ON TRAIN-TRACK SPLITTING SEQUENCES

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Abstract

We present a structure theorem for the subsurface projections of train-track splitting sequences. For the proof we introduce induced tracks*,* efficient position*, and* wide curves*. As a consequence of the structure theorem, we prove that train-track sliding and splitting sequences give quasi-geodesics in the train-track graph; this generalizes a result of Hamenstädt.*

Contents

1. Introduction

Thurston, in his revolution of geometric topology, introduced *train tracks* to the study of surface diffeomorphisms and hyperbolic 3-manifolds. The geometry and combinatorics of individual tracks have been carefully studied (see [20], [21], [25]). Equally important is the dynamical idea of splitting a train track. This leads to a deep connection between splitting sequences of tracks and the curves they carry on the one hand, and Teichmüller geodesics and measured foliations on the other.

Another notion that has recently proved useful, for example, in the resolution of the ending lamination conjecture (see [3], [19]), is subsurface projection. A combinatorial or geometric object τ in a surface S has a projection $\pi_X(\tau)$ contained in

DUKE MATHEMATICAL JOURNAL
Vol. 161, No. 9, DOI 10.1215/00127094-1593344
Received 30 August 2010. Revision received 13 October 2011.
2010 *Mathematics Subject Classification*. Primary 57M60; Secondary 20F65.
Masur's work partially supported by National Science Foundation grant DMS-0905907.
Mosher's work partially supported by National Science Foundation grant DMS-1006248.
Schleimer's work partially supported by Engineering and Physical Sciences Research Council (U.K.) grant EP/I028870/1.

the given essential subsurface $X \subset S$. Subsurface projection arises naturally in the study of the metric properties of entities such as the marking graph and the pants graph (see [16]) and also when obtaining coarse control over Teichmüller geodesics (see [22], [23]).

In this light, it becomes important to understand how a splitting sequence of tracks $\{\tau_i\}$ interacts with the subsurface projection map π_X. We give a structure theorem (Theorem 5.3) that explains this interaction in great detail. As a consequence we obtain the following.

THEOREM 5.5
For any surface S with $\xi(S) \geq 1$, there is a constant $Q = Q(S)$ with the following property. For any sliding and splitting sequence $\{\tau_i\}_{i=0}^N$ of birecurrent train tracks in S and for any essential subsurface $X \subset S$, if $\pi_X(\tau_N) \neq \emptyset$, then the sequence $\{\pi_X(\tau_i)\}_{i=0}^N$ is a Q-unparameterized quasi-geodesic in the curve complex $\mathcal{C}(X)$.

Using the structure theorem we also generalize, via a very different proof, a result of Hamenstädt.

THEOREM 6.2 ([10, Corollary 3])
For any surface S with $\xi(S) \geq 1$, there is a constant $Q = Q(S)$ with the following property. If $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence in the train-track graph $\mathcal{T}(S)$, injective on slide subsequences, then $\{\tau_i\}$ is a Q-quasi-geodesic.

Our techniques have further applications. Theorem 5.5 is used in Masur and Schleimer's proof (see [18]) that the disk complex is Gromov-hyperbolic. Maher and Schleimer [14] use our structure theorem and the local finiteness of the train-track graph to prove the *stability* of disk sets in the curve complex. As a consequence, they prove that the *graph of handlebodies* has infinite diameter, and they also produce, in every genus, a pseudo-Anosov map so that no nontrivial power extends over any nontrivial compression body. Additionally, our notion of efficient position is used by Gadre [7] to show that harmonic measures on $\mathcal{PMF}(S)$ for distributions with finite support on $\mathcal{MCG}(S)$ are singular with respect to Lebesgue measure on $\mathcal{PMF}(S)$.

For the proof of Theorem 5.3 we introduce *induced tracks*, *efficient position*, and *wide curves*. For any essential subsurface $X \subset S$ and track $\tau \subset S$ there is an induced track $\tau|X$. Induced tracks generalize the notion of subsurface projection of curves. Efficient position of a curve with respect to a track τ is a simultaneous generalization of curves carried by τ and curves dual to τ (called *hitting τ efficiently* in [21]). Efficient position of ∂X allows us to pin down the location of the induced track $\tau|X$. Wide curves are our combinatorial analogue of curves of definite modulus in a Riemann surface. The structure theorem (Theorem 5.3) then implies Theorem 5.5; this, together with subsurface projection, controls the motion of a splitting sequence through the complex of curves $\mathcal{C}(X)$.

The quasi-geodesic behavior of splitting sequences in the train-track graph (Theorem 6.2) is a direct consequence of Theorem 6.1. Note that Theorem 6.1 requires a delicate induction, conceptually similar to the hierarchy machine developed in [16]. We do not deduce Theorem 6.1 directly from the results of [16]; in particular, it is not known if splitting sequences fellow-travel resolutions of hierarchies.

After this work was submitted, we discovered a paper of Takarajima [24] introducing *quasi-transverse* curves. Quasi-transversality and efficient position (Definition 2.3) are equivalent concepts for simple curves. Takarajima goes on to give an intricate proof that the quasi-transverse position exists, relying on a lexicographic ordering of various combinatorial geodesic curvatures; this also implies the existence statement of our Theorem 4.1. We give a completely independent and somewhat simpler proof. Note, however, that Takarajima's existence proof is in principle constructive, while ours is not.

2. Background

We provide the definitions needed for Theorem 5.3 and its corollaries.

2.1. Coarse geometry

Suppose that $Q \geq 1$ is a real number. For real numbers r, s we write $r \leq_Q s$ if $r \leq Qs + Q$, and we say that r is *quasi-bounded* by s. We write $r =_Q s$ if $r \leq_Q s$ and $s \leq_Q r$; this is called a *quasi-equality*.

For a metric space $(\mathcal{X}, d_\mathcal{X})$ and finite diameter subsets $A, B \subset \mathcal{X}$, define $d_\mathcal{X}(A, B) = \operatorname{diam}_\mathcal{X}(A \cup B)$. Following Gromov [8], a relation $f: \mathcal{X} \to \mathcal{Y}$ of metric spaces is a *Q-quasi-isometric embedding* if, for all $x, y \in \mathcal{X}$, we have $d_\mathcal{X}(x, y) =_Q d_\mathcal{Y}(f(x), f(y))$. (Here $f(x) \subset \mathcal{Y}$ is the set of points related to x.) If, additionally, the Q-neighborhood of $f(\mathcal{X})$ equals $\mathcal{Y}$, then f is a *Q-quasi-isometry* and $\mathcal{X}$ and $\mathcal{Y}$ are *quasi-isometric*.

If $[m, n]$ is an interval in $\mathbb{Z}$ and if $f: [m, n] \to \mathcal{Y}$ is a quasi-isometric embedding, then f is a *Q-quasi-geodesic*. Now suppose that $Q \geq 1$ is a real number, that $[m, n]$ and $[p, q]$ are intervals in $\mathbb{Z}$, and that $f: [m, n] \to \mathcal{Y}$ is a relation. Then f is a *Q-unparameterized quasi-geodesic* if there is a strictly increasing function $\rho: [p, q] \to [m, n]$ so that $f \circ \rho$ is a Q-quasi-geodesic and, for all $i \in [p, q-1]$, the diameter of $f([\rho(i), \rho(i+1)])$ is at most Q.

2.2. Surfaces, arcs, and curves

Let $S = S_{g,n}$ be a compact, connected, orientable surface of genus g with n boundary components. The *complexity* of S is $\xi(S) = 3g - 3 + n$. A *curve* in S is an embedding of the circle into S. An *arc* in S is a proper embedding of the interval $[0,1]$ into S. A curve or arc $\alpha \subset S$ is *trivial* if α separates S and one component of $S-\alpha$ is a disk; otherwise, α is *essential*. A curve α is *peripheral* if α separates S and one component of $S-\alpha$ is an annulus; otherwise, α is *nonperipheral*. A connected subsurface $X \subset S$ is *essential* if every component of ∂X is essential in S and if X is neither a pair of pants ($S_{0,3}$) nor a peripheral annulus (the core curve is peripheral). Note that X inherits an orientation from S. This, in turn, induces an orientation on ∂X so that X is to the left of ∂X.

Define $\mathcal{C}(S)$ to be the set of isotopy classes of essential, nonperipheral curves in S. Define $\mathcal{A}(S)$ to be the set of proper isotopy classes of essential arcs in S. Let $\mathcal{AC}(S) = \mathcal{C}(S) \cup \mathcal{A}(S)$. If $\alpha, \beta \in \mathcal{AC}(S)$, then the *geometric intersection number* (see [5, p. 46]) of α and β is

$$i(\alpha,\beta) = \min\{|a \cap b| : a \in \alpha, b \in \beta\}.$$

A finite subset $\Delta \subset \mathcal{AC}(S)$ is a *multicurve* if $i(\alpha,\beta) = 0$ for all $\alpha, \beta \in \Delta$.

