\hat{\omega})$$

for any $\gamma \in \Gamma(S)$ or $\Gamma(S, z_i, z_j)$ and $\hat{\omega} \in \Omega(S, \mathbf{z})$ by picking arbitrarily representatives of the relevant equivalence classes. Furthermore, (6) implies that the actions of $\mathrm{Mod}(S, \mathbf{z})$ and $\mathrm{Mod}(S)$ satisfy

$$\int_{\bar{\phi}(\gamma)} \mathrm{Re}(\bar{\phi} \cdot \bar{\omega}) = \int_{\gamma} \mathrm{Re}(\bar{\omega}) \qquad \text{and} \qquad \int_{\hat{\phi}(\gamma)} \mathrm{Re}(\hat{\phi} \cdot \hat{\omega}) = \int_{\gamma} \mathrm{Re}(\hat{\omega}). \tag{7}$$

A 1-form $\omega \in \widetilde{\Omega}(S)$ also determines a measured foliation on S. The foliation is denoted $\mathscr{F}(\mathrm{Re}(\omega))$ as it is obtained by integrating the line field $\ker(\mathrm{Re}(\omega))$. The measure $|\mathrm{Re}(\omega)|$ is obtained by integrating the absolute value of $\mathrm{Re}(\omega)$. Passing to the quotient by $\mathrm{Diff}_0(S, \mathbf{z})$ and $\mathrm{Diff}_0(S)$, we obtain well-defined points $|\mathrm{Re}(\hat{\omega})| \in \mathscr{MF}(S, \mathbf{z})$ and $|\mathrm{Re}(\bar{\omega})| \in \mathscr{MF}(S)$, respectively.

This determines a map

$$|\mathrm{Re}| : \Omega(S, \mathbf{z}) \to \mathscr{MF}(S, \mathbf{z})$$

defined by

$$|\mathrm{Re}|(\hat{\omega}) = |\mathrm{Re}(\hat{\omega})|.$$

LEMMA 2.4

$|\mathrm{Re}|$ *is continuous and* $\mathrm{Mod}(S, \mathbf{z})$*-equivariant.*

Proof
Continuity is well known (see [22]). The idea is that given $\alpha \in \mathscr{C}^0(S, \mathbf{z})$, as $\omega \in \widetilde{\Omega}(S)$ varies, the geodesic representatives vary continuously. Since the geodesic representatives realize $\mathbf{i}(\alpha, |\mathrm{Re}(\omega)|)$, it easily follows that this quantity varies continuously, proving continuity of $|\mathrm{Re}|$.

To see the equivariance, we need only compare the various definitions. Fixing $\hat{\phi} \in \mathrm{Mod}(S, \mathbf{z})$ and $\hat{\omega} \in \Omega(S, \mathbf{z})$, we must show

$$\hat{\phi} \cdot |\mathrm{Re}(\hat{\omega})| = |\mathrm{Re}(\hat{\phi} \cdot \hat{\omega})|.$$

The action on $\mathscr{MF}(S, \mathbf{z})$ is determined by the action on $\mathscr{C}^0(S, \mathbf{z})$ via intersection numbers according to the equation

$$\mathbf{i}\big(\alpha, \hat{\phi} \cdot |\mathrm{Re}(\hat{\omega})|\big) = \mathbf{i}\big(\hat{\phi}^{-1}(\alpha), |\mathrm{Re}(\hat{\omega})|\big)$$

for every $\alpha \in \mathscr{C}^0(S, \mathbf{z})$. Therefore, we fix any $\alpha \in \mathscr{C}^0(S, \mathbf{z})$, and we must prove

$$\mathbf{i}\big(\hat{\phi}^{-1}(\alpha), |\mathrm{Re}(\hat{\omega})|\big) = \mathbf{i}\big(\alpha, |\mathrm{Re}(\hat{\phi} \cdot \hat{\omega})|\big).$$

Arbitrarily orienting α (i.e., coherently orienting all representatives of α) and picking any representative ϕ of $\hat{\phi}$, we obtain

$$\begin{aligned}
\mathbf{i}\big(\alpha, |\mathrm{Re}(\hat{\phi} \cdot \hat{\omega})|\big) = \mathbf{i}\big(\alpha, |\mathrm{Re}(\widehat{\phi \cdot \omega})|\big) &= \inf_{\alpha_0 \in \alpha} \int_{\alpha_0} |\mathrm{Re}(\phi \cdot \omega)| \\
&= \inf_{\alpha_0 \in \alpha} \int_{\alpha_0} \big|\phi^{-1*}\big(\mathrm{Re}(\omega)\big)\big| = \inf_{\alpha_0 \in \alpha} \int_{\alpha_0} \phi^{-1*}|\mathrm{Re}(\omega)| \\
&= \inf_{\alpha_0 \in \alpha} \int_{\phi^{-1}(\alpha_0)} |\mathrm{Re}(\omega)| = \inf_{\beta_0 \in \hat{\phi}^{-1}(\alpha)} \int_{\beta_0} |\mathrm{Re}(\omega)| \\
&= \mathbf{i}\big(\hat{\phi}^{-1}(\alpha), |\mathrm{Re}(\hat{\omega})|\big).
\end{aligned}$$

This proves equivariance and completes the proof of the lemma. □

3. Periods and arationality

Given $\hat{\omega} \in \Omega(S, \mathbf{z})$, we define the *periods of* $\hat{\omega}$ by

$$\mathrm{Per}(\hat{\omega}) = \left\{ \int_\alpha \mathrm{Re}(\hat{\omega}) \,\middle|\, \forall\, \alpha \in \Gamma(S) \right\}.$$

For each $i \neq j$ between 1 and n, we define the ij*-relative periods of* $\hat{\omega}$ by

$$\mathrm{Per}_{ij}(\hat{\omega}) = \left\{ \int_\alpha \mathrm{Re}(\hat{\omega}) \,\middle|\, \forall\, \alpha \in \Gamma(S, z_i, z_j) \right\}.$$

We note that $\text{Per}(\hat{\omega})$ depends only on $\pi(\hat{\omega}) = \bar{\omega}$, whereas $\text{Per}_{ij}(\hat{\omega})$ actually depends on $\hat{\omega}$.

Our interest in the periods and relative periods comes from the following.

PROPOSITION 3.1
Suppose that $\omega \in \widetilde{\Omega}(S)$*, suppose that* $|\text{Re}(\bar{\omega})| \in \mathcal{AF}(S)$*, and suppose that for every* $i \neq j$*, we have*

$$\text{Per}_{ij}(\hat{\omega}) \not\subset \text{Per}(\hat{\omega}).$$

Then $|\text{Re}(\hat{\omega})| \in \mathcal{AF}(S, \mathbf{z})$.

Proof
We apply Lemma 2.1 and therefore need only check that for every $i \neq j$ the points z_i and z_j are not connected by a leaf cycle of $\mathcal{F}\big(\ker(\text{Re}(\omega))\big)$.

Suppose, to the contrary, that there is a leaf cycle δ with endpoints z_i and z_j. If ϵ is any path from z_i to z_j, then we can build a closed curve $\alpha = \delta \cup \epsilon$ by concatenating these two paths. Because δ is a leaf cycle, the integral of $\text{Re}(\omega)$ over δ is zero, and so

$$\int_\alpha \text{Re}(\omega) = \int_\delta \text{Re}(\omega) + \int_\epsilon \text{Re}(\omega) = \int_\epsilon \text{Re}(\omega).$$

This implies $\text{Per}_{ij}(\omega) \subset \text{Per}(\omega)$, which is a contradiction. □

The following subspace of Ω is needed for technical reasons (see Section 4). Define the subspace $\widetilde{\Omega}_*(S, \mathbf{z}) \subset \widetilde{\Omega}(S)$ to be

$$\widetilde{\Omega}_*(S, \mathbf{z}) = \big\{\omega \,\big|\, \text{Zeros}(\omega) \cap \mathbf{z} = \emptyset\big\}.$$

The group $\text{Diff}_0(S, \mathbf{z})$ leaves $\widetilde{\Omega}_*(S, \mathbf{z})$ invariant, and we let $\Omega_*(S, \mathbf{z})$ denote the image in $\Omega(S, \mathbf{z})$.

LEMMA 3.2
$\Omega_*(S, \mathbf{z})$ *is path connected and dense in* $\Omega(S, \mathbf{z})$.

Proof
The space $\Omega(S, \mathbf{z})$ is the complement of the zero section of a complex vector bundle over the Teichmüller space. Since Teichmüller space is path connected, so is $\Omega(S, \mathbf{z})$. The space $\Omega_*(S, \mathbf{z})$ is the complement of a subspace with real codimension 2: the subspace $\{z_i \in \text{Zeros}(\omega)\}$ is codimension 2 since for any fixed $\omega \in \widetilde{\Omega}(S)$, z_i and $\text{Zeros}(\omega)$ are both zero dimensional, and the zeros (considered as a function from $\widetilde{\Omega}(S)$ to the $(2g-2)$-fold product $S \times \cdots \times S$ modulo the action by the symmetric group) vary continuously. It follows that $\Omega_*(S, \mathbf{z})$ is dense and path connected. □

We now come to a more interesting subspace. Define

$$\widetilde{\Omega}_{\text{best}}(S,\mathbf{z}) = \left\{ \omega \in \widetilde{\Omega}_*(S,\mathbf{z}) \;\middle|\; \begin{array}{l} |\text{Re}(\bar{\omega})| \text{ is arational, and for every} \\ i \neq j\,,\ \ \text{Per}_{ij}(\hat{\omega}) \not\subset \text{Per}(\hat{\omega}) \end{array} \right\}.$$

By construction, $\widetilde{\Omega}_{\text{best}}(S,\mathbf{z})$ is invariant by $\text{Diff}_0(S,\mathbf{z})$, and we define $\Omega_{\text{best}}(S,\mathbf{z})$ to be the image in $\Omega_*(S,\mathbf{z})$.

PROPOSITION 3.3
We have $|\text{Re}|(\Omega_{\text{best}}(S,\mathbf{z})) \subset \mathscr{AF}(S,\mathbf{z})$.

Proof
This is immediate from Proposition 3.1 and the definition of $\Omega_{\text{best}}(S,\mathbf{z})$. □

In order for this subspace to be useful, we need the following.

PROPOSITION 3.4
$\Omega_{\text{best}}(S,\mathbf{z})$ *is nonempty and dense in* $\Omega(S,\mathbf{z})$.

Proof
First, note that the set of $\omega \in \widetilde{\Omega}(S)$ for which $|\text{Re}(\bar{\omega})|$ is arational is a dense subset. Indeed, for *any* $\omega \in \widetilde{\Omega}(S)$, the set of θ for which $|\text{Re}(e^{i\theta}\omega)|$ fails to be arational is countable—there are only countably many directions with a saddle connection (see also [27]).

