Dehn twists have roots

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We show that every Dehn twist in the mapping class group of a closed, connected, orientable surface of genus at least two has a nontrivial root.

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Let S_g denote a closed, connected, orientable surface of genus g. We denote by $\mathrm{Mod}(S_g)$ its mapping class group: the group of homotopy classes of orientation preserving homeomorphisms of S_g. In this note, we demonstrate:

Fact *If $g \geq 2$, then every Dehn twist in $\mathrm{Mod}(S_g)$ has a nontrivial root.*

It follows from the classification of elements in $\mathrm{Mod}(S_1) \cong \mathrm{SL}(2,\mathbb{Z})$ that Dehn twists are primitive in the mapping class group of the torus.

For Dehn twists about separating curves, the fact is well-known: if c is a separating curve then a square root of the left Dehn twist T_c is obtained by twisting one side of c through an angle of π. We construct roots of Dehn twists about nonseparating curves in two ways.

Geometric construction Fix $g \geq 2$. Let P be a regular $(4g-2)$–gon. Glue opposite sides to obtain a surface $Q \cong S_{g-1}$. The rotation of P about its center through angle $2\pi k/(2g-1)$ induces a periodic map f_k of Q. Notice that f_k fixes the points $x, y \in Q$ that are the images of the vertices of P and induces a rotation through angle $-4\pi k/(2g-1)$ about each. Let R be the surface obtained from Q by removing small open disks centered at x and y. Define $f = f_{g^2}^{-1}|R$.

Modify f by an isotopy supported in a collar of ∂R so that $f|\partial R$ is the identity and f restricts to a $(g-1)/(2g-1)$–right Dehn twist in each annulus. Identify the components of ∂R to obtain a surface $S \cong S_g$. The image of ∂R in S is a nonseparating curve; call it d. We see that $(f\,T_d)^{2g-1} = T_d$, as desired.

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Algebraic construction Let $c_1, \ldots, c_k$ be curves in S_g where c_i intersects c_{i+1} once for each i, and all other pairs of curves are disjoint. If k is odd, then a regular neighborhood of $\bigcup c_i$ has two boundary components, say d_1 and d_2, and we have a relation in $\mathrm{Mod}(S_g)$:

$$(T_{c_1}^2 T_{c_2} \cdots T_{c_k})^k = T_{d_1} T_{d_2}.$$

This relation comes from the Artin group of type B_n, in particular, the factorization of the central element in terms of standard generators. The relation also follows from the D_{2p} case of [2, Proposition 2.12(i)]. If $k = 2g-1$ the curves d_1 and d_2 are isotopic nonseparating curves; call this isotopy class d. Using the fact that T_d commutes with each T_{c_i}, we see that

$$[(T_{c_1}^2 T_{c_2} \cdots T_{c_{2g-1}})^{1-g} T_d]^{2g-1} = T_d.$$

In the remainder of the paper, we find roots for several analogues of Dehn twists.

Roots of half-twists We denote by $S_{0,n}$ a two-sphere with n punctures (or cone points). Let d be a curve in $S_{0,2g+2}$ with two punctures on one side and $2g$ on the other. On the side of d with two punctures we perform a left half-twist. On the other side of d we perform a $(g-1)/(2g-1)$–right Dehn twist by arranging the punctures so that one puncture is in the middle and the other punctures rotate around this central puncture. The $(2g-1)$–st power of the composition is a left half-twist about d. Thus, we have roots of half-twists in $\mathrm{Mod}(S_{0,2g+2})$ for $g \geq 2$. Forgetting the central puncture gives roots of half-twists in $\mathrm{Mod}(S_{0,2g+1})$.

In the geometric construction, reflection through the center of the polygon P induces a hyperelliptic involution of the surface S. In the algebraic construction there is a hyperelliptic involution preserving each curve c_i. In either case there is an induced orbifold double covering $S_g \to S_{0,2g+2}$ and the root of the Dehn twist descends to the given root of the half-twist in $\mathrm{Mod}(S_{0,2g+2})$ [1, Theorem 1 plus Corollary 7.1].

Roots of elementary matrices If we consider the map $\mathrm{Mod}(S_g) \to \mathrm{Sp}(2g,\mathbb{Z})$ given by the action of $\mathrm{Mod}(S_g)$ on $H_1(S_g,\mathbb{Z})$ we also see that elementary matrices have roots in $\mathrm{Sp}(2g,\mathbb{Z})$:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

By stabilizing we obtain cube roots of elementary matrices in $\mathrm{Sp}(2g,\mathbb{Z})$ for $g \geq 2$.

Roots of Nielsen transformations Let F_n denote the free group generated by elements $x_1, \ldots, x_n$. Let $\mathrm{Aut}(F_n)$ denote the group of automorphisms of F_n, and assume $n \geq 2$. A Nielsen transformation is an element of $\mathrm{Aut}(F_n)$ conjugate to the one given by $x_1 \mapsto x_1 x_2$ and $x_k \mapsto x_k$ for $2 \leq k \leq n$. The following automorphism is the square root of a Nielsen transformation in $\mathrm{Aut}(F_n)$ for $n \geq 3$:

$$\begin{aligned} x_1 &\mapsto x_1 x_3 \\ x_2 &\mapsto x_3^{-1} x_2 x_3 \\ x_3 &\mapsto x_3^{-1} x_2 \end{aligned}$$

Passing to quotients, this gives a square root of a Nielsen transformation in $\mathrm{Out}(F_n)$ and, multiplying by $-\mathrm{Id}$, a square root of an elementary matrix in $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$. Also, our roots of Dehn twists in $\mathrm{Mod}(S_g)$ can be modified to work for once-punctured surfaces, thus giving "geometric" roots of Nielsen transformations in $\mathrm{Out}(F_n)$.

Other roots If $f \in \mathrm{Mod}(S_g)$ is a root of a Dehn twist T_d, then f commutes with T_d. Since $f T_c f^{-1} = T_{f(c)}$ for any curve c, we see that f fixes d. In the complement of d, the class f must be periodic. This line of reasoning translates to $\mathrm{GL}(n, \mathbb{Z})$ and $\mathrm{Aut}(F_n)$: roots correspond to torsion elements in $\mathrm{GL}(n-1, \mathbb{Z})$ and $\mathrm{Aut}(F_{n-1})$, respectively. In all cases, one can show that the degree of the root is equal to the order of the torsion element.

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References

[1] **J S Birman**, **H M Hilden**, *On the mapping class groups of closed surfaces as covering spaces*, from: "Advances in the theory of Riemann surfaces (Proc. Conf., Stony Brook, N.Y., 1969)", (L V Ahlfors et al, editors), Ann. of Math. Studies 66, Princeton Univ. Press (1971) 81–115 MR0292082

[2] **C Labruère**, **L Paris**, *Presentations for the punctured mapping class groups in terms of Artin groups*, Algebr. Geom. Topol. 1 (2001) 73–114 MR1805936

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