If $T \subset S$ is a subsurface with ∂T a union of smooth arcs, meeting perpendicularly at their endpoints, then define

$$\operatorname{index}(T) = \chi(T) - \frac{c^+(T)}{4} + \frac{c^-(T)}{4},$$

where $c^\pm(T)$ is the number of outward (inward) corners of ∂T. Note that index is additive: $\operatorname{index}(T \cup T') = \operatorname{index}(T) + \operatorname{index}(T')$ as long as the interiors of T and T' are disjoint.

2.3. *Train tracks*

For a detailed discussion of train tracks, see [21], [25], and [20]. A *pretrack* $\tau \subset S$ is a properly embedded graph in S with additional structure. The vertices of τ are called *switches*; every switch x is equipped with a tangent $v_x \in T^1_x S$. We require every switch to have valence 3 (higher valence is dealt with in [21]). The edges of τ are called *branches*. All branches are smoothly embedded in S. All branches incident to a fixed switch x have derivative $\pm v_x$ at x.

An immersion $\rho\colon \mathbb{R} \to S$ is a *train-route* (or simply a *route*) if

- $\rho(\mathbb{R}) \subset \tau$, and
- $\rho(n)$ is a switch if and only if $n \in \mathbb{Z}$.

The restriction $\rho|[0,\infty)$ is a *half-route*. If ρ factors through $\mathbb{R}/m\mathbb{Z}$, then ρ is a *train-loop*. We require, for every branch b, a train-route travelling along b.

For each branch b and each point p in the interior of b, a component b' of $b - \{p\}$ is a *half-branch*. Two half-branches $b', b'' \subset b$ are equivalent if $b' \cap b''$ is again a half-branch. Every switch divides the three incident half-branches into a pair of *small*

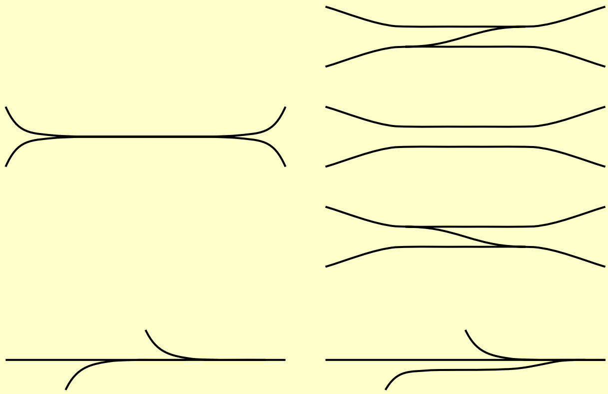

Figure 1. Top: a large branch admits a left, central, or right splitting. Bottom: a mixed branch admits a slide.

half-branches on one side and a single *large* half-branch on the other. A branch b is *large* (*small*) if all of its half-branches are large (small); if b has a large and a small half-branch, then b is called *mixed*.

Let $\mathcal{B} = \mathcal{B}(\tau)$ be the set of branches of τ. A function $w\colon \mathcal{B} \to \mathbb{R}_{\geq 0}$ is a *transverse measure* on τ if w satisfies the *switch conditions*: for every switch $x \in \tau$ we have $w(a) + w(b) = w(c)$, where a', b' are the small half-branches and where c' is the large half-branch meeting x. Let $P(\tau)$ be the projectivization of the cone of transverse measures; define $V(\tau)$ to be the vertices of the polyhedron $P(\tau)$.

We may *split* a pretrack along a large branch or *slide* it along a mixed branch (see Figure 1). (Slides are called *shifts* in [21].) The inverse of a split or slide is called a *fold*. Note that the inverse of a slide may be obtained via a slide followed by an isotopy.

Suppose that $\tau \subset S$ is a pretrack. Let $N = N(\tau) \subset S$ be a *tie neighborhood* of τ; so N is a union of rectangles $\{R_b \mid b \in \mathcal{B}\}$ foliated by vertical intervals (the *ties*). At a switch, the upper and lower thirds of the vertical side of the large rectangle are identified with the vertical side of the small rectangles, as shown in Figure 2. Since N is a union of rectangles, it follows that $\operatorname{index}(N) = 0$. The *horizontal boundary* $\partial_h N$ is the union of $\partial_h R_b$, for $b \in \mathcal{B}$, while the *vertical boundary* is $\partial_v N = \overline{\partial N - \partial_h N}$.

Let $N = N(\tau)$ be a tie neighborhood. Let T be a *complementary region* of τ: a component of the closure of $S - N$. Define the horizontal and vertical boundary of T to be $\partial_h T = \partial T \cap \partial_h N$ and $\partial_v T = \partial T \cap \partial_v N$. Note that all corners of T are outward, so $\operatorname{index}(T) = \chi(T) - (1/4)|\partial\partial_h T|$.

Suppose that $\tau \subset S$ is a pretrack. The subsurface *filled* by τ is the union of N with all complementary regions T of τ that are disks or peripheral annuli.

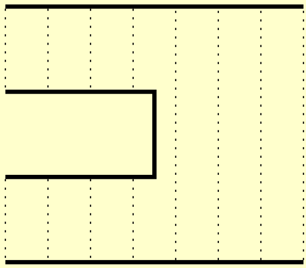

Figure 2. The local model for $N(\tau)$ near a switch, with horizontal and vertical boundary in the correct orientation. The dotted lines are ties.

Definition 2.1
Suppose that $\tau \subset S$ is a pretrack and that $N = N(\tau)$. We say that τ is a *train track* if τ is compact, every component of ∂N has at least one corner, and every complementary region T of τ has negative index.

Definition 2.2
In a *sliding and splitting sequence* $\{\tau_i\}$ of train tracks, each τ_{i+1} is obtained from τ_i by a slide or a split.

2.4. Carrying, duality, and efficient position
Suppose that $\tau \subset S$ is a train track. If σ is also a track, contained in $N = N(\tau)$ and transverse to the ties, then we write $\sigma \prec \tau$ and say that σ is *carried* by τ. For example, if τ is a fold of σ, then σ is carried by τ.

A properly embedded arc or curve $\beta \subset N$ is *carried* by τ if β is transverse to the ties and $\partial\beta \cap \partial_h N = \emptyset$. Thus if β is carried, then $\partial\beta \subset \partial_v N$. Again, we write $\beta \prec \tau$ for carried arcs and curves.

Definition 2.3
Suppose that $\alpha \subset S$ is a properly embedded arc or curve. Then α is in *efficient position* with respect to τ, denoted $\alpha \dashv \tau$, if
- every component of $\alpha \cap N$ is a tie or is carried by τ, and
- every region of $S - (N \cup \alpha)$ has negative index or is a rectangle.

Suppose that $\alpha \dashv \tau$. If $\alpha \subset N$, then α is carried, $\alpha \prec \tau$. If no component of $\alpha \cap N$ is carried, then α is *dual* to τ and we write $\alpha \pitchfork \tau$. If $\Delta \subset \mathcal{AC}(S)$ is a multicurve, then we write $\Delta \prec \tau$, $\Delta \pitchfork \tau$, or $\Delta \dashv \tau$ if all elements of Δ are disjointly and simultaneously carried, dual, or in efficient position, respectively.

Remark 2.4
Duality here is called *hitting efficiently* by Penner and Harer [21, p. 19]. Note that $\alpha \pitchfork \tau$ if and only if α is carried by some extension of the *dual track* τ^*, also defined in [21]. Likewise, if $\alpha \dashv \tau$ and $\alpha \cap N$ consists of carried arcs, then α is carried by some extension of τ.

An index argument proves the following.

LEMMA 2.5
If α is a properly embedded curve or arc in efficient position with respect to a train-track $\tau \subset S$, then α is essential and nonperipheral in S. □

One of the goals of this paper is to prove the converse of Lemma 2.5; this is done in Theorem 4.1. Following Lemma 2.5, we may define $\mathcal{C}(\tau) = \{\alpha \mid \alpha \prec \tau\}$ and $\mathcal{C}^*(\tau) = \{\alpha \mid \alpha \pitchfork \tau\}$. Notice that if $\sigma \prec \tau$ are tracks, then $\mathcal{C}(\sigma) \subset \mathcal{C}(\tau)$ and $\mathcal{C}^*(\tau) \subset \mathcal{C}^*(\sigma)$.

A branch $b \in \mathcal{B}(\tau)$ is *recurrent* if there is some $\alpha \prec \tau$ that meets R_b. The track τ is *recurrent* if every branch is recurrent. *Transverse recurrence* is defined by replacing carrying by duality (see [21, p. 20]). The track τ is *birecurrent* if τ is recurrent and transversely recurrent (see [21, Section 1.3]). In a slight departure from Penner and Harer's terminology (see [21, p. 27]), we will call a birecurrent track τ *complete* if all complementary regions have index $-1/2$. (When $S = S_{1,1}$, there is, instead, a single complementary region with index -1.)