We therefore fix $\omega \in \widetilde{\Omega}_*(S,\mathbf{z})$ such that $|\text{Re}(\bar{\omega})|$ is arational and prove that the intersection $\Omega_{\text{best}}(S,\mathbf{z}) \cap F_{\bar{\omega}}$ is dense in $F_{\bar{\omega}}$. The proposition follows from this and Lemma 3.2.

Density in $F_{\bar{\omega}}$ comes from basic genericity considerations: we show that the relative periods vary by a translation of $\mathbb{R}$ in a controlled way as one moves around within the fibers, while the periods do not change. Since the sets of periods and relative periods are countable, this easily implies the result. We now explain this more precisely.

We consider a $\mathbf{B}$-trivialization, $f : S \times \mathbf{B} \to S$. Let $f^\omega : \mathbf{B} \to \widetilde{\Omega}(S)$ and $\hat{f}^\omega : \mathbf{B} \to \Omega(S,\mathbf{z})$ be the associated maps as in Section 2.7.

We choose the specific $\mathbf{B} = B_1 \times \cdots \times B_n$ and $\mathbf{U} = U_1 \times \cdots \times U_n$ so that for each $j = 1, \ldots, n$, the natural coordinate ζ_j based at z_j is defined on U_j (see Section 2.6). Moreover, we require that ζ_j map B_j diffeomorphically onto a square in $\mathbb{C}$. That is, there exists $\epsilon > 0$ such that

$$\zeta_j(B_j) = (-\epsilon,\epsilon)^2 = \left\{ x + iy \in \mathbb{C} \,\middle|\, x, y \in (-\epsilon,\epsilon) \right\}.$$

Observe that $U_j \cap \mathrm{Zeros}(\omega) = \emptyset$. Since $\mathrm{Zeros}(f^\omega(\mathbf{b})) = \mathrm{Zeros}(f_{\mathbf{b}}^*(\omega))$, $f_{\mathbf{b}}$ is the identity outside $\bigcup_i U_i$, and $f_{\mathbf{b}}(z_i) = b_i \in B_i \subset U_i$, it follows that $\hat{f}^\omega(\mathbf{B}) \subset \Omega_*(S, \mathbf{z})$.

CLAIM
There exists a dense subset $\mathbf{E} \subset \mathbf{B}$ *such that* $\hat{f}^\omega(\mathbf{E}) \subset \Omega_{\mathrm{best}}(S, \mathbf{z})$. *Equivalently,* $f_{\mathbf{b}}^*(\omega) \in \widetilde{\Omega}_{\mathrm{best}}(S, \mathbf{z})$ *for all* $\mathbf{b} \in \mathbf{E}$.

Since $\hat{f}^\omega(\mathbf{B})$ is a neighborhood of $\hat{\omega}$ in $F_{\bar{\omega}}$, this claim implies that there exists a point of $\Omega_{\mathrm{best}}(S, \mathbf{z})$ arbitrarily close to $\hat{\omega}$. Since $\hat{\omega}$ was an arbitrary point of a dense subset, this proves the proposition.

Proof
Let $\gamma : [0, 1] \to S$ be any path from z_i to z_j, representing an element of $\Gamma(S, z_i, z_j)$. The following says that the change in γ-period from ω to $f_{\mathbf{b}}^*(\omega)$ is independent of γ and is given by a simple function defined on $\mathbf{B}$.

SUBCLAIM
We have

$$\int_\gamma \mathrm{Re}(f_{\mathbf{b}}^* \omega) - \int_\gamma \mathrm{Re}(\omega) = \mathrm{Re}\big(\zeta_j(b_j)\big) - \mathrm{Re}\big(\zeta_i(b_i)\big).$$

Proof
For any $\mathbf{b} \in \mathbf{B}$, fix a path $\sigma : [0, 1] \to \mathbf{B}$ going from $\mathbf{z}$ to $\mathbf{b}$, writing $\sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t))$. This determines a map

$$H : [0, 1] \times [0, 1] \to S$$

by $H(t, u) = f_{\sigma(u)}(\gamma(t))$.

The restriction of H to the boundary of $[0, 1] \times [0, 1]$ determines four paths. The bottom path is $H(t, 0) = \gamma(t)$. The top path is $H(t, 1) = f_{\sigma(1)}(\gamma(t)) = f_{\mathbf{b}}(\gamma(t))$. The left side path, oriented up, is $H(0, u) = f_{\sigma(u)}(\gamma(0)) = f_{\sigma(u)}(z_i) = \sigma_i(u)$, and the right side path, also oriented up, is $H(1, u) = f_{\sigma(u)}(\gamma(1)) = f_{\sigma(u)}(z_j) = \sigma_j(u)$. Because $\mathrm{Re}(\omega)$ is closed, the integral over the boundary of $[0, 1] \times [0, 1]$ of $H^*(\mathrm{Re}(\omega))$ is zero, and so

$$0 = \int_{H(\partial([0,1]\times[0,1]))} \mathrm{Re}(\omega) = \int_{f_{\mathbf{b}}(\gamma)} \mathrm{Re}(\omega) + \int_{\sigma_i} \mathrm{Re}(\omega) - \int_{\sigma_j} \mathrm{Re}(\omega) - \int_\gamma \mathrm{Re}(\omega).$$

Since σ_i connects z_i to b_i within B_i and σ_j connects z_j to b_j within B_j, we have

$$\int_{\sigma_i} \mathrm{Re}(\omega) = \mathrm{Re}\big(\zeta_i(b_i)\big) \qquad \text{and} \qquad \int_{\sigma_j} \mathrm{Re}(\omega) = \mathrm{Re}\big(\zeta_j(b_j)\big).$$

Combining this with the previous equation and the descriptions of the four paths involved in that equation, we obtain

$$\int_\gamma \mathrm{Re}(f_{\mathbf{b}}^*\omega) = \int_{f_{\mathbf{b}}(\gamma)} \mathrm{Re}(\omega) = \int_\gamma \mathrm{Re}(\omega) + \mathrm{Re}\big(\zeta_j(b_j)\big) - \mathrm{Re}\big(\zeta_i(b_i)\big),$$

and this proves the subclaim. □

We now see that

$$\mathrm{Per}_{ij}\big(\hat{f}^\omega(\mathbf{b})\big) = \mathrm{Per}_{ij}\big(\widehat{f_{\mathbf{b}}^*(\omega)}\big) = \mathrm{Per}_{ij}(\widehat{\omega}) + \mathrm{Re}\big(\zeta_j(b_j)\big) - \mathrm{Re}\big(\zeta_i(b_i)\big).$$

That is, the subsets of $\mathbb{R}$, $\mathrm{Per}_{ij}(\hat{\omega})$, and $\mathrm{Per}_{ij}(\hat{f}^\omega(\mathbf{b}))$, differ exactly by a translation by $\mathrm{Re}(\zeta_j(b_j)) - \mathrm{Re}(\zeta_i(b_i))$. Since the set of periods and relative periods are all countable sets and since $\mathrm{Per}(\hat{f}^\omega(\mathbf{b})) = \mathrm{Per}(\hat{\omega})$, it follows that for almost all $\mathbf{b}$ we have

$$\mathrm{Per}_{ij}\big(\hat{f}^\omega(\mathbf{b})\big) \cap \mathrm{Per}\big(\hat{f}^\omega(\mathbf{b})\big) = \emptyset.$$

In particular, setting

$$\mathbf{E} = \Big\{\mathbf{b} \,\Big|\, \mathrm{Per}_{ij}(\hat{f}^\omega(\mathbf{b})) \cap \mathrm{Per}(\hat{f}^\omega(\mathbf{b})) = \emptyset\Big\},$$

we have found the required set, and the claim is proved. □

This completes the proof of Proposition 3.4. □

COROLLARY 3.5
$|\mathrm{Re}|(\Omega_{\mathrm{best}}(S, \mathbf{z}))$ is dense in $\mathscr{AF}(S, \mathbf{z})$.

Proof
Since $\Omega_{\mathrm{best}}(S, \mathbf{z})$ is invariant by $\mathrm{Mod}(S, \mathbf{z})$, Lemma 2.4 implies that $|\mathrm{Re}|(\Omega_{\mathrm{best}}(S, \mathbf{z}))$ is also $\mathrm{Mod}(S, \mathbf{z})$-invariant. Now, $\mathrm{Mod}(S, \mathbf{z})$ acts minimally on $\mathbb{P}\mathscr{MF}(S, \mathbf{z})$ (see [18, Theorem 6.7]), and so it follows that the image of $|\mathrm{Re}|(\Omega_{\mathrm{best}}(S, \mathbf{z}))$ in $\mathbb{P}\mathscr{MF}(S, \mathbf{z})$ is dense. This implies that the same is true of $|\mathrm{Re}|(\Omega_{\mathrm{best}}(S, \mathbf{z}))$ in $\mathscr{MF}(S, \mathbf{z})$ and so also in $\mathscr{AF}(S, \mathbf{z})$. □

4. Paths in $\mathscr{AF}(S, \mathbf{z})$

In this section, we prove the key ingredient, which produces an abundance of paths in $\mathscr{AF}(S, \mathbf{z})$.

THEOREM 4.1
There is an open cover of $\mathscr{U}$ of $\Omega_*(S, \mathbf{z})$ with the property that for any $U \in \mathscr{U}$ and any $\hat{\omega}, \hat{\eta} \in U \cap \Omega_{\mathrm{best}}(S, \mathbf{z})$, there is a path in $\mathscr{AF}(S, \mathbf{z})$ connecting $|\mathrm{Re}(\hat{\omega})|$ and $|\mathrm{Re}(\hat{\eta})|$.

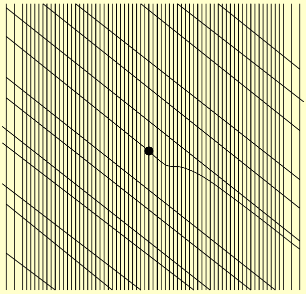

Figure 1. Closing up to a simple closed curve through z_1

4.1. Twisting pairs

Fix a point $\omega \in \widetilde{\Omega}_*(S, \mathbf{z})$. We say that a pair of simple closed curves α and β on S are a *twisting pair* for ω if

- α and β meet transversely and minimally,
- α and β fill S,
- $\mathbf{z} \subset \alpha \cap \beta$, and
- α and β are transverse to $\mathscr{F}(\mathrm{Re}(\omega))$, and $\alpha \cap \mathrm{Zeros}(\omega) = \beta \cap \mathrm{Zeros}(\omega) = \emptyset$.

LEMMA 4.2

For any $\omega \in \widetilde{\Omega}_*(S, \mathbf{z})$, there is a twisting pair α, β for ω.