LEMMA 2.6
Suppose that $\sigma \subset S$ is a birecurrent track. Then $\mathcal{C}^*(\sigma)$ has infinite diameter inside of $\mathcal{C}(S)$.

Proof
Let τ be a complete track extending σ (see [21, Corollary 1.4.2]). Section 3.4 of [21] and a dimension count give a lamination $\lambda \pitchfork \tau$ so that $i(\lambda, \alpha) \neq 0$ for all $\alpha \in \mathcal{C}(S)$. Now an argument of Kobayashi [13], refined by Luo [15, p. 124], implies that $\mathcal{C}^*(\tau) \subset \mathcal{C}^*(\sigma)$ has infinite diameter. □

2.5. Vertex cycles
When $\alpha \prec \tau$ is a curve there is a transverse measure w_α defined by taking $w_\alpha(b) = |\alpha \cap t|$, where t is any tie of the rectangle R_b. Conversely, for any integral transverse measure w there is a multicurve α_w—take $w(b)$-many horizontal arcs in R_b and glue endpoints as dictated by the switch conditions.

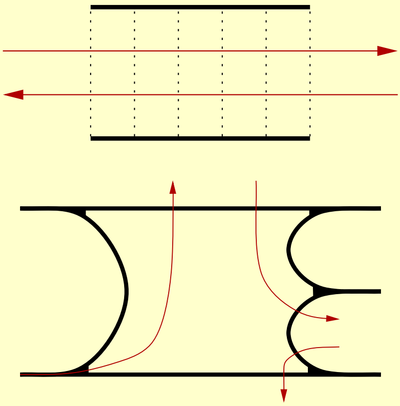

Figure 3. Above: arcs of $\Delta \cap R_b$ to the right of each other. The vertical dotted line are ties; the heavy horizontal lines are arcs of $\partial_h N$. Below: arcs meeting the complementary region T, all to the right of each other.

Note that if $v \in V(\tau)$, then there is a minimal integral measure w projecting to v. Since v is an extreme point of $P(\tau)$, deduce that α_w is an embedded curve. We call α_w a *vertex cycle* of τ and henceforth use $V(\tau)$ to denote the set of vertex cycles.

2.6. Wide curves

Let $N = N(\tau)$ be a tie neighborhood.

Definition 2.7

A multicurve $\Delta \dashv \tau$ is *wide* if there is an orientation of the components of Δ such that

- for every $b \in \mathcal{B}(\tau)$, all arcs of $\Delta \cap R_b$ are to the right of each other (see the top of Figure 3), and
- for every complementary region T of τ, all arcs of $\Delta \cap T$ are to the right of each other (see the bottom of Figure 3).

It follows from the definition that if $\Delta \dashv \tau$ is wide, then for any branch $b \in \mathcal{B}(\tau)$ the intersection $\Delta \cap R_b$ has at most two components.

LEMMA 2.8

Every vertex cycle $\alpha \in V(\tau)$ is wide.

For an even more precise characterization of vertex cycles, see [20, Lemma 3.11.3]. The proof employs *curve surgery*, a technique used several times in the paper.

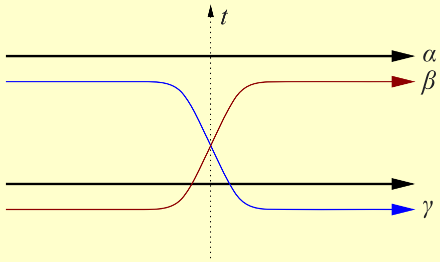

Figure 4. Surgery when adjacent intersections have the same sign.

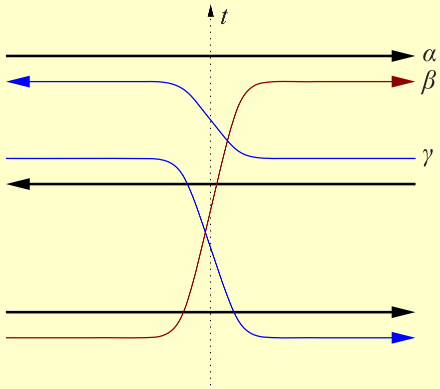

Figure 5. Surgery when the three intersections have alternating sign.

Proof of Lemma 2.8

We prove the contrapositive. Suppose that α is not wide. Each of the two orientations of α leads to three possible cases.

Suppose that there is a branch $b \subset \tau$ and an oriented tie $t \subset R_b$, where x and y are consecutive (along t) points of $\alpha \cap t$ such that the signs of intersection at x and y are equal. Let $[x, y]$ be the subarc of t bounded by x and y. Surger α along $[x, y]$ to form curves $\beta, \gamma \prec \tau$, as in Figure 4. Thus $w_\alpha = w_\beta + w_\gamma$, and α is not a vertex cycle.

Suppose instead that x, y, z are consecutive (along t) points of $\alpha \cap t$ with alternating sign. In this case there is again a surgery along $[x, z]$ producing curves β and γ. See Figure 5 for one of the possible arrangements of α, β and γ. Again, $w_\alpha = w_\beta + w_\gamma$ is a nontrivial sum and α is not a vertex cycle.

In the remaining case $w_\alpha(a) \leq 2$ for all $a \in \mathcal{B}$, and there are branches $b, c \in \mathcal{B}$ where the arcs of $\alpha \cap R_b$ are to the right of each other while the arcs of $\alpha \cap R_c$ are to the left of each other. (See Figure 6 for the two ways α may be carried by τ.)

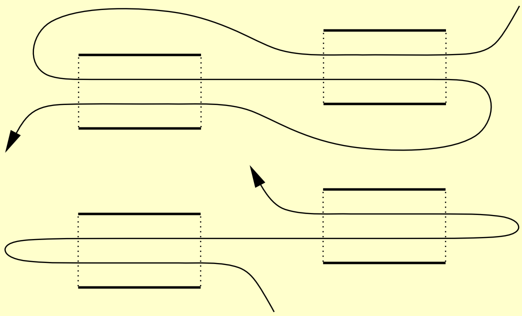

Figure 6. The curve α meets R_b and R_c twice.

If α is carried as in the top line of Figure 6, then surger α in both rectangles R_b and R_c as in Figure 4. This shows that w_α is a nontrivial sum and so α is not a vertex cycle. Now suppose that α is carried as in the bottom line of Figure 6. Note that the closure of $\alpha - (R_b \cup R_c)$ is a union of four arcs. Two of these, β' and γ', meet both R_b and R_c. Since S is orientable, no tie-preserving isotopy of N throws β' onto γ'. Let $\alpha' \cup \alpha'' = \alpha - (\beta' \cup \gamma')$. Create an embedded curve β by taking two parallel copies of β' and joining them to α' and α''. Similarly create γ by joining two parallel copies of γ' to the arcs α' and α''. It follows that $w_\beta \neq w_\gamma$. Since $2w_\alpha = w_\beta + w_\gamma$, again, α is not a vertex cycle. □

2.7. *Combing*

Suppose that $\alpha \prec \tau$ has $w_\alpha(b) \leq 1$ for every branch $b \subset \tau$. Orient and transversely orient α to agree with the orientation of the surface S. We think of the orientation as pointing in the x-direction and the transverse orientation pointing in the y-direction. A half-branch $b \subset \tau$, with $\alpha \cap b$ being a single switch, *twists to the right* if any train-route through b locally has positive slope. Otherwise b twists to the left. If all branches on one side of α twist to the right, then that side of α has a *right-combing*, and similarly for a *left-combing* (see Figure 8 for an example where both sides are combed to the left).

2.8. *Curve complexes and subsurface projection*

For more information on the curve complex, see [15] and [16]. Impose a simplicial structure on $\mathcal{AC}(S)$, where $\Delta \subset \mathcal{AC}(S)$ is a simplex if and only if Δ is a multicurve. The complex of curves $\mathcal{C}(S)$ and the arc complex $\mathcal{A}(S)$ are the subcomplexes spanned by curves and arcs, respectively. Note that if the complexity $\xi(S)$ is at least 2, then $\mathcal{C}(S)$ is connected (see [11, Proposition 2]). For surfaces of lower complexity, we alter the simplicial structure of $\mathcal{C}(S)$.

Define the Farey tessellation $\mathcal{F}$ to have vertex set $\mathbb{Q} \cup \{\infty\}$. A collection of *slopes* $\Delta \subset \mathcal{F}$ spans a simplex if $ps - rq = \pm 1$ for all $p/q, r/s \in \Delta$. If $S = S_{1,1}$ or $S_{0,4}$, then we take $\mathcal{C}(S) = \mathcal{F}$; that is, there is an edge between curves that intersect in exactly one point (for $S_{1,1}$) or two points (for $S_{0,4}$). Note that for surfaces with $\xi(S) \geq 1$, the inclusion of $\mathcal{C}(S)$ into $\mathcal{AC}(S)$ is a quasi-isometry.