Proof

Pick two distinct points $e^{i\theta_1}, e^{i\theta_2} \in S^1 \subset \mathbb{C}$, neither of which is equal to 1, and let $\mathscr{F}_j = \mathscr{F}(\mathrm{Re}(e^{i\theta_j}\omega))$ and $\mu_j = |\mathrm{Re}(e^{i\theta_j}\omega)|$ for $j = 1, 2$. We choose $e^{i\theta_1}, e^{i\theta_2}$ so that $(\mathscr{F}_1, \mu_1)$ and $(\mathscr{F}_2, \mu_2)$ are uniquely ergodic and arational. According to [27], this is true for almost every $e^{i\theta} \in S^1$. We also assume, as we may, that the leaves through z_1 do not pass through $\mathrm{Zeros}(\omega) = \mathrm{Zeros}(e^{i\theta_j}\omega)$ for $j = 1, 2$.

For each $j = 1, 2$, we construct a sequence of simple closed curves $\{\gamma_k^j\}_{k=1}^{\infty}$ which approximate leaves of $\mathscr{F}_j$. Start with a ray in the leaf of $\mathscr{F}_j$ emanating from z_1. Take an initial segment of length at least k which comes sufficiently close to z_1 so that it can be smoothly closed up to a simple closed curve γ_k^j transverse to $\mathscr{F}(\mathrm{Re}(\omega))$ (see Figure 1) For this, one can work in the natural coordinate ζ_1 based at z_1 for ω (as described in Section 2.6) and appeal to the fact that the ray is dense in S by arationality of $\mathscr{F}_j$.

Given any $\epsilon > 0$, we can also assume that the curve γ_k^j makes an angle at most ϵ with $\mathscr{F}_j$ at every point, provided k is sufficiently large. In particular, if ϵ is sufficiently small, it follows that γ_k^1 and γ_k^2 will intersect transversely and minimally.

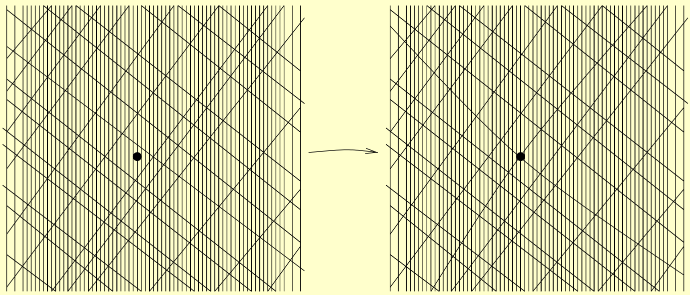

Figure 2. Perturbing γ_k^1 and γ_k^2 to α and β passing through z_i

Observe that by construction, $\mathbf{i}(\gamma_k^j, \mu_j) \to 0$ as $k \to \infty$, and hence by unique ergodicity and arationality, $\gamma_k^j \to \mathbb{P}(\mu_j)$ in $\mathbb{P}\mathscr{MF}(S)$ for each $j = 1, 2$. So by taking k even larger if necessary, we may assume that γ_k^1 and γ_k^2 fill S.

Finally, because both rays are dense and transverse to each other, their points of intersection are dense. Hence, by taking k larger still, we can guarantee that the set of intersection points of γ_k^1 and γ_k^2 are ϵ-dense. In particular, each z_i is within ϵ of a point of intersection of γ_k^1 and γ_k^2. For sufficiently small ϵ (again, working in a natural coordinate), we can perturb γ_k^1 and γ_k^2 to simple closed curves α and β which are a twisting pair for ω (see Figure 2). □

4.2. The group of a twisting pair.
Now, let α, β be a twisting pair for $\omega \in \widetilde{\Omega}_*(S)$. Our first goal is to define isotopies $D_{\alpha,t}$ and $D_{\beta,t}$ supported on annular neighborhoods of α and β, respectively, which push the set $\mathbf{z}$ once around α and β, respectively, at a constant speed as measured with respect to $\mathrm{Re}(\omega)$. We use these isotopies to construct paths in $\Omega_{\mathrm{best}}(S, \mathbf{z})$ which are used in the proof of Theorem 4.1. We now describe this in more detail.

Let $\epsilon > 0$ be such that the ϵ-neighborhood $N_\epsilon(\alpha)$ is an annulus, and the foliation $\mathscr{F}(\mathrm{Re}(\omega))$ restricted to this annulus is the product foliation $N_\epsilon(\alpha) \cong S^1 \times [0, 1]$. More precisely, we have a diffeomorphism

$$f_\alpha : N_\epsilon(\alpha) \to S^1 \times [0, 1]$$

which we assume takes α to $S^1 \times \{1/2\}$. Moreover, we can choose f_α so that

$$f_\alpha^*(ds) = \frac{1}{c}\,\mathrm{Re}(\omega),$$

where $c = \mathbf{i}(\alpha, |\mathrm{Re}(\omega)|)$ and ds is the 1-form coming from the factor $S^1 = \mathbb{R}/\mathbb{Z}$, where it is defined by the standard coordinate s on $\mathbb{R}$. Let $D_{\alpha,t} : S \to S$, $t \in [0, 1]$, be an isotopy supported on $N_\epsilon(\alpha)$ defined as follows.

Let $\psi : [0,1] \to [0,1]$ be a smooth function identically zero in a neighborhood of 0 and 1 and equal to 1 at 1/2. Define

$$D^0_{\alpha,t} : S^1 \times [0,1] \to S^1 \times [0,1]$$

by

$$D^0_{\alpha,t}(s,x) = \big(s + t\psi(x), x\big).$$

Then define $D_{\alpha,t}$ to be the identity outside $N_\epsilon(\alpha)$ and equal to

$$f_\alpha^{-1} D^0_{\alpha,t} f_\alpha$$

on $N_\epsilon(\alpha)$.

Likewise, we can define

$$D_{\beta,t} : S \to S$$

using β in place of α.

PROPOSITION 4.3
If $\omega \in \widetilde{\Omega}_{\mathrm{best}}(S, \mathbf{z})$, then $D^*_{\alpha,t}(\omega)$, $D^*_{\beta,t}(\omega) \in \widetilde{\Omega}_{\mathrm{best}}(S, \mathbf{z})$ for all $t \in [0,1]$.

Proof
Observe that $\delta_i(t) = D_{\alpha,t}(z_i)$, $t \in [0,1]$, is a parameterization of α starting at z_i with

$$\int_{\delta_i([0,t])} \mathrm{Re}(\omega) = t \int_\alpha \mathrm{Re}(\omega). \tag{8}$$

A computation similar to the one done in the proof of Proposition 3.4 tells us that for any $\gamma \in \Gamma(S, z_i, z_j)$, we have

$$\int_\gamma D^*_{\alpha,t}\big(\mathrm{Re}(\omega)\big) - \int_\gamma \mathrm{Re}(\omega) = \int_{\delta_j([0,t])} \mathrm{Re}(\omega) - \int_{\delta_i([0,t])} \mathrm{Re}(\omega).$$

According to (8), this becomes

$$\int_\gamma D^*_{\alpha,t}\big(\mathrm{Re}(\omega)\big) - \int_\gamma \mathrm{Re}(\omega) = t\int_\alpha \mathrm{Re}(\omega) - t\int_\alpha \mathrm{Re}(\omega) = 0.$$

It follows that

$$\mathrm{Per}_{ij}(\omega) = \mathrm{Per}_{ij}\big(D^*_{\alpha,t}(\omega)\big), \quad \forall t \in [0,1]. \tag{9}$$

Similarly, for $D_{\beta,t}$ we obtain

$$\mathrm{Per}_{ij}(\omega) = \mathrm{Per}_{ij}\big(D^*_{\beta,t}(\omega)\big), \quad \forall t \in [0,1]. \tag{10}$$

Finally, observe that for all $t \in [0,1]$,

$$\left|\mathrm{Re}(\overline{D^*_{\alpha,t}(\omega)})\right| = |\mathrm{Re}(\bar{\omega})| = \left|\mathrm{Re}(\overline{D^*_{\beta,t}(\omega)})\right|.$$

From this, equations (9) and (10), and the definition of $\widetilde{\Omega}_{\mathrm{best}}(S,\mathbf{z})$, the proposition follows. □

The diffeomorphisms $D_\alpha = D_{\alpha,1}$ and $D_\beta = D_{\beta,1}$ are in $\mathrm{Diff}_{0,\mathbf{z}}(S,\mathbf{z})$. In fact, D_α can be alternatively described as a Dehn twist in one component of $\partial N_\epsilon(\alpha)$ composed with an inverse Dehn twist in the other, and similarly for D_β. As usual, we let $\hat{D}_\alpha, \hat{D}_\beta \in \mathscr{B}(S,\mathbf{z}) < \mathrm{Mod}(S,\mathbf{z})$ denote the associated mapping classes.

As described in Section 2.6, equation (5), the left action is described by the equations

$$D_\alpha^{-1} \cdot \omega = D^*_\alpha(\omega)$$

and

$$D_\beta^{-1} \cdot \omega = D^*_\beta(\omega).$$

Let $\mathscr{G}(\hat{D}_\alpha, \hat{D}_\beta)$ be the Cayley graph of $\langle \hat{D}_\alpha, \hat{D}_\beta \rangle < \mathscr{B}(S,\mathbf{z})$ with respect to the generators $\hat{D}_\alpha, \hat{D}_\beta$.

PROPOSITION 4.4
Suppose that α, β is a twisting pair for ω. If $\hat{\omega} \in \Omega_{\mathrm{best}}(S,\mathbf{z})$, then there is a $\langle \hat{D}_\alpha, \hat{D}_\beta \rangle$-equivariant continuous map

$$P_{\hat{\omega}} : \mathscr{G}(\hat{D}_\alpha, \hat{D}_\beta) \to \Omega_{\mathrm{best}}(S,\mathbf{z})$$

sending $\mathrm{Id} \in \langle \hat{D}_\alpha, \hat{D}_\beta \rangle$ *to* $\hat{\omega}$.

Proof
Proposition 4.3 implies that $t \mapsto D^*_{\alpha,t}(\omega)$ is a path from ω to $D^*_\alpha(\omega)$ in $\widetilde{\Omega}_{\mathrm{best}}(S,\mathbf{z})$. Projecting down to $\Omega_{\mathrm{best}}(S,\mathbf{z})$, we obtain a path from $\hat{\omega}$ to $\hat{D}_\alpha^{-1} \cdot \hat{\omega}$. Likewise, we obtain a path from $\hat{\omega}$ to $\hat{D}_\beta^{-1} \cdot \hat{\omega}$.

We now define

$$P_{\hat{\omega}} : \mathscr{G}(\hat{D}_\alpha, \hat{D}_\beta) \to \Omega_{\mathrm{best}}(S,\mathbf{z}).$$

We do this first on the edge from Id to $\hat{D}_\alpha^{-1}$ by sending it to the path from $\hat{\omega}$ to $\hat{D}_\alpha^{-1} \cdot \hat{\omega}$ and sending the edge from Id to $\hat{D}_\beta^{-1}$ to the path from $\hat{\omega}$ to $\hat{D}_\beta^{-1} \cdot \hat{\omega}$. There is now a unique way to extend this to a $\langle \hat{D}_\alpha, \hat{D}_\beta \rangle$-equivariant continuous map. □

LEMMA 4.5
If α, β is a twisting pair for ω, then $\langle \hat{D}_\alpha, \hat{D}_\beta \rangle < \mathscr{B}(S, \mathbf{z})$ contains a pseudo-Anosov mapping class ψ.