Suppose now that $X \cong S_{0,2}$ is an annulus. Define $\mathcal{A}(X)$ to be the set of all essential arcs in X, up to isotopy fixing ∂X pointwise. For $\alpha, \beta \in \mathcal{A}(X)$, define

$$i(\alpha, \beta) = \min\{|(a \cap b) - \partial X| : a \in \alpha, b \in \beta\}.$$

As usual, multicurves give simplices for $\mathcal{A}(X)$.

If α, β are vertices of $\mathcal{C}(S)$, $\mathcal{A}(S)$, or $\mathcal{AC}(S)$, then define $d_S(\alpha, \beta)$ to be the minimal number of edges in a path, in the 1-skeleton, connecting α to β; the containing complex will be clear from context. Note that if α β are distinct arcs of $\mathcal{A}(X)$, when X is an annulus, then $d_X(\alpha, \beta) = 1 + i(\alpha, \beta)$ (see [16, (2.3)]).

As usual, suppose that $\xi(S) \geq 1$. Fix an essential subsurface $X \subset S$ with $\xi(X) < \xi(S)$. We suppose that X is either a nonperipheral annulus or a surface of complexity at least 1. (The case of an essential annulus inside of S_1 is excluded.) Following [16], we will define the *subsurface projection* relation $\pi_X : \mathcal{AC}(S) \to \mathcal{AC}(X)$. Let S^X be the cover of S corresponding to the inclusion $\pi_1(X) < \pi_1(S)$. The surface S^X is not compact; however, there is a canonical (up to isotopy) homeomorphism between X and the Gromov compactification of S^X. This identifies the arc and curve complexes of X with those of S^X. Fix $\alpha \in \mathcal{AC}(S)$. Let α^X be the preimage of α in S^X.

Place every nonperipheral curve and essential arc of α^X into the set $\pi_X(\alpha) \subset \mathcal{AC}(X)$. If there are none such, then $\pi_X(\alpha) = \emptyset$, and we say that α *misses* X. If $\pi_X(\alpha) \neq \emptyset$, then α *cuts* X.

Suppose that $\alpha, \beta \in \mathcal{C}(S)$. If $\pi_X(\alpha)$ and $\pi_X(\beta)$ are nonempty, define

$$d_X(\alpha, \beta) = \operatorname{diam}_X\big(\pi_X(\alpha) \cup \pi_X(\beta)\big).$$

Likewise define the distance $d_X(A, B)$ between finite sets $A, B \subset \mathcal{C}(S)$. When τ is a track, we use the shorthand $\pi_X(\tau)$ for the set $\pi_X(V(\tau))$. If σ is also a track, we write $d_X(\tau, \sigma)$ for the distance $d_X(\pi_X(\tau), \pi_X(\sigma))$.

We end with a lemma connecting the subsurface projection of carried (or dual) curves to the behavior of wide curves.

LEMMA 2.9

Suppose that $X \subset S$ is an essential surface and that τ is a track. If some $\alpha \prec \tau$ ($\alpha \pitchfork \tau$) cuts X, then there is a vertex cycle $\beta \prec \tau$ (wide dual $\beta \pitchfork \tau$) cutting X.

Proof

Some multiple of $\alpha \prec \tau$ is a sum of vertices: $m \cdot w_\alpha = \sum n_i w_i$, where w_i is the integral transverse measure for the vertex $\beta_i \in V(\tau)$. Via a sequence of tie-preserving isotopies of N, we may arrange for all of the β_i to realize their geometric intersection with each other. Note that there is an isotopy representative of α contained inside a small neighborhood of the union $B = \bigcup \beta_i$.

To prove the contrapositive, suppose that none of the β_i cut X. It follows that X may be isotoped in S to be disjoint from B. Thus α misses X, as desired. A similar discussion applies when one has $\alpha \pitchfork \tau$. □

3. Induced tracks

Suppose that $\tau \subset S$ is a train track. Suppose that $X \subset S$ is an essential subsurface with $\xi(X) < \xi(S)$. Let S^X be the corresponding cover of S. Let τ^X be the preimage of τ in S^X (note that the pretrack τ^X satisfies all of the axioms of a train track except compactness).

Define $\mathcal{AC}(\tau^X)$ to be the set of essential arcs and essential, nonperipheral curves properly embedded in the Gromov compactification of S^X with interior a train-route or train-loop carried by τ^X. A bit of caution is required here—inessential arcs and peripheral curves may be carried by τ^X but these are not admitted into $\mathcal{AC}(\tau^X)$. Define $\mathcal{A}(\tau^X), \mathcal{C}(\tau^X) \subset \mathcal{AC}(\tau^X)$ to be the subsets of arcs and curves, respectively. Define $\mathcal{AC}^*(\tau^X)$ to be the set of dual essential arcs and dual essential, nonperipheral curves, up to isotopies fixing τ^X setwise.

3.1. Induced tracks for nonannuli

If X is not an annulus, define $\tau|X$, the *induced track*, to be the union of the branches of τ^X crossed by an element of $\mathcal{C}(\tau^X)$.

LEMMA 3.1
If X is not an annulus, then the induced track $\tau|X$ is compact.

Proof
Note that train-routes in τ^X that are mapped properly to S^X are uniform quasi-geodesics in S^X (see [20, Proposition 3.3.3]). Thus there is a compact core $X' \subset S^X$, homeomorphic to X, such that any route meeting $S^X - X'$ has one endpoint on the Gromov boundary of S^X. It follows that $\tau|X \subset X'$. □

Note that $\tau|X$ may not be a train track; $N = N(\tau|X)$ may have smooth boundary components and complementary regions with nonnegative index. However, since all complementary regions of τ^X have negative index, it follows that if a complementary region T of $\tau|X$ has nonnegative index, then T is a peripheral annulus meeting a smooth component of ∂N.

The definition of $\tau|X$ implies that $\tau|X$ is recurrent. Carrying, duality, efficient position, and wideness with respect to an induced track are defined as in Section 2.4. Define $\mathcal{C}(\tau|X) \subset \mathcal{C}(X)$, the subset of curves carried by $\tau|X$. Note that $\mathcal{C}(\tau|X) = \mathcal{C}(\tau^X)$. Define $\mathcal{AC}^*(\tau|X) \subset \mathcal{AC}(X)$ to be the subset of arcs and curves dual to $\tau|X$. Note that $\mathcal{AC}^*(\tau|X) \supset \mathcal{AC}^*(\tau^X)$.

Now, $\tau|X$ fails to be transversely recurrent exactly when it carries a peripheral curve. We say that a branch $b \subset \tau|X$ is *transversely recurrent with respect to arcs and curves* if there is $\alpha \in \mathcal{AC}^*(\tau|X)$ meeting b. Then $\tau|X$ is *transversely recurrent with respect to arcs and curves* if every branch b is.

LEMMA 3.2
Suppose that τ is transversely recurrent in S. Then $\tau|X$ is transversely recurrent with respect to arcs and curves in X. Furthermore, suppose that $\tau|X$ is transversely recurrent with respect to arcs and curves in X. If $\sigma \subset \tau|X$ is a train track, then σ is transversely recurrent in X.

Proof
The first claim follows from the definitions. An index argument proves the second claim. □

Here is our second curve surgery argument.

LEMMA 3.3
Suppose that τ is a track and that $X \subset S$ is an essential subsurface, yet not an annulus. For every $\alpha \in \mathcal{A}(\tau^X)$, at least one of the following holds.
- *There is an arc $\beta \in \mathcal{A}(\tau^X)$ such that β is wide and $i(\alpha, \beta) = 0$.*
- *There is a curve $\gamma \in \mathcal{C}(\tau|X)$ such that $i(\alpha, \gamma) \leq 2$.*

The statement also holds replacing $\mathcal{A}, \mathcal{C}$ by $\mathcal{A}^*, \mathcal{C}^*$.

Proof
The proof is modeled on that of Lemma 2.8. If $\alpha \prec \tau^X$ is wide, we are done. If not, as α is a quasi-geodesic (see [20, Proposition 3.3.3]), orient α so that α is wide outside of a compact core for S^X. Now we induct on the total number of arcs of intersection between α and rectangles $R_b \subset N(\tau^X)$ meeting the compact core.

Let t be a tie of R_b. Orient t. Suppose that x, y are consecutive (along t) points of $\alpha \cap t$. Suppose that the sign of intersection at x equals the sign at y. Let $[x, y]$ be the subarc of t bounded by x and y. As in Lemma 2.8, surger α along $[x, y]$ to form an arc β' and a curve γ. (See Figure 4 with β' substituted for β.)

Note that γ is essential in S^X by an index argument. If γ is nonperipheral, then the second conclusion holds. So suppose that γ is peripheral. Then α is obtained by Dehn twisting β' about γ. So β' is properly isotopic to α and has smaller intersection with R_b; thus we are done by induction.

Suppose instead that x, y, z are consecutive (along t) points of $\alpha \cap t$, with alternating sign. Surger α along $[x, z]$ to form an arc β' and a curve γ. (See Figure 5, with β' substituted for β, for one of the possible arrangements of α, β', and γ.) Again, γ is essential. If γ is nonperipheral, then the second conclusion holds and we are done. If γ is peripheral, then, as α and β' differ by a half-twist about γ, we find that β' is properly isotopic to α. Since β' has smaller intersection with R_b, we are done by induction.