Proof
Since α, β are a twisting pair, they fill S. This implies that $\partial N_\epsilon(\alpha)$, $\partial N_\epsilon(\beta)$ fill $(S, \mathbf{z})$, and so for any $\delta \in \mathscr{C}^0(S, \mathbf{z})$, $\mathbf{i}(\delta, \partial N_\epsilon(\alpha)) \neq 0$ or $\mathbf{i}(\delta, \partial N_\epsilon(\beta)) \neq 0$. Since $\hat{D}_\alpha$ and $\hat{D}_\beta$ are multitwists in $N_\epsilon(\alpha)$ and $N_\epsilon(\beta)$, respectively, it follows that $\hat{D}_\alpha^k(\delta) \neq \delta$ or $\hat{D}_\beta^k(\delta) \neq \delta$ for all $k \neq 0$.

In particular, there is no curve δ with a finite $\langle \hat{D}_\alpha, \hat{D}_\beta \rangle$-orbit. That is, a finite index pure subgroup of $\langle \hat{D}_\alpha, \hat{D}_\beta \rangle$ is an *irreducible* subgroup of $\mathscr{B}(S, \mathbf{z}) < \mathrm{Mod}(S, \mathbf{z})$. According to a theorem of Ivanov, [24, Theorem 5.9], there exists a pseudo-Anosov mapping class $\psi \in \langle \hat{D}_\alpha, \hat{D}_\beta \rangle$. □

Given $\omega \in \widetilde{\Omega}_{\text{best}}(S)$ and a twisting pair α, β for ω, we let $\langle \hat{D}_\alpha, \hat{D}_\beta \rangle$ be the associated group, and we let ψ be a pseudo-Anosov element guaranteed by Lemma 4.5. We let μ_s and μ_u denote stable and unstable foliations for ψ, respectively (unique up to scalar multiple).

LEMMA 4.6
There exists a path connecting $|\mathrm{Re}(\hat{\omega})|$ to μ_s in $\mathscr{A}\mathscr{F}(S, \mathbf{z})$.

Proof
It is more convenient to work in the space $\mathbb{P}\mathscr{M}\mathscr{F}(S, \mathbf{z})$, and we write $\mathbb{P} : \mathscr{M}\mathscr{F}(S, \mathbf{z}) \to \mathbb{P}\mathscr{M}\mathscr{F}(S, \mathbf{z})$ for the quotient map. Since the fibers of Φ are homeomorphic to $\mathbb{R}_+$ and since there exists a section of $\mathbb{P}$, a path between projective classes $\mathbb{P}(|\mathrm{Re}(\hat{\omega})|)$ and $\mathbb{P}(\mu_s)$ easily implies the existence of a path between any representatives $|\mathrm{Re}(\hat{\omega})|$ and μ_s. The advantage to working in $\mathbb{P}\mathscr{M}\mathscr{F}(S, \mathbf{z})$ is that we can appeal to the dynamics as described in Section 2.5.

Consider the path in $\mathscr{G}(\hat{D}_\alpha, \hat{D}_\beta)$ given by $[\mathrm{Id}, \psi] \cup [\psi, \psi^2] \cup [\psi^2, \psi^3] \cup \cdots$, where $[\mathrm{Id}, \psi]$ is a geodesic from Id to ψ, and $[\psi^k, \psi^{k+1}]$ is the image of this geodesic under ψ^k. We can parameterize this as

$$f : [0, 1) \to [\mathrm{Id}, \psi] \cup [\psi, \psi^2] \cup \cdots$$

sending $[0, 1/2]$ linearly onto the first segment, $[1/2, 3/4]$ linearly onto the second segment, and so on.

CLAIM
The map $h = \mathbb{P} \circ |\mathrm{Re}| \circ P_{\hat{\omega}} \circ f : [0, 1) \to \mathbb{P}\mathscr{MF}(S, \mathbf{z})$ *extends to a continuous map defined on* $[0, 1]$ *by setting* $h(1) = \mathbb{P}(\mu_s)$.

Proof
Let $\{x_k\}$ be any sequence in $[0, 1)$ tending to 1; we show that $h(x_k) \to \mathbb{P}(\mu_s)$. Lemma 2.2 implies that $\mathbb{P}(\mu_u) \notin h([0, 1))$. Since

$$h\big([0, 1)\big) = \bigcup_{j=0}^{\infty} \psi^j \left(h\left(\left[0, \frac{1}{2}\right]\right)\right),$$

it follows that for each k, there is a $j(k)$ such that $f(x_k) \in [\psi^{j(k)}, \psi^{j(k)+1}]$. Since $x_k \to 1$, it must be that $j(k) \to \infty$ as $k \to \infty$.

Now, let V be any neighborhood of $\mathbb{P}(\mu_s)$ in $\mathbb{P}\mathscr{MF}(S, \mathbf{z}) - \{\mathbb{P}(\mu_u)\}$. Since $h([0, 1/2])$ is compact, there exists $J > 0$ such that for all $j \geq J$, $\psi^j(h([0, 1/2])) \subset V$. By the previous paragraph, there exists $K > 0$ such that for all $k \geq K$, $j(k) \geq J$, which implies

$$h(x_k) = \mathbb{P} \circ |\mathrm{Re}| \circ P_{\hat{\omega}} \circ f(x_k) \in \mathbb{P} \circ |\mathrm{Re}| \circ P_{\hat{\omega}}([\psi^{j(k)}, \psi^{j(k)+1}]).$$

Since we also have

$$\begin{aligned} \mathbb{P} \circ |\mathrm{Re}| \circ P_{\hat{\omega}}([\psi^{j(k)}, \psi^{j(k)+1}]) &= \mathbb{P} \circ |\mathrm{Re}| \circ P_{\hat{\omega}} \left(\psi^{j(k)}([\mathrm{Id}, \psi])\right) \\ &= \psi^{j(k)}\left(\mathbb{P} \circ |\mathrm{Re}| \circ P_{\hat{\omega}} \circ f\left(\left[0, \frac{1}{2}\right]\right)\right) \\ &= \psi^{j(k)}\left(h\left(\left[0, \frac{1}{2}\right]\right)\right) \subset V, \end{aligned}$$

it follows that $h(x_k) \in V$. This completes the proof of the claim. □

The image of $h : [0, 1] \to \mathbb{P}\mathscr{MF}(S, \mathbf{z})$ is contained in $\mathbb{P}\mathscr{AF}(S, \mathbf{z})$, and it connects $\mathbb{P}(|\mathrm{Re}(\hat{\omega})|)$ to $\mathbb{P}(\mu_s)$, as required. This completes the proof of Lemma 4.6. □

4.3. The open cover
The definition of a twisting pair for $\omega \in \widetilde{\Omega}_*(S)$ had only one condition that involved ω. Namely, we required that each of α, β be transverse to $\mathscr{F}(\mathrm{Re}(\omega))$. It is not surprising then that a twisting pair for ω is also a twisting pair for elements of $\widetilde{\Omega}_*(S)$ which are sufficiently close to ω.

LEMMA 4.7
Given $\omega \in \widetilde{\Omega}_*(S)$ and a twisting pair α, β for ω, there is a neighborhood U' of ω such that for all $\eta \in U'$, α, β is a twisting pair for η.

Proof
By definition, the underlying foliation $\mathscr{F}(\mathrm{Re}(\omega))$ is obtained by integrating $\ker(\mathrm{Re}(\omega))$. Thus, the condition that a curve γ be transverse to $\mathscr{F}(\mathrm{Re}(\omega))$ is equivalent to requiring that $\mathrm{Re}(\omega)$ restricted to γ be nonvanishing. Perturbing the 1-form $\mathrm{Re}(\omega)$ slightly preserves the property that it is nonvanishing on γ since γ is compact. Therefore, there is a neighborhood U' of ω in $\widetilde{\Omega}_*(S)$ such that if $\eta \in U'$, then the restriction of $\mathrm{Re}(\eta)$ to both α and β is nonvanishing. It follows that α, β is a twisting pair for η, as required. □

We can now give the following.

Proof of Theorem 4.1
Fix $\nu \in \widetilde{\Omega}_*(S)$, and let α, β be a twisting pair for ν. Lemma 4.7 provides a neighborhood U', so that for all $\omega \in U'$, α, β is also a twisting pair for ω. We let $\hat{\psi}$ be a pseudo-Anosov mapping class in $\langle \hat{D}_\alpha, \hat{D}_\beta \rangle$ as in Lemma 4.5, and we let μ_s be its stable foliation.

It follows from the discussions in Sections 2.4 and 2.6 that we can locally find a continuous section of $\widetilde{\Omega}_*(S) \to \Omega_*(S, \mathbf{z})$. In particular, there is a neighborhood U of $\hat{\nu}$ and a continuous section $\sigma : U \to \widetilde{\Omega}_*(S)$ with $\sigma(U) \subset U'$. Now, given $\hat{\omega}, \hat{\eta} \in U \cap \Omega_{\mathrm{best}}(S, \mathbf{z})$, since α, β is a twisting pair for both ω and η, Lemma 4.6 guarantees paths from $|\mathrm{Re}(\hat{\omega})|$ to μ_s and from $|\mathrm{Re}(\hat{\eta})|$ to μ_s in $\mathscr{AF}(S, \mathbf{z})$. Therefore, we can connect $|\mathrm{Re}(\hat{\omega})|$ and $|\mathrm{Re}(\hat{\eta})|$ by a path in $\mathscr{AF}(S, \mathbf{z})$. Since ν was an arbitrary point of $\widetilde{\Omega}_*(S)$, we have constructed the open cover. □

4.4. The main theorem for $\mathbf{z} \neq \emptyset$

We now put all the ingredients together to prove the main theorem for surfaces of genus at least 2 and nonempty marked point set.

Proof of Theorem 1.1 for $\mathbf{z} \neq \emptyset$
Corollary 3.5 implies that $|\mathrm{Re}|(\Omega_{\mathrm{best}}(S, \mathbf{z}))$ is dense in $\mathscr{AF}(S, \mathbf{z})$. Therefore, to prove that $\mathscr{AF}(S, \mathbf{z})$ is connected, we show that for any two points $\hat{\omega}, \hat{\eta} \in \Omega_{\mathrm{best}}(S, \mathbf{z})$, there is a path connecting $|\mathrm{Re}|(\hat{\omega}) = |\mathrm{Re}(\hat{\omega})|$ to $|\mathrm{Re}|(\hat{\eta}) = |\mathrm{Re}(\hat{\eta})|$ in $\mathscr{AF}(S, \mathbf{z})$. This produces a dense path-connected subset of $\mathscr{AF}(S, \mathbf{z})$, and so $\mathscr{AF}(S, \mathbf{z})$ is connected.

Let $\hat{\omega}, \hat{\eta} \in \Omega_{\mathrm{best}}(S, \mathbf{z})$ be any two points. According to Lemma 3.2, there is a path δ in $\Omega_*(S, \mathbf{z})$ connecting $\hat{\omega}$ to $\hat{\eta}$. Because δ is compact, the cover $\mathscr{U}$ restricted to δ has a finite subcover. From this we can produce a finite set $U_0, \ldots, U_k \in \mathscr{U}$

such that $\hat{\omega} \in U_0$, $\hat{\eta} \in U_k$, and $U_j \cap U_{j+1} \neq \emptyset$ for all $j = 0, \ldots, k-1$. For each $j = 0, \ldots, k-1$, by appealing to Proposition 3.4 we can therefore find some element in the intersection

$$\hat{\omega}_j \in U_j \cap U_{j+1} \cap \Omega_{\text{best}}(S, \mathbf{z}).$$

By Theorem 4.1, for every $j = 0, \ldots, k-2$ since $\hat{\omega}_j$ and $\hat{\omega}_{j+1}$ are in $U_{j+1} \cap \Omega_{\text{best}}(S, \mathbf{z})$, there is a path in $\mathscr{AF}(S, \mathbf{z})$ connecting $|\mathrm{Re}(\hat{\omega}_j)|$ and $|\mathrm{Re}(\hat{\omega}_{j+1})|$. Likewise, there is a path connecting $|\mathrm{Re}(\hat{\omega})|$ to $|\mathrm{Re}(\hat{\omega}_0)|$ and $|\mathrm{Re}(\hat{\eta})|$ to $|\mathrm{Re}(\hat{\omega}_{k-1})|$. Concatenating these paths, we obtain a path connecting $|\mathrm{Re}(\hat{\omega})|$ to $|\mathrm{Re}(\hat{\eta})|$, as required. □

4.5. Cut points: $\mathbf{z} \neq \emptyset$

For the applications to Kleinian groups, we need to see that $\mathscr{AF}(S, \mathbf{z})/\sim$ has no cut points, meaning no points whose removal disconnects. Together with the proof in Section 4.4, the following proves Theorem 1.4 for the case $\mathbf{z} \neq \emptyset$.

THEOREM 4.8
If $\mathbf{z} \neq \emptyset$, *then* $\mathscr{AF}(S, \mathbf{z})/\sim$ *has no cut points.*

In what follows, we let $[\mu]$ denote the equivalence class in $\mathscr{AF}(S, \mathbf{z})/\sim$ of the measured foliation $\mu \in \mathscr{AF}(S, \mathbf{z})$.

Proof
The dense path-connected subset of $\mathscr{AF}(S, \mathbf{z})$ descends to a dense path-connected subset W in $\mathscr{AF}(S, \mathbf{z})/\sim$. To prove the theorem, it suffices to verify that given any point of $\mathscr{AF}(S, \mathbf{z})/\sim$, there is a dense path-connected subset W_0 of the complement. Let $|\mu| \in \mathscr{AF}(S, \mathbf{z})/\sim$ be an arbitrary point. If $|\mu| \notin W$, then we may take $W = W_0$, so we assume that $|\mu| \in W$.

Points of W are of two types:

(1) the stable foliations of the pseudo-Anosov mapping classes coming from Lemma 4.6, and
(2) the points in the image of $|\mathrm{Re}|(\Omega_{\text{best}}(S, \mathbf{z}))$.

It follows from Lemma 2.2 that these two subsets of W are disjoint.

Suppose first that $\mu = |\mathrm{Re}|(\hat{\omega}) \in |\mathrm{Re}|(\Omega_{\text{best}}(S, \mathbf{z}))$, so we are in case (2). We can define

$$\Omega'_{\text{best}}(S, \mathbf{z}) = \left\{ \hat{\eta} \in \Omega_{\text{best}}(S, \mathbf{z}) \,\middle|\, [|\mathrm{Re}|(\bar{\eta})] \neq [|\mathrm{Re}|(\bar{\omega})] = [\mu] \right\}.$$

The space $\Omega'_{\text{best}}(S, \mathbf{z})$ also has the property that it is dense in $\Omega(S, \mathbf{z})$ and has dense image in $\mathscr{AF}(S, \mathbf{z})$. This set can be used in place of $\Omega_{\text{best}}(S, \mathbf{z})$ to build a path-connected dense set with image W_0 in $\mathcal{AF}(S, \mathbf{z})/\sim$. By construction, $[\mu] \notin W_0$, and so it provides the required connected dense subset.

Now we assume that $\mu = \mu_s$ is the stable foliation for a pseudo-Anosov ψ as found in Lemma 4.6 and that $[\mu] \in W$. Such a foliation $[\mu_s] \in W$ must arise from the open cover $\mathcal{U}$ constructed in the proof of Theorem 4.1 as follows. For some $U \in \mathcal{U}$, we showed that

$$|\mathrm{Re}|\big(\Omega_{\mathrm{best}}(S, \mathbf{z}) \cap U\big) \cup \{\mu_s\}$$

is path connected. Since the unstable foliation μ_u for ψ is the stable foliation for ψ^{-1}, it follows that the same proof shows that

$$|\mathrm{Re}|\big(\Omega_{\mathrm{best}}(S, \mathbf{z}) \cap U\big) \cup \{\mu_u\}$$

is path connected. In particular, we see that

$$W_0 = \big(W - \{[\mu_s]\}\big) \cup \{[\mu_u]\}$$

is a dense path-connected subset disjoint from $[\mu_s]$, as required. □

Remark 4.9

By connecting an appropriately chosen pair of points to both the stable and the unstable foliations of the pseudo-Anosov mapping class ψ, we can construct maps of circles into $\mathcal{AF}(S, \mathbf{z})$. From these, one can find embedded circles.

5. Closed surfaces

The proof of Theorem 1.1 in the case $\mathbf{z} = \emptyset$ uses the case $\mathbf{z} \neq \emptyset$. The argument involves branched covers, so we begin with some elementary observations.

5.1. Branched covers

Let $\mathbf{z}' \subset S'$ be a finite set. We say that a branched cover $f : S \to S'$ is *properly branched over* $\mathbf{z}'$ if f is a covering map from the complement of $f^{-1}(\mathbf{z}')$ in S to the complement of $\mathbf{z}'$ in S' and if the restriction to any point of $f^{-1}(\mathbf{z}')$ has no neighborhood on which f is injective. Said differently, $\mathbf{z}'$ is precisely the branching locus in S', and f nontrivially branches at every point of $f^{-1}(\mathbf{z}')$.

PROPOSITION 5.1

Suppose that $f : S \to S'$ is a branched cover, properly branched over $\mathbf{z}' \subset S'$. Then there is an embedding

$$f^* : \mathcal{MF}(S', \mathbf{z}') \to \mathcal{MF}(S).$$

Moreover, $f^*(\mathcal{AF}(S', \mathbf{z}')) \subset \mathcal{AF}(S)$.

Proof
Given a measured foliation $(\mathscr{F}, \mu)$ on $(S', \mathbf{z}')$, there is a natural way of defining a measured foliation on S, denoted $f^*(\mathscr{F}, \mu)$, as follows. We define the underlying foliation $f^*(\mathscr{F})$ so that the leaves are precisely the preimages of leaves on S'. The transverse measure, denoted $f^*(\mu)$, is defined by declaring the measure of an arc on S to be the measure of its image in S'. The f-image of a leaf cycle for $f^*(\mathscr{F})$ can be used to construct one for $\mathscr{F}$, and so $f^*(\mathscr{AF}(S', \mathbf{z}')) \subset \mathscr{AF}(S)$.

We must therefore show that f^* is an embedding. One proof of this appeals to the theory of train tracks. We give a different proof using quadratic differentials.

Fix a complex structure on S' and one on S so that $f : S \to S'$ is holomorphic. According to the work of Hubbard and Masur [22] and Gardiner [20], the space $Q(S', \mathbf{z}')$ of integrable meromorphic quadratic differentials on S' with the only poles at $\mathbf{z}'$ is naturally homeomorphic to $\mathscr{MF}(S', \mathbf{z}')$ (see also Marden and Strebel [31]). The homeomorphism is given by sending a quadratic differential $q \in Q(S', \mathbf{z}')$ to the measure class of its vertical foliation $[\mathscr{F}(q), \mu(q)]$. Likewise, the space of holomorphic quadratic differentials $Q(S)$ is naturally homeomorphic to $\mathscr{MF}(S)$.

The pullback

$$f^* : Q(S', \mathbf{z}') \to Q(S)$$

is an embedding, and one checks that

$$f^*\big(\mathscr{F}(q)\big) = \mathscr{F}\big(f^*(q)\big) \qquad \text{and} \qquad f^*\big(\mu(q)\big) = \mu\big(f^*(q)\big).$$

It follows that

$$f^* : \mathscr{MF}(S', \mathbf{z}') \to \mathscr{MF}(S)$$

is an embedding, as required. □

5.2. Graph of involutions
Let σ be an involution of S with nonempty fixed point set. We write

$$f_\sigma : S \to S_\sigma = S/\langle\sigma\rangle$$

for the quotient. If f is properly branched over $\mathbf{z}_\sigma \subset S_\sigma$ ($\mathbf{z}_\sigma \neq \emptyset$) and S_σ has genus at least 2, then we say that σ is an *allowable involution*.

Fix an allowable involution σ; we define the *graph of* σ*–involutions* $\mathfrak{G}_\sigma(S)$ as follows. The vertex set of $\mathfrak{G}_\sigma(S)$ is in a one-to-one correspondence with the $\mathrm{Mod}(S)$ conjugates of σ. If σ_0 and σ_1 are conjugates of σ, then we connect the associated vertices (also denoted σ_0 and σ_1) by an edge if and only if

- $G = \langle\sigma_0, \sigma_1\rangle$ is a finite group and

- the quotient $f_G : S \to S_G = S/G$ is properly branched over $\mathbf{z}_G$ and $(S_G, \mathbf{z}_G)$ admits a pseudo-Anosov mapping class.

THEOREM 5.2
For each closed surface S of genus at least 4, there exists an allowable involution σ such that $\mathfrak{G}_\sigma(S)$ is connected.

We postpone the proof of this fact and use it to prove Theorem 1.1 for $\mathbf{z} = \emptyset$.

Proof of Theorem 1.1 for $\mathbf{z} = \emptyset$
We are assuming that S has genus at least 4, and so according to Theorem 5.2 there exists an allowable involution σ such that $\mathfrak{G}_\sigma(S)$ is connected.