All that remains is the case that α meets every rectangle R_b in at most a pair of arcs of opposite orientation. For every branch b where α meets R_b twice, choose a subarc t_b of a tie in R_b so that $\alpha \cap t_b = \partial t_b$. We call t_b a *chord* for α. For every t_b there is a subarc $\alpha_b \subset \alpha$ such that $\partial t_b = \partial \alpha_b$. A chord t_b is *innermost* if there is no chord t_c with α_c strictly contained in α_b. Let t_b be the first innermost chord. Let α' be the component of $\alpha - \alpha_b$ before α_b. Build a route β by taking two copies of α' and joining them to α_b. Since t_b is the first innermost chord, the intersection $\beta \cap R_c$ is a single arc or a pair of arcs depending on whether it is α_b or α' that meets R_c. Thus β is wide. Also, β is essential; otherwise, $t_b \cup \alpha_b$ bounds a disk with index $1/2$, a contradiction. By construction, $i(\alpha, \beta) = 0$ and Lemma 3.3 is proved. □

3.2. *Induced tracks for annuli*

Suppose that $X \subset S$ is an annulus. Define $\tau|X$ to be the union of branches $b \subset \tau^X$ so that some element of $\mathcal{A}(\tau^X)$ travels along b. (Note that $\tau|X$, if nonempty, is not compact.) Define $\mathcal{A}(\tau|X) = \mathcal{A}(\tau^X)$ and also the duals $\mathcal{A}^*(\tau|X) \supset \mathcal{A}^*(\tau^X)$.

Define $V(\tau|X)$ in $\mathcal{A}(\tau|X)$ to be the set of wide carried arcs. Define $V^*(\tau|X)$ dually.

LEMMA 3.4

Suppose that $X \subset S$ is an essential annulus. If $\mathcal{A}^{(*)}(\tau|X)$ is nonempty, then $V^{(*)}(\tau|X)$ is nonempty. Let $N = N(\tau|X)$. If $\gamma \dashv \tau|X$ is a wide essential arc, then γ meets each rectangle of N and each region of $S^X - N$ in at most a single arc.

For example, if $\gamma \prec \tau|X$ is a wide essential arc, then γ embeds into $\tau|X$.

Proof of Lemma 3.4

We prove the second conclusion; the first is similar. Suppose that R is either a rectangle or a region such that $\gamma \cap R$ is a pair of arcs to the right of each other. Let δ be an arc embedded in the interior of R such that

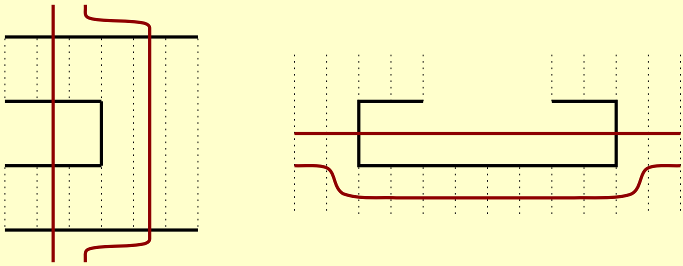

Figure 7. Pieces of N are shown, with vertical and horizontal boundary in the correct orientation; the dotted lines are ties. The left and right pictures show a vertical and horizontal rectangle swap, respectively.

- $\delta \cap \gamma = \partial\delta$, and
- δ meets both components of $\gamma \cap R$.

Let γ' be the component of $\gamma - \partial\delta$ such that $\partial\gamma' = \partial\delta$. If $\gamma' \cup \delta$ bounds a disk in S^X, then this disk has index $1/2$ and we contradict efficient position. If $\gamma' \cup \delta$ bounds an annulus, then γ was not essential, another contradiction. □

Suppose that α is the core curve of the annulus X.

LEMMA 3.5

If α is not carried by $\tau|X$, then $V(\tau|X) = \mathcal{A}(\tau|X)$. If α is not dual to $\tau|X$, then $V^*(\tau|X) = \mathcal{A}^*(\tau|X)$. □

LEMMA 3.6

Suppose that $\alpha \prec \tau|X$. One side of α is combed if and only if both sides are combed in the same direction if and only if some isotopy representative of α is dual to $\tau|X$. □

4. Finding efficient position

After discussing the various sources of nonuniqueness, we prove in Theorem 4.1 that efficient position exists.

Let $N = N(\tau)$; suppose that $\alpha \dashv \tau$. A rectangle $T \subset S - (N \cup \alpha)$ is *vertical* if ∂T has a pair of opposite sides meeting α and $\partial_v N$, respectively. Define *horizontal* rectangles similarly. Figure 7 depicts the two kinds of *rectangle swap*.

Now suppose that $\alpha \prec \tau$, that every rectangle $R_b \subset N$ meets α in at most a single arc, and that one side of α is combed. Let A be a small regular neighborhood of α. Then an *annulus swap* interchanges α and the component of ∂A on the combed side (see Figure 8).

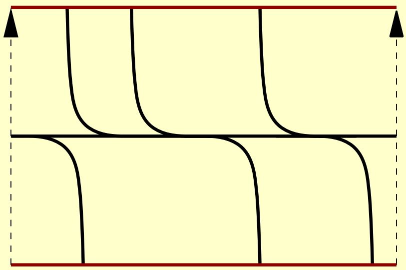

Figure 8. Both sides of α are combed (to the left). Thus both boundary components of the annulus shown are dual to τ and both differ from the carried core curve by an annulus swap.

THEOREM 4.1
Suppose that $\xi(S) \geq 1$ and that $\tau \subset S$ is a birecurrent train track. Suppose that $\Delta \subset \mathcal{AC}(S)$ is a multicurve. Then efficient position for Δ with respect to τ exists and is unique up to rectangle swaps, annulus swaps, and isotopies of S preserving the foliation of $N(\tau)$ by ties.

Remark 4.2
When $S = S_1$ is a torus, [9, Lemma 14] proves the existence of efficient position for curves with respect to Reebless bigon tracks. Uniqueness of efficient position follows from a slight generalization of Section 4.1 using *bigon swaps*.

4.1. Uniqueness of efficient position
Suppose that α and β are isotopic curves and in efficient position with respect to τ. We induct on $i(\alpha, \beta)$. For the base case, suppose that $|\alpha \cap \beta| = 0$. Then α and β cobound an annulus $A \subset S$ so that ∂A has no corners (see [4, Lemma 2.4]). Since $N = N(\tau)$ is a union of rectangles, the intersection $A \cap N$ is also a union of rectangles. Thus $\text{index}(N \cap A) = 0$. By the hypothesis of efficient position, any region $T \subset \overline{A - N}$ has nonpositive index and has all corners outwards. By the additivity of index, it follows that $\text{index}(T) = 0$. It follows that each region T is either an annulus without corners or a rectangle.

Suppose that some region T is an annulus without corners. Then we must have $T = A$. For if ∂T meets ∂N, then ∂N has a component without corners, contrary to assumption. Since $T = A$, it follows that α and β are isotopic in the complement of N, and we are done.

So we may assume that all regions of $A - N$ are rectangles. (In particular, $A \cap N \neq \emptyset$.) Note that if a region R is a horizontal rectangle, then there is no obstruction to doing a rectangle swap across R. After doing all such swaps, we may assume that $A - N$ contains no horizontal rectangles.

We now abuse terminology slightly by assuming that the position of N determines that of τ. So if A contains vertical rectangles, then there are switches of τ contained in A. This implies that A contains half-branches of τ. Let b' be a half-branch in A, meeting ∂A. If b' is large, then there is a vertical rectangle swap removing three half-branches from A. After doing all such swaps, we may assume that any such b' is small. If R is a vertical rectangle meeting ∂A, then R has two horizontal sides. If neither of these meets a switch on its interior, then again there is a swap removing three half-branches from A.

After doing all such swaps, if there are still vertical rectangles in A, then we proceed as follows. Every vertical rectangle must have a horizontal side that properly contains the horizontal side of another vertical rectangle. (For example, in Figure 8, number the rectangles above the core curve R_0, R_1, R_2 from left to right. Note that the left horizontal side of R_i strictly contains the right horizontal side of R_{i-1}.) It follows that the union of these vertical rectangles gives an annulus swap which we perform. Thus, we are reduced to the situation where A contains no horizontal or vertical rectangles.

If $A \subset N$, then α and β are both carried. For any tie $t \subset N$, any component $t' \subset t \cap A$ is an essential arc in A. (To see this, suppose that t' is inessential. Let $B \subset A$ be the bigon cobounded by t' and $\alpha' \subset \alpha$, say. Since α is carried, α' is transverse to the ties. We define a continuous involution on α'; for every tie s and for every component $s' \subset s \cap B$, transpose the endpoints of s'. As this involution is fixed-point-free, we have reached a contradiction.) It follows that A is foliated by subarcs of ties, and we are done.

There is one remaining possibility in the base case of our induction: $A \cap N \neq \emptyset$, $A \not\subset N$, and A contains no switches of τ. Thus every region of $A \cap N$ and of $A - N$ is a rectangle meeting both α and β. Any region R of $A \cap N$ is foliated by (subarcs of) ties and, as above, all ties meet R essentially. Thus R gives a parallelism between (carried arcs) ties of α and β. It follows that A gives an isotopy between α and β, sending ties to ties. This completes the proof of uniqueness when $|\alpha \cap \beta| = 0$.

For the induction step, assume that $|\alpha \cap \beta| > 0$. Since α is isotopic to β, the bigon criterion (see [4, Lemma 2.5]) implies that there is a disk $B \subset S$ with exactly two outward corners x and y such that $B \cap (\alpha \cup \beta) = \partial B$. Suppose that x is a *dual intersection*: an intersection of a tie of α and a carried arc of β. (See Figure 9.)

Let $\alpha' = \alpha \cap B$, and let $\beta' = \beta \cap B$. Orient β' away from x. Let $z \in \alpha'$ be immediately adjacent to x. Without loss of generality, we may assume that z is to the left of β', near x. Let ρ be the half-route starting at z, initially agreeing with β, and turning left at every switch. If $\rho \subset B$, then eventually ρ repeats a branch b in the same direction; it follows that there is a curve $\gamma \prec \tau$ contained in B contradicting Lemma 2.5. However, if ρ exits B through α' (β'), then we contradict efficient position of α (β).

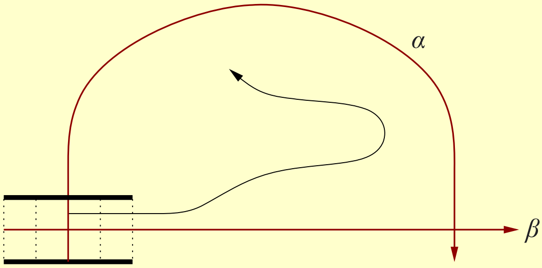

Figure 9. The left corner is a dual intersection between a tie of α and a carried arc of β. If the half-route ρ exits through α or β, then a bigon or nontrivial trigon is created.

It follows that the corner x either lies in $S - N$ or is the intersection of carried arcs of α and β. The same holds for y. In either case, we remove from B a small neighborhood of x and of y: when the corner lies in N, we use a subarc of a tie to do the cutting. The result B' is a rectangle with the components of $\partial_h B$ contained in α and β, respectively. As $\operatorname{index}(B') = 0$, the argument given in the case of an annulus gives a sequence of rectangle swaps moving α across B. This reduces $|\alpha \cap \beta|$ by two and so completes the induction step.

The proof when α and β are arcs follows the above, omitting any mention of annulus swaps.

Finally, suppose that Δ, Γ are isotopic multicurves, both in efficient position. We may isotope Γ to Δ, as above, being careful to always use innermost bigons. This completes the proof that efficient position is unique.

We end this section with a useful corollary.

COROLLARY 4.3
Suppose that $\Gamma \subset \mathcal{AC}(S)$ is a finite collection of arcs and curves in efficient position. Then we may perform a sequence of rectangle swaps to realize the pairwise geometric intersection numbers.

Proof
Let $\Gamma = \{\gamma_i\}_{i=1}^k$. By induction, the curves of $\Gamma' = \Gamma - \{\gamma_k\}$ realize their pairwise geometric intersection numbers. If γ_k meets some $\gamma_i \in \Gamma'$ nonminimally, then by the bigon criterion (see [5, p. 46]), there is an innermost bigon between γ_k and some $\gamma_j \in \Gamma'$. We now may reduce the intersection number following the proof of uniqueness of efficient position. □

4.2. Existence of efficient position

Our hypotheses are weaker, and thus our discussion is more detailed, but the heart of the matter is inspired by [15, pp. 122–123].

We assume that τ fills S. This is because, if τ did not fill, we could replace S by the subsurface it does fill. Since τ is transversely recurrent, for any $\epsilon, L > 0$ there is a finite area hyperbolic metric on the interior of S and an isotopy of τ with the following property. Every branch of τ has length at least L and every train-route $\rho \prec \tau$ has geodesic curvature less than ϵ at every point (see [21, Theorem 1.4.3]).

Let $\tau^{\mathbb{H}}$ be the lift of τ to $\mathbb{H} = \mathbb{H}^2$, the universal cover of S. Every train-route $\rho \prec \tau^{\mathbb{H}}$ cuts $\mathbb{H}$ into a pair $H^{\pm}(\rho)$ of open *half-planes*. Fix a route $\rho \prec \tau^{\mathbb{H}}$ and an integer $n \in \mathbb{Z}$. Suppose that $b' \subset \tau^{\mathbb{H}}$ is the half-branch with $b' \cap \rho = \rho(n)$. We say the branch b, containing b', is *rising* or *falling* with respect to ρ as the large half-branch at the switch $\rho(n)$ is contained in $\rho|[n, \infty)$ or contained in $\rho|(-\infty, n]$.

CLAIM 4.4

For any route $\rho \prec \tau^{\mathbb{H}}$, one side of ρ has infinitely many rising branches while the other side has infinitely many falling branches.

Proof

Note that there are infinitely many half-branches on both sides of ρ; if not, then $\partial_h N(\tau)$ would have a component without corners, contrary to assumption. Suppose that there are only finitely many rising branches along ρ. Then there is a curve $\gamma \prec \tau$ such that $w_\gamma(b) \leq 1$ for every branch b and such that the two sides of γ are combed in opposite directions. Thus τ is not recurrent, a contradiction. The same contradiction is obtained if there are only finitely many falling branches along ρ. □

CLAIM 4.5

For any route $\rho \prec \tau^{\mathbb{H}}$ and for any family of half-routes $\{\beta_n\}$, if $\beta_n \cap \rho = \rho(n)$, then $\lim_{n\to\infty} \beta_n(\infty) = \rho(\infty)$.

Proof

Let $x = \rho(\infty) \in \partial_\infty \mathbb{H}$. Suppose that $\{\beta_n\}$ is a subsequence where the first branch of each β_n is falling. Let $P_n = \rho|(-\infty, n] \cup \beta_n$, oriented away from $\rho(-\infty)$. Note that $P_n(\infty) = \beta_n(\infty)$. Recall that ρ and P_n are both uniformly close to geodesics (see [21, pp. 61–62]). Thus $P_n(\infty) \to x$ as $n \to \infty$.

Suppose instead that $\{\beta_n\}$ is a subsequence where the first branch of each β_n is rising. Let $P_n = \beta_n \cup \rho|[n, \infty)$, oriented towards x; so $P_n(\infty) = x$ for all n. Note that $P(-\infty) = \beta_n(\infty)$. Since all complementary regions of $\tau^{\mathbb{H}}$ have negative index, none of the P_n may cross each other. It follows that either the P_n exit compact subsets of

$\mathbb{H}$, and we are done, or the P_n converge (see [21, Theorem 1.5.4]) to P, a train-route with $P(\infty) = x$. Since P does not cross any P_n, deduce that P and ρ are disjoint. But this contradicts [20, Corollary 3.3.4]: train-routes that share an endpoint must share a half-route. □

Given distinct points $x, y, z \in S^1 = \partial_\infty \mathbb{H}$, arranged counterclockwise, let (y, z), (z, y) be the two components of $S^1 - \{y, z\}$; we chose notation so that $x \in (z, y)$. Let $[y, z]$ be the closure of (y, z).

CLAIM 4.6
For any distinct $z, y \in S^1$, there is a train-route ρ such that one of the intervals $\partial_\infty H^\pm(\rho)$ is contained in (z, y).

Proof
The endpoints of train-routes are dense in $S^1 = \partial_\infty \mathbb{H}$. Fix $x \in (z, y)$ so that x is the endpoint of a train-route γ. Since there are infinitely many rising branches along γ (Claim 4.4), the claim follows from the rising case of Claim 4.5. □

Let $H_{x,y} \subset \mathbb{H}$ be the convex hull of $(x, y) \subset \partial_\infty \mathbb{H}$. Let $\mathcal{H}_{x,y}$ be the union of all open half-planes $H(\rho)$ such that $\partial_\infty H(\rho) \subset (x, y)$. Since train-routes have geodesic curvature less than ϵ at every point, we have the following.

CLAIM 4.7
The union $\mathcal{H}_{x,y}$ is contained in a δ-neighborhood of $H_{x,y}$, where δ may be taken as small as desired by choosing appropriate ϵ, L. □

A set $X \subset \mathbb{H}$ is *ϵ'-convex* if every pair of points in X can be connected by a path in X which has geodesic curvature less than ϵ' at every point.

CLAIM 4.8
$\mathbb{H} - \mathcal{H}_{x,y}$ is closed and ϵ'-convex, where ϵ' may be taken as small as desired by choosing appropriate ϵ, L.