Let $f_\sigma : S \to S_\sigma = S/\langle\sigma\rangle$ be the corresponding quotient properly branched over $\mathbf{z}_\sigma$. According to Proposition 5.1, we have an embedding

$$f_\sigma^* : \mathscr{AF}(S_\sigma, \mathbf{z}_\sigma) \to \mathscr{AF}(S).$$

It follows from the case $\mathbf{z} \neq \emptyset$ of Theorem 1.1 that the subspace

$$X_\sigma = f_\sigma^*\big(\mathscr{AF}(S_\sigma, \mathbf{z}_\sigma)\big)$$

is connected. Since the involution σ_0 associated to any vertex of $\mathfrak{G}_\sigma(S)$ is a conjugate of σ, we also see that the associated space X_{σ_0} is connected.

Observe that $\mathfrak{G}_\sigma(S)$ admits an obvious action by $\mathrm{Mod}(S)$. Therefore, the set

$$\mathscr{X}(\sigma) = \bigcup_{\sigma_0 \in \mathrm{Vert}(\mathfrak{G}_\sigma(S))} X_{\sigma_0} \subset \mathscr{AF}(S)$$

is $\mathrm{Mod}(S)$-invariant and hence dense.

Now, suppose that $\{\sigma_0, \sigma_1\}$ is an edge of $\mathfrak{G}_\sigma(S)$. The hypothesis implies that $G = \langle\sigma_0, \sigma_1\rangle$ is a finite group and $S_G = S/G$ admits a pseudo-Anosov mapping class. A power of this can be lifted to a pseudo-Anosov mapping class ψ in $\mathrm{Mod}(S)$. Moreover, because the branched covering f_G factors through the branched coverings

$$S \longrightarrow S_{\sigma_0} \longrightarrow S_G \qquad \text{and} \qquad S \longrightarrow S_{\sigma_1} \longrightarrow S_G,$$

we can assume that ψ is also a lift of pseudo-Anosov mapping classes $\psi_0 \in \mathrm{Mod}(S_{\sigma_0}, \mathbf{z}_{\sigma_0})$ and $\psi_1 \in \mathrm{Mod}(S_{\sigma_1}, \mathbf{z}_{\sigma_1})$. As such, the stable fixed point of ψ lies in $X_{\sigma_0} \cap X_{\sigma_1}$. In particular, since these spaces are both connected and nontrivially intersect, it follows that $X_{\sigma_0} \cup X_{\sigma_1}$ is connected.

Inductively we see that any two vertices σ_0 and σ_1 which are connected by an edge path have their associated sets X_{σ_0} and X_{σ_1} in the same connected component

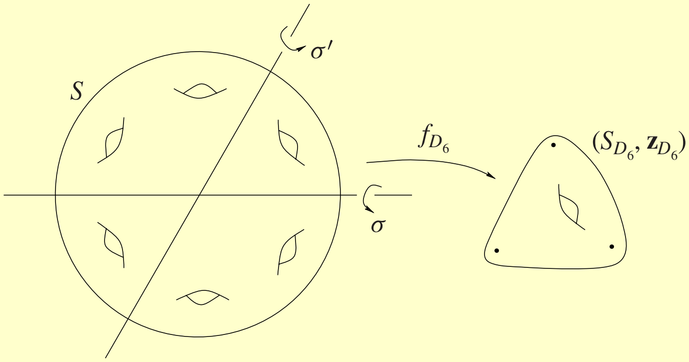

Figure 3. Generators σ and σ' for D_6 and the quotient of S

of $\mathscr{X}(\sigma)$. Connectivity of $\mathfrak{G}_\sigma(S)$ means that every two vertices σ_0 and σ_1 are connected by an edge path, and so $\mathscr{X}(\sigma)$ is connected. Therefore, $\overline{\mathscr{X}(\sigma)} = \mathscr{AF}(S)$ is connected. □

5.3. Proof of Theorem 5.2

The proof of Theorem 5.2 divides into two cases depending on whether the genus is even or odd. Both proofs are essentially the same, except for the descriptions of the involutions. We first describe the involution and the proof for the case of even genus (with corresponding figures for the case of genus 6) and explain the proof in detail. For odd genus, we simply describe the involution, with the remainder of the proof left as an easy exercise.

Suppose that the genus g of S is even. The dihedral group D_g of order g acts on S with quotient $f_{D_g} : S \to S_{D_g} = S/D_g$ having genus 1 properly branched over $\mathbf{z}_{D_g}$ with $|\mathbf{z}_{D_g}| = 3$ (see Figure 3 for the case of genus 6).

The group is generated by involutions σ and σ' which are conjugate in D_g and hence also in Mod(S). The quotient $f_\sigma : S \to S_\sigma = S/\langle\sigma\rangle$ has genus $g/2$ and is properly branched over $\mathbf{z}_\sigma$ with $|\mathbf{z}_\sigma| = 2$ (see Figure 4 for the case $g = 6$).

We consider the involution graph $\mathfrak{G}_\sigma(S)$. According to the previous paragraph, σ and σ' are both vertices of $\mathfrak{G}_\sigma(S)$, and $\{\sigma, \sigma'\}$ is an edge.

THEOREM 5.3
For this choice of σ, $\mathfrak{G}_\sigma(S)$ is connected.

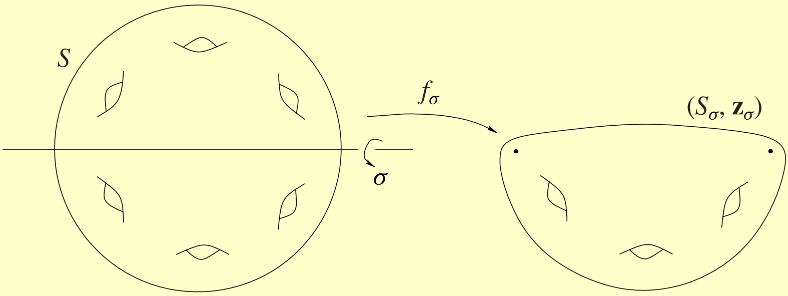

Figure 4. The involution σ and the quotient of S

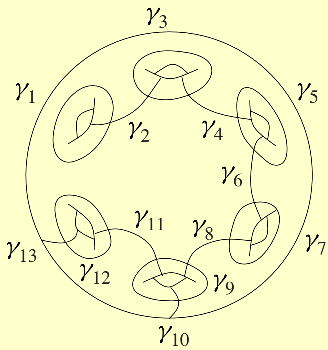

Figure 5. Twisting curves for the Humphries generators

Proof

Recall that the Humphries generating set for the mapping class group (see [23]) consists of $2g + 1$ Dehn twists $T_{\gamma_1}, \ldots, T_{\gamma_{2g+1}}$. The curves $\gamma_1, \ldots, \gamma_{2g+1}$ can be chosen as in Figure 5. The relevant features are the following:

(1) γ_g is invariant by σ;

(2) γ_{g-2} is invariant by σ';

(3) for every $i \neq g$, there exists an element of $\phi_i \in \mathrm{Mod}(S)$ commuting with σ taking γ_{g-2} to γ_i.

That we may arrange for the last property is perhaps least obvious but is an easy exercise given Figure 5.

CLAIM

There is an edge path in $\mathfrak{G}_\sigma(S)$ connecting the vertex σ to the vertex $T_{\gamma_i}\sigma T_{\gamma_i}^{-1}$ for any $i = 1, \ldots, 2g + 1$.

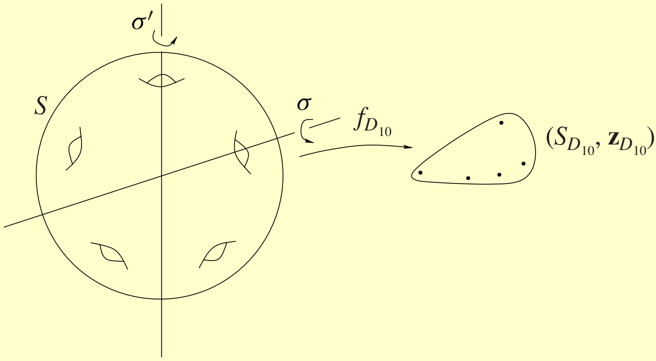

Figure 6. Generators σ and σ' for D_{10} and the quotient of S

Proof
According to property (1), T_{γ_g} commutes with σ. It follows that $T_{\gamma_g}\sigma T_{\gamma_g}^{-1} = \sigma$, so there is a constant path from σ to $T_{\gamma_g}\sigma T_{\gamma_g}^{-1}$.

According to (3), for every $i \neq g$ there exists an element $\phi_i \in \mathrm{Mod}(S)$ commuting with σ taking γ_{g-2} to γ_i. Then $\phi_i T_{\gamma_{g-2}}\phi_i^{-1} = T_{\gamma_i}$. Since $T_{\gamma_{g-2}}$ commutes with σ' by (2), it follows that T_{γ_i} commutes with $\phi_i\sigma'\phi_i^{-1}$. From this, we see the following edges:

$$\{\phi_i\sigma\phi_i^{-1}, \phi_i\sigma'\phi_i^{-1}\} = \{\sigma, \phi_i\sigma'\phi_i^{-1}\},$$
$$\{T_{\gamma_i}\sigma T_{\gamma_i}^{-1}, T_{\gamma_i}(\phi_i\sigma'\phi_i^{-1})T_{\gamma_i}^{-1}\} = \{T_{\gamma_i}\sigma T_{\gamma_i}^{-1}, \phi_i\sigma'\phi_i^{-1}\}.$$

It follows that σ and $T_{\gamma_i}\sigma T_{\gamma_i}^{-1}$ are connected by an edge path (of length 2), proving the claim. □

Let $\mathscr{G}(\mathrm{Mod}(S))$ denote the Cayley graph of $\mathrm{Mod}(S)$ with respect to the generating set $\{T_{\gamma_1}, \ldots, T_{\gamma_{2g+1}}\}$. It follows from the claim that there is a continuous equivariant map from $\mathscr{G}(\mathrm{Mod}(S))$ to $\mathfrak{G}_\sigma(S)$. Since $\mathrm{Mod}(S)$ acts transitively on the vertices (by construction) and since $\mathscr{G}(\mathrm{Mod}(S))$ is connected, it follows that $\mathfrak{G}_\sigma(S)$ is connected, as required. This completes the proof of Theorem 5.3. □

When the genus g is odd, we again choose an involution σ for which it and a conjugate σ' generate a dihedral group, this time of order $2g$. The involutions σ and σ' are shown in Figure 6 for the case of genus 5.

THEOREM 5.4
With this choice of σ, $\mathfrak{G}_\sigma(S)$ is connected.

We omit the proof, which is similar to that of Theorem 5.3. One need only verify that the Humphries generators for $\mathrm{Mod}(S)$ can be chosen with properties similar to those used in the proof of Theorem 5.3. Theorems 5.3 and 5.4 now imply Theorem 5.2. □

5.4. Cut points: $\mathbf{z} = \emptyset$

To complete the proof of Theorem 1.4 in the case $\mathbf{z} = \emptyset$, we must prove the counterpart to Theorem 4.8.

THEOREM 5.5

The space $\mathscr{AF}(S)/\sim$ *has no cut points.*

Proof

We continue to denote the equivalence class of $\mu \in \mathscr{AF}(S)$ as $[\mu]$. As in the proof of Theorem 4.8, we show that for every point $[\mu] \in \mathscr{AF}(S)/\sim$, there is a dense connected subset W_0 of $(\mathscr{AF}(S)/\sim) - \{[\mu]\}$.

We first extract the following key ingredients from the proof of Section 5.3. The space $\mathscr{AF}(S)$ contains a countable collection of connected subsets $\{\tilde{W}_j\}_{j=1}^{\infty}$ with the property that for any i, j, there is a chain $\tilde{W}_j = \tilde{W}_{j_1}, \dots, \tilde{W}_{j_n} = \tilde{W}_i$, so that $\tilde{W}_{j_k} \cap \tilde{W}_{j_{k+1}}$ contains at least two points (the stable and unstable fixed points of some pseudo-Anosov mapping class) and so that

$$\tilde{W} = \bigcup_{j=1}^{\infty} \tilde{W}_j$$

is dense in $\mathscr{AF}(S)$.

Each of the sets $\tilde{W}_j$ is naturally homeomorphic to one of the connected spaces $\mathscr{AF}(S', \mathbf{z}')$. In particular, we see that the image of $\tilde{W}_j$ in $\mathscr{AF}(S)/\sim$ is a connected set W_j with no cut points. Moreover, the collection $\{W_j\}_{j=1}^{\infty}$ has the same property for $\mathscr{AF}(S)/\sim$ as $\{\tilde{W}_j\}_{j=1}^{\infty}$ has for $\mathscr{AF}(S)$ as described in the previous paragraph. We set

$$W = \bigcup_{j=1}^{\infty} W_j.$$

To find the required set W_0, we first note that if $[\mu] \notin W$, then we can take $W_0 = W$. We therefore assume that $[\mu] \in W$.

Since W_j is connected and has no cut points for each $j \geq 1$, we see that $W_j - \{[\mu]\}$ is connected for every $j \geq 1$. Moreover, if $W_i \cap W_j$ contains at least two points, then

$$(W_i - \{[\mu]\}) \cap (W_i - \{[\mu]\})$$

contains at least 1 point. It follows that

$$W_0 = \bigcup_{j=1}^{\infty}(W_j - \{[\mu]\}) = \left(\bigcup_{j=1}^{\infty} W_j\right) - \{[\mu]\} = W - \{[\mu]\}$$

is connected. Since W is dense in $\mathscr{AF}(S)/\sim$, it follows that W_0 is dense in $(\mathscr{AF}(S)/\sim) - \{[\mu]\}$, as required. □

6. Hyperbolic 3-manifolds

For the purposes of this section, we take $\Sigma = \Sigma_{g,n}$ to be a compact orientable surface of genus g with n boundary components. Here we work exclusively with laminations instead of foliations as these are more natural in this context. We write λ to denote a measured lamination (or its projective class) and $|\lambda|$ for the supporting (geodesic) lamination.

Given a hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ and $\epsilon > 0$, we let M_ϵ be the ϵ-thick part of M. We fix $\epsilon > 0$ less than the 3-dimensional Margulis constant so that the ϵ-thin part, $M - M_\epsilon$, is a disjoint union of Margulis tubes and parabolic cusps. The Margulis tubes are homeomorphic to open solid tori, and each of the parabolic cusps is homeomorphic to either $A \times (0, \infty)$ or $T \times (0, \infty)$, where A and T denote an open annulus and torus, respectively (e.g., see [41] or [2]). We further define $M^0 = M^0_\epsilon = M_\epsilon \cup \{\text{Margulis tubes}\}$. The boundary of M^0 is thus a union of annuli and tori, and M^0 is homotopy equivalent to M.

For $\Gamma = \pi_1(M)$ finitely generated, it follows from [40] that there exists a compact core $N \subset M^0$ for which the inclusion is a homotopy equivalence. Indeed, from [33] and [29], one may choose such an N so that

$$P = N \cap \partial M^0$$

is a union of incompressible annuli and tori called the parabolic locus of N. Moreover, N can be chosen so that each component $U \subset \overline{M^0 - N}$ is a neighborhood of the unique end of M^0 it contains and so that the inclusion $U \cap N \hookrightarrow U$ induces a homotopy equivalence. Observe that $U \cap N$ is a component of $\partial N - P$; we say that this component faces U. We call such an N a relative compact core.

The space of equivalence classes of orientable hyperbolic 3-manifolds marked by a relative homotopy equivalence to $(\Sigma, \partial\Sigma)$ is denoted

$$\mathrm{AH}(\Sigma, \partial\Sigma) = \left\{ f : (\Sigma, \partial\Sigma) \xrightarrow{\simeq} (M_0, \partial M_0) \,\middle|\, M = \mathbb{H}^3/\Gamma \right\}/\sim.$$

Here $(f : (\Sigma, \partial\Sigma) \to (M^0, \partial M^0)) \sim (h : (\Sigma, \partial\Sigma) \to (L^0, \partial L^0))$ if there exists an isometry $\varphi : M \to L$ and a relative homotopy $\varphi \circ f \simeq h$. Using the holonomy homomorphism, we can identify a point of $\mathrm{AH}(\Sigma, \partial\Sigma)$ with a conjugacy class of homomorphisms to $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$. We thus view $\mathrm{AH}(\Sigma, \partial\Sigma)$ as a subspace of the space of conjugacy classes of homomorphisms to $\mathrm{PSL}_2(\mathbb{C})$ which defines a topology on $\mathrm{AH}(\Sigma, \partial\Sigma)$ (the algebraic topology). We abuse notation and simply denote points of $\mathrm{AH}(\Sigma, \partial\Sigma)$ as M, with the marking understood unless clarification is necessary.

For $M \in \mathrm{AH}(\Sigma, \partial\Sigma)$, the work of Bonahon [8] and Thurston [42] implies that $M^0 \cong \Sigma \times \mathbb{R}$, and N can be chosen to correspond to $\Sigma \times [-1, 1]$ under this homeomorphism. We choose ϵ sufficiently small so that, given any simple closed curve $\gamma \subset \Sigma$, the geodesic representative of γ lies in M^0 (which is possible since the geodesic representative lies on a pleated surface and so cannot penetrate too far into any cusp).

Given $M \in \mathrm{AH}(\Sigma, \partial\Sigma)$ and a simple closed curve $\gamma \in \mathscr{C}^0(\Sigma)$, one can measure the length of γ in M, which is the length of the geodesic representative, unless none exists (in which case the length is zero). In [10], Brock proved that this naturally extends to a continuous function.

THEOREM 6.1 (Brock [10, Theorem 2])
There is a continuous function

$$\underline{\mathrm{length}} : \mathrm{AH}(\Sigma, \partial\Sigma) \times \mathscr{ML}(\Sigma) \to \mathbb{R}$$

homogeneous in the second argument which extends the above-mentioned length function on $\mathrm{AH}(\Sigma, \partial\Sigma) \times \mathscr{C}^0(\Sigma)$.

In what follows, we assume that $M \in \mathrm{AH}(\Sigma, \partial\Sigma)$ has no accidental parabolics, meaning that the only parabolics in $\pi_1(M)$ are represented by peripheral loops in Σ. Given $M \in \mathrm{AH}(\Sigma, \partial\Sigma)$, we let $C(M)$ denote the convex core of M. Fix a component $U \subset \overline{M^0 - N}$ providing a neighborhood of an end of M^0. This end is called geometrically finite if $U \cap C(M)$ is compact and called simply degenerate otherwise.

The work of Thurston and Bonahon greatly clarifies the simply degenerate ends, as we now describe. A sequence of simple closed curves $\{\gamma_i\}_{i=1}^{\infty}$ is said to exit the end if the geodesic representative γ_i^* of γ_i in M is contained in U for every i and if, for every compact $K \subset M$, there are at most finitely many i for which $\gamma_i^* \cap K \neq \emptyset$ (i.e., the geodesics γ_i^* lie further and further out in U).

The work of Thurston and Bonahon associates a unique lamination $|\lambda| \in \mathscr{EL}(\Sigma)$ to a simply degenerate end with the following property. First, there exists a sequence of simple closed curves $\{\gamma_i\}_{i=1}^{\infty}$ exiting the end, so that in $\mathbb{P}\mathscr{ML}(\Sigma)$ we have $\lim_{i\to\infty} \gamma_i = \lambda'$ for some $\lambda' \in \mathbb{P}\mathscr{ML}(\Sigma)$ with $|\lambda'| = |\lambda|$. Second, for every sequence $\{\gamma_i\}_{i=1}^{\infty}$ exiting the end, up to subsequence, $\lim_{i\to\infty} \gamma_i = \lambda'$ with $|\lambda'| = |\lambda|$.

This lamination $|\lambda|$ is called the ending lamination associated to the simply degenerate end.

Using the orientations on M and on Σ, the ends of M^0 can be labeled as either positive or negative. We define $\mathscr{E}^+(M)$ and $\mathscr{E}^-(M)$ to be the ending laminations associated to the positive and negative ends of M^0, respectively, if the end is simply degenerate. If either or both of the ends are geometrically finite, then we let $\mathscr{E}^{\pm}(M) \in \mathscr{T}(\Sigma)$ be the (finite-type) conformal structure at infinity associated to the geometrically finite end.

We say that M is doubly degenerate if both ends are simply degenerate. We denote the space of doubly degenerate manifolds $\mathrm{DD}(\Sigma, \partial\Sigma) \subset \mathrm{AH}(\Sigma, \partial\Sigma)$. Fixing a conformal structure $Y \in \mathscr{T}(\Sigma)$, we denote the associated Bers slice

$$B_Y = \left\{ M \in \mathrm{AH}(\Sigma, \partial\Sigma) \,\middle|\, M \text{ is geometrically finite, and } \mathscr{E}^-(M) = Y \right\}.$$

The singly degenerate points on the boundary of the Bers slice are denoted

$$\partial_0 B_Y = \left\{ M \in \overline{B_Y} \,\middle|\, \mathscr{E}^-(M) = Y,\ \mathscr{E}^+(M) \in \mathscr{EL}(\Sigma) \right\}.$$

We thus have maps

$$\mathscr{E} : \mathrm{DD}(\Sigma, \partial\Sigma) \to \mathscr{EL}(\Sigma) \times \mathscr{EL}(\Sigma) - \Delta$$

and

$$\mathscr{E}^+ : \partial_0 B_Y \to \mathscr{EL}(\Sigma).$$

A result of Thurston (see [42], [43]) states that the length function can be used to characterize the ending laminations (as a set). This is described by the following corollary of Theorem 6.1.