Proof
This is proved in detail in [15, pp. 122–123]. □

CLAIM 4.9
The point x is an accumulation point of $\partial(\mathbb{H} - \mathcal{H}_{x,y})$.

Proof
Pick a sequence of subintervals $(x_n, y_n) \subset (x, y)$ such that $x_n, y_n \to x$ as $n \to \infty$. By Claim 4.6, for every n there is a route ρ_n and a half-plane $H_n = H(\rho_n)$ such that $\partial_\infty H_n \subset (x_n, y_n)$. It follows that $H_n \subset \mathcal{H}_{x,y}$. Let r_n be any bi-infinite geodesic perpendicular to $\partial H_{x,y}$ and meeting H_n. Thus $r_n \to x$ as $n \to \infty$. By Claim 4.7, the intersection $r_n \cap \partial(\mathbb{H} - \mathcal{H}_{x,y})$ is nonempty, and we are done. □

The next lemma is not needed for the proof of Theorem 4.1; we state it and give the proof in order to introduce necessary techniques and terminology.

LEMMA 4.10
For any nonparabolic point $x \subset S^1$ there is a sequence of train-routes $\{\rho_n\}$ with associated half-planes $\{H(\rho_n)\}$ forming a neighborhood basis for x.

Proof
Let y, z be arbitrary points of S^1 such that x, y, z are ordered counterclockwise. It suffices to construct a train-route separating x from (y, z).

First, assume that x is the endpoint of a route ρ. Claim 4.4 implies that there are infinitely many rising branches $\{a_m\}$ on one side of ρ and infinitely many falling branches $\{c_n\}$ on the other side. Run half-routes α_m and γ_n through a_m and c_n so each half-route meets ρ in a single switch. By Claim 4.5, the endpoints converge: $\alpha_m(\infty), \gamma_n(\infty) \to x$. Thus sufficiently large m, n give a train-route

$$\alpha_m \cup \rho|_{[m,n]} \cup \gamma_n$$

that separates x from (y, z), as desired.

For the general case, consider

$$\mathcal{K} = \mathbb{H} - (\mathcal{H}_{z,x} \cup \mathcal{H}_{x,y}).$$

Note that x is an accumulation point of $\mathcal{K}$ (by Claim 4.9 and because $\mathcal{H}_{z,x}$ cannot contain points of $\partial(\mathbb{H} - \mathcal{H}_{x,y})$). Fix any base point $w \in \mathcal{K}$. By Claim 4.8, the set $\mathcal{K}$ is ϵ'-convex. Thus there is a path $r \subset \mathcal{K}$ from w to x which has geodesic curvature less than ϵ' at every point. Since x is not a parabolic point, the projection of r to S recurs to the thick part of S; thus r meets infinitely many branches $\{b_n\}$ of $\tau^{\mathbb{H}}$.

Suppose that b is a branch of $\tau^{\mathbb{H}}$ lying in $\mathcal{K}$. If the two sides of b meet $\mathcal{H}_{z,x}$ and $\mathcal{H}_{x,y}$, then b is a *bridge* of $\mathcal{K}$. If the sides of b meet neither $\mathcal{H}_{z,x}$ nor $\mathcal{H}_{x,y}$, then b is an *interior branch* of $\mathcal{K}$. If exactly one side of b lies in $\mathcal{K}$, then b is a *boundary branch*. If both sides lie in $\mathcal{H}_{z,x}$ (or both sides lie in $\mathcal{H}_{x,y}$), then b is an *exterior branch*. (See Figure 10 for an illustration of the case $z = y$.)

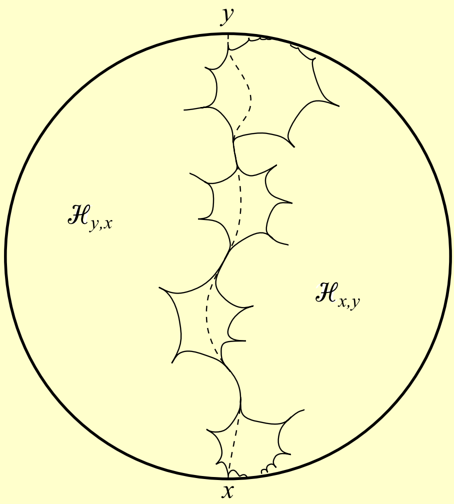

Figure 10. The path r runs from y to x. To simplify the figure, no interior branches are shown.

By convexity, the path r is disjoint from the exterior branches. After a small isotopy, the path r is also disjoint from the boundary branches, meets the interior branches transversely, and still has geodesic curvature less than ϵ' at every point.

Now, if r travels along a bridge, then there are routes $\rho^{\pm}$ cutting off half-planes $H^{\pm}$ lying in $\mathcal{H}_{z,x}$ and $\mathcal{H}_{x,y}$, respectively. Then either x is the endpoint of a train-route or a cut and paste of $\rho^{\pm}$ gives the desired route Γ separating x from (y, z). In either case, we are done.

So suppose that r only meets interior branches $\{b_n\}$ of $\mathcal{K}$. Let γ_n be any train-route travelling along b_n. If any of the γ_n land at x, then we are done as above. So suppose instead none of the γ_n land at x. By fixing orientations and passing to a subsequence, we may assume that $\gamma_n(\infty) \to x$ as $n \to \infty$. There are now two cases. Suppose that, for infinitely many n, we find that $\gamma_n(-\infty) \in [y, z]$. Then passing to a further subsequence, we have that $\gamma_n \to \Gamma$, where $\Gamma(\infty) = x$ (see [21, Theorem 1.5.4]); thus x is the endpoint of a train-route and we are done as above. The other possibility is that, for some sufficiently large n, both endpoints $\gamma_n(\pm\infty)$ lie in (z, y). Since b_n is an interior branch, γ_n separates x from (y, z) and Lemma 4.10 is proved. □

4.3. *Finding invariant efficient position*

Fix $\alpha \in \mathcal{C}(S)$. (The case of $\alpha \in \mathcal{A}(S)$ is dealt with in Section 4.4.) Let α' be a component of the lift of α to the universal cover $\mathbb{H}$. Let $\pi_1(\alpha)$ be the cyclic subgroup (of the deck group) preserving α'. Let $\{x, y\} = \partial_\infty \alpha' \subset S^1$. We take

$$\mathcal{K} = \mathbb{H} - (\mathcal{H}_{y,x} \cup \mathcal{H}_{x,y}).$$

By construction, $\mathcal{K}$ is $\pi_1(\alpha)$-invariant. By Claims 4.8 and 4.9, the set $\mathcal{K}$ is closed, ϵ'-convex, and has $\{x, y\} \subset \partial_\infty \mathcal{K}$. By Lemma 4.10, the only nonparabolic points of $\partial_\infty \mathcal{K}$ are x and y. As in the proof of Lemma 4.10, we find a bi-infinite path $r \subset \mathcal{K}$ connecting y to x, with geodesic curvature less than ϵ' at every point. (See Figure 10.)

Let $H(r)$ be the open half-plane to the right of r. If we remove the union

$$\bigcup_{g \in \pi_1(\alpha)} g \cdot H(r)$$

from $\mathbb{H}$, then, as with Claim 4.8, what remains is closed and ϵ''-convex for some small ϵ''. It follows that we may homotope the path r to become a $\pi_1(\alpha)$-invariant smooth path, contained in $\mathcal{K}$ and transverse to the interior branches, and avoiding the exterior branches of $\mathcal{K}$. A further equivariant isotopy ensures that r also avoids the boundary branches of $\mathcal{K}$. Orient r from y to x.

Remark 4.11
Suppose that $\gamma \prec \tau^{\mathbb{H}}$ is a train-route that separates x from y. Note that if there exists a nonidentity element $g \in \pi_1(\alpha)$ such that γ and $g \cdot \gamma$ meet, then r is carried by $\tau^{\mathbb{H}}$; thus $\alpha \prec \tau$, and we are done. We will henceforth assume that train-routes separating x from y are disjoint from their nontrivial translates.

Let b be any interior branch of $\mathcal{K}$, and let γ be any train-route travelling along b. Since b is interior, γ must separate x from y. Orient γ from $\mathcal{H}_{y,x}$ to $\mathcal{H}_{x,y}$. (Thus if γ and r meet once, then the tangent vectors to r and γ, in that order, form a positive frame.) The orientation of γ gives an orientation to b. Moreover, as b is an interior branch, a cut-and-paste argument shows that the orientation on b is independent of our choice of γ. Orient all interior branches in this fashion, and note that these orientations agree at interior switches.

We say that $p \in r \cap b$ has *positive* or *negative sign* as the tangent vectors to r and b (in that order) form a positive or negative frame. Suppose that there are $\mathsf{N} \in \mathbb{N}$ orbits of points of negative sign, under the action of $\pi_1(\alpha)$. We now induct on N.

Suppose that N is zero. Any bigon between r and a train-route is contained in $\mathcal{K}$ and so contributes one point of positive and one point of negative sign. So if there are no points of negative sign, then there are no bigons and r is in efficient position with respect to $\tau^{\mathbb{H}}$. Recall that r is $\pi_1(\alpha)$-invariant. So $\beta \subset S$, the image of r under the universal covering map, is an immersed curve in S homotopic to α. If β is embedded, then we are done. If not, then the bigon criterion for immersed curves (see [12, Theorem 2.7]) implies that β must have either a monogon or a bigon of self-intersection. If β has a monogon B of self-intersection, then, since index is additive, τ must be disjoint from B. Thus we can homotope β to remove B while fixing τ pointwise. If β has a bigon B of self-intersection, then, as in the proof of uniqueness in Section 4.1,

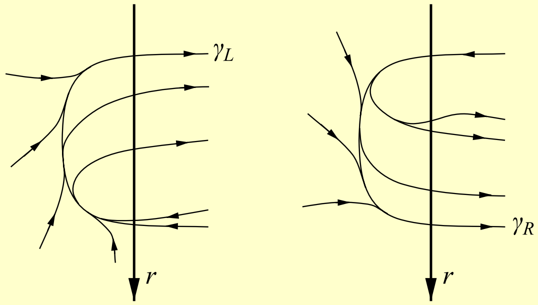

Figure 11. Left: the lowest point shown has negative sign. The paths r and γ_L form a bigon. Right: the corresponding figure for γ_R.

we may remove B via a sequence of rectangle swaps. After removing all monogons and bigons of self-intersection, the curve β is embedded and in efficient position.

Suppose that N is positive. Let b be a branch with a point $p \in r \cap b$ of negative sign. Let γ_R (γ_L) be the half-route starting at p, travelling in the direction of b, and thereafter turning only right (left). Each of γ_R and γ_L must have at least one bigon with r, as their points at infinity lie in (x, y). There are now two (essentially identical) cases (see Figure 11).

- There is a bigon B between r and γ_R, to the right of r, such that the corners of B appear in the same order along r and γ_R.
- There is a bigon B between r and γ_L, to the right of r, such that the corners of B appear in opposite order along r and γ_L.

If neither case holds, then any half-route ending at p must originate in (x, y), contradicting the fact that p has negative sign. Note that, by Remark 4.11, γ_R (γ_L) is disjoint from its nontrivial $\pi_1(\alpha)$ translates. It follows that the bigon B is also disjoint from its nontrivial translates. Finally, we may equivariantly isotope r across $\pi_1(\alpha) \cdot B$. Since the arc of $\gamma_R \cap \partial B$ (resp., $\gamma_L \cap \partial B$) is combed outside of B, this isotopy reduces N by at least 1 (again, see Figure 11). This completes the proof of Theorem 4.1 when α is a curve.

4.4. *Efficient position for arcs and multicurves*

Now suppose that $\alpha \subset S$ is an essential arc. Let α' be a lift of α to $\mathbb{H}$, the universal cover of S. Note that $\{x, y\} = \partial_\infty \alpha'$ is a pair of parabolic points. Construct $\mathcal{K} = \mathbb{H} - (\mathcal{H}_{y,x} \cup \mathcal{H}_{x,y})$ as before. The proof now proceeds as above, omitting any mention of $\pi_1(\alpha)$, equivariance, or annulus swaps.

Finally, suppose that Δ is a multicurve. As shown in Section 4.2, we may isotope, individually, every $\alpha \in \Delta$ into efficient position. By Corollary 4.3, all $\alpha \in \Delta$ may be realized disjointly in efficient position. This completes the proof of Theorem 4.1. $\square$

5. The structure theorem

5.1. Bounding diameter

We now bound the diameters of the sets of wide arcs and curves carried by the induced track.

LEMMA 5.1

Suppose that τ is a birecurrent track and that $X \subset S$ is an essential annulus. If $\tau|X \neq \emptyset$, then the diameter of $V(\tau|X) \cup V^*(\tau|X)$ inside of $\mathcal{A}(X)$ is at most 8.

Proof

In the proof, we use V and V^* to represent $V(\tau|X)$ and $V^*(\tau|X)$. Since $\tau|X \neq \emptyset$, it follows that $\mathcal{A}(\tau|X)$ is nonempty. The first conclusion of Lemma 3.4 now implies that V is nonempty.

CLAIM

The set V^* of wide dual arcs is nonempty.

Proof

By Lemma 2.6, there is a dual curve $\beta \in \mathcal{C}^*(\tau)$ such that $i(\alpha, \beta) > 0$. Thus there is a lift $\beta' \subset S^X$ with closure an essential arc. Since $\tau|X \subset \tau^X$, it follows that $\beta' \in \mathcal{A}^*(\tau|X)$. The first conclusion of Lemma 3.4 now implies that V^* is nonempty. □

CLAIM

If $\beta \in V$ and $\gamma \in V^*$, then $i(\beta, \gamma) \leq 3$.

Proof

Suppose that $i(\beta, \gamma) = n \geq 4$. Let $\{\gamma_i\}_{i=1}^{n-1}$ be the components of $\gamma - \beta$ with both endpoints on β. Let R_i be the components of $S^X - (\beta \cup \gamma)$ with compact closure. We arrange matters so that opposite sides of R_i are on γ_i and γ_{i+1}. Let R be the union of the R_i. Since $\operatorname{index}(R) = 0$, every region T of the closure of $R - N(\tau|X)$ also has index zero and so is a rectangle. If T meets both γ_i and γ_{i+1}, then γ was not wide, a contradiction. As $n - 1 \geq 3$, any region T meeting γ_2 is a compact rectangle component of the closure of $S^X - N(\tau|X)$. An index argument implies that τ^X, and thus $\tau \subset S$, has a complementary region with nonnegative index, a contradiction. □

Since V and V^* are nonempty, it follows that $\operatorname{diam}(V \cup V^*) \leq 8$. □

Now suppose that X is not an annulus. Prompted by Lemma 2.8, we define

$$W(\tau^X) = \{\alpha \in \mathcal{AC}(\tau^X) \mid \alpha \text{ is wide}\}.$$

Define $W^*(\tau^X)$ similarly, replacing $\mathcal{AC}(\tau^X)$ by $\mathcal{AC}^*(\tau^X)$.

LEMMA 5.2

There is a constant $\mathsf{K}_1 = \mathsf{K}_1(S)$ with the following property. Suppose that τ is a track and that $X \subset S$ is an essential subsurface (not an annulus) with $\pi_X(\tau) \neq \emptyset$. Then the diameter of $W(\tau^X) \cup W^*(\tau^X)$ inside of $\mathcal{AC}(X)$ is at most K_1. Furthermore, if ∂X, after isotopy into efficient position and with the induced orientation, is not wide, then either $\mathcal{C}(\tau|X)$ or $\mathcal{C}^*(\tau|X)$ has diameter at most 2 in $\mathcal{C}(X)$.

Proof

In the proof, we use $W, W^*, \mathcal{AC}, \mathcal{AC}^*$ to represent $W(\tau^X)$ and so on. Since $\pi_X(\tau) \neq \emptyset$, there is some vertex cycle $\alpha \in V(\tau)$ such that α cuts X. Since α is wide (see Lemma 2.8), there is a lift $\alpha' \prec \tau^X$ which is also wide; deduce that W is nonempty.

CLAIM

The set W^* of wide duals is nonempty.

Proof

By Lemma 2.6, there is a dual curve $\alpha \in \mathcal{C}^*(\tau)$ cutting X. By Lemma 2.9, there is a wide dual β that also cuts X. Thus there is a lift $\beta' \subset S^X$ with closure an essential wide arc or wide essential, nonperipheral curve. So $\beta' \in W^*$, as desired. ☐

Now isotope ∂X into efficient position. Let X' be the compact component of the preimage of X under the covering map $S^X \to S$. Note that $\partial X'$ is in efficient position with respect to τ^X. Note that the covering map $S^X \to S$ induces a homeomorphism between X' and X. Let $N^X = N(\tau^X) \subset S^X$ be the preimage of $N = N(\tau)$. Let $N' = X' \cap N^X$. Again, the covering map induces a homeomorphism between N' and $N \cap X$.

Suppose that ∂X, with its induced orientation, is not wide. If ∂X fails to be wide in $S - N(\tau)$, then there is a properly embedded, essential arc $\gamma \subset X$ disjoint from $N(\tau)$. Lift γ to $\gamma' \subset X'$. Adjoin to γ' geodesic rays in $S^X - X'$ to obtain an essential, properly embedded arc $\gamma'' \subset S^X$. Note that $i(\gamma'', \alpha) = 0$ for every $\alpha \in \mathcal{AC}$; only intersections in X' contribute to geometric intersection number as computed in S^X. This implies that $\operatorname{diam}_X(\mathcal{C}(\tau|X)) \leq 2$. Furthermore, $i(\gamma'