COROLLARY 6.2 (Thurston)
For $M \in \mathrm{DD}(\Sigma, \partial\Sigma)$, $\underline{\mathrm{length}}_M(\lambda) = 0$ *if and only if*

$$|\lambda| \in \left\{ \mathscr{E}^+(M), \mathscr{E}^-(M) \right\}.$$

Similarly, for $M \in \partial_0 B_Y$, $\underline{\mathrm{length}}_M(\lambda) = 0$ *if and only if* $|\lambda| = \mathscr{E}^+(M)$.

Surjectivity of $\mathscr{E}$ onto $\mathscr{EL}(\Sigma) \times \mathscr{EL}(\Sigma) - \Delta$ (and $\mathscr{E}^+$ onto $\mathscr{EL}(\Sigma)$) follows from the Bers simultaneous uniformization theorem [3, Theorem 1], Thurston's double limit theorem [43, Theorem 4.1] (see also Otal [38, Theorem 5.0.1]), and Corollary 6.2 (to guarantee the correct ending laminations; see also the proof of Theorem 6.5). A slightly stronger version of the double limit theorem which we need is given by the following.

THEOREM 6.3 (Thurston)
Suppose that $\{M_i\} \in \mathrm{AH}(\Sigma, \partial\Sigma)$ *has the property that for some* $K > 0$,

$$\underline{\text{length}}_{M_i}(\lambda_i) + \underline{\text{length}}_{M_i}(\mu_i) \le K$$

for some $\{\lambda_i\}, \{\mu_i\} \subset \mathscr{ML}(\Sigma)$. *If* $\lambda_i \to \lambda$, $\mu_i \to \mu$, *and* λ, μ *fills* S, *then up to subsequence,* M_i *converges to some manifold* $M \in \mathrm{AH}(\Sigma, \partial\Sigma)$.

This is precisely [43, Theorem 6.3] but also follows from a diagonal argument, the density theorem in [12], and the double limit theorem as stated in [38, Theorem 5.0.1].

Our discussion culminates in the following simplified version of the ending lamination theorem (see [35], [12]) for $\mathrm{DD}(\Sigma, \partial\Sigma)$ and $\partial_0 B_Y$.

THEOREM 6.4 (Brock, Canary, and Minsky [12])
$\mathscr{E}$ *is a bijection on* $\mathrm{DD}(\Sigma, \partial\Sigma)$, *and* $\mathscr{E}^+$ *is a bijection on* $\partial_0 B_Y$.

We can now assemble the pieces from the discussion above to prove that the ending laminations actually serve as a continuous parameterization of $\mathrm{DD}(\Sigma, \partial\Sigma)$.

THEOREM 6.5
The map $\mathscr{E} : \mathrm{DD}(\Sigma, \partial\Sigma) \to \mathscr{EL}(\Sigma) \times \mathscr{EL}(\Sigma) - \Delta$ *is a homeomorphism.*

Proof
As just noted, Theorem 6.4 implies that $\mathscr{E}$ is a bijection. We are therefore left to prove that $\mathscr{E}$ and $\mathscr{E}^{-1}$ are continuous.

We first show that $\mathscr{E}$ is continuous. Suppose that $\{M_i\} \subset \mathrm{DD}(\Sigma, \partial\Sigma)$ is any sequence that converges to some $M \in \mathrm{DD}(\Sigma, \partial\Sigma)$. It suffices to show that there is a subsequence, also called $\{M_i\}$, such that $\mathscr{E}^\pm(M_i) \to \mathscr{E}^\pm(M)$ as $i \to \infty$. Throughout, when passing to subsequences, we always reindex so that the index set is the set of positive integers.

We start by picking a subsequence of measured laminations $\{\lambda_i^\pm\}$ with $|\lambda_i^\pm| = \mathscr{E}^\pm(M_i)$ which converge to some measured laminations $\lambda^\pm$. We wish to show that $\lambda^\pm$ are measures supported on $\mathscr{E}^\pm(M)$.

Theorem 6.1 and Corollary 6.2 imply that

$$\lim_{i\to\infty} \underline{\text{length}}_{(M_i)}(\lambda_i^\pm) = \underline{\text{length}}_M(\lambda^\pm) = 0.$$

Corollary 6.2 then implies that one of the following happens:

$$\lim_{i\to\infty} \mathscr{E}^\pm(M_i) \to \mathscr{E}^\pm(M) \qquad \text{or} \qquad \lim_{i\to\infty} \mathscr{E}^\pm(M_i) \to \mathscr{E}^\mp(M).$$

CLAIM
We have $\lim_{i\to\infty} \mathscr{E}^{\pm}(M_i) \to \mathscr{E}^{\pm}(M)$.

Proof
The argument we give here is a very minor modification of an argument given by Brock and Bromberg in the proof of [11, Theorem 8.1].

We suppose that

$$\lim_{i\to\infty} \mathscr{E}^{\pm}(M_i) \to \mathscr{E}^{\mp}(M)$$

and arrive at a contradiction.

Since there are no accidental parabolics, the work of Canary [13] implies that the convergence $M_i \to M$ is strong, and so, in particular, there are relative compact cores $N_i \subset M_i^0$ and $N \subset M^0$ and K_i-bi-Lipschitz maps

$$F_i : N_i \to N$$

compatible with the markings with $K_i \to 1$ as $i \to \infty$. Let $\overline{M_i^0 - N_i} = U_i^- \sqcup U_i^+$ denote the corresponding neighborhoods of the negative and positive ends of M_i^0, respectively. Similarly, we write $\overline{M^0 - N} = U^- \sqcup U^+$. We remark that F_i takes the component of $N_i - P_i$ facing $U_i^{\pm}$ to the component of $N - P$ facing $U^{\pm}$.

Let $\{\gamma_i\} \subset \mathscr{C}(\Sigma)$ be a sequence exiting the end of M^0 defined by U^- so that

$$\lim_{i\to\infty} \gamma_i = \mathscr{E}^-(M)$$

in the quotient of $\mathbb{P}\mathscr{M}\mathscr{L}(\Sigma)$ obtained by forgetting measures. It follows from Klarreich's work [28] that this convergence also takes place in $\overline{\mathscr{C}}(\Sigma)$.

Strong convergence further implies that, after passing to a subsequence if necessary, we may assume that the geodesic representative of γ_i in M_i lies in U_i^-.

Similarly, for each i, we can construct a sequence of curves $\{\delta_j(i)\}_{j=1}^{\infty}$ exiting the end of M_i^0 defined by U_i^+ and hence converging to $\mathscr{E}^+(M_i)$. Since $\mathscr{E}^+(M_i) \to \mathscr{E}^-(M)$, we can choose a diagonal sequence $\{\delta_i\}_{i=1}^{\infty}$ with $\delta_i = \delta_{j(i)}(i)$ so that

$$\lim_{i\to\infty} \delta_i = \mathscr{E}^-(M)$$

in $\overline{\mathscr{C}}(\Sigma)$.

Next, observe that if $\{\alpha_i\}_{i=1}^{\infty} \subset \mathscr{C}(\Sigma)$ is any sequence with α_i a vertex of the geodesic $[\gamma_i, \delta_i] \subset \mathscr{C}(\Sigma)$, then we also have

$$\lim_{i\to\infty} \alpha_i = \mathscr{E}^-(M).$$

Now, fix any i. Two consecutive vertices of $[\gamma_i, \delta_i]$ represent a pair of disjoint essential simple closed curves and so can be realized in a single pleated surface in M_i (see [42], [14]). The geodesic $[\gamma_i, \delta_i]$ thus gives rise to a finite set of pleated surfaces $X_i(1), \ldots, X_i(d_i)$, where $d_i = d(\gamma_i, \delta_i)$ with the property that

(1) γ_i is realized on $X_i(1)$,

(2) δ_i is realized on $X_i(d_i)$, and

(3) $X_i(j) \cap X_i(j+1) \neq \emptyset$ for each $j = 1, \ldots, d_i - 1$.

Therefore, there is a vertex $\alpha_i \in [\gamma_i, \delta_i]$ realized on one of these pleated surfaces in M_i, call it X_i, which nontrivially intersects N_i.

Once again appealing to strong convergence, we see that pleated surfaces Y_i in M realizing α_i must intersect the (compact) R-neighborhood of N for some $R > 0$. Compactness of pleated surfaces implies that Y_i converges to a pleated surface Y in M which realizes the limit $\mathscr{E}^-(M)$ of α_i. This is a contradiction, and it follows that $\mathscr{E}^{\pm}(M_i) \to \mathscr{E}^{\pm}(M)$, verifying the claim. □

All that remains is to show that $\mathscr{E}^{-1}$ is continuous, and this will complete the proof of Theorem 6.5. We assume that $\{M_i\} \subset \mathrm{DD}(\Sigma, \partial\Sigma)$ with

$$\lim_{i\to\infty} \mathscr{E}^{\pm}(M_i) = \mathscr{E}(M)$$

and prove that, after passing to a subsequence if necessary, we have

$$\lim_{i\to\infty} M_i = M.$$

This shows that $\mathscr{E}^{-1}$ is continuous.

Let $\{\lambda_i^{\pm}\}_{i=1}^{\infty}$ be any measured laminations with $|\lambda_i^{\pm}| = \mathscr{E}^{\pm}(M_i)$. After scaling the transverse measures and passing to a subsequence if necessary, we may assume that

$$\lim_{i\to\infty} \lambda_i^{\pm} = \lambda^{\pm},$$

where $|\lambda^{\pm}| = \mathscr{E}^{\pm}(M)$.

Theorem 6.3 implies that, after passing to further subsequence if necessary, $\{M_i\}$ converges to some $M' \in \mathrm{AH}(\Sigma, \partial\Sigma)$. Since $\underline{\mathrm{length}}_{M_i}(\lambda_i^{\pm}) = 0$, it follows that $\underline{\mathrm{length}}_{M'}(\lambda^{\pm}) = 0$. Therefore, $\{E^+(M), \mathscr{E}^-(M)\} = \{\mathscr{E}^+(M'), \mathscr{E}^-(M')\}$, so either $\mathscr{E}^{\pm}(M) = \mathscr{E}^{\pm}(M')$ and we are done by Theorem 6.4, or else $\mathscr{E}^{\pm}(M) = \mathscr{E}^{\mp}(M')$. The argument given above implies the latter situation cannot occur, and so the proof is complete. □

In a similar fashion, one obtains the following.

THEOREM 6.6
The map $\mathscr{E}^{+} : \partial_0 B_Y \to \mathscr{E}\mathscr{L}(\Sigma)$ *is a homeomorphism.*

The proof is similar to the previous proof but simpler, and we leave it to the reader. We note that strong convergence in this setting follows from Anderson and Canary [1].

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Leininger
Department of Mathematics, University of Illinois, Urbana-Champaign, Illinois 61801, USA;
clein@math.uiuc.edu

Schleimer
Department of Mathematics, University of Warwick, Coventry CV4 7AL, United Kingdom;
s.schleimer@warwick.ac.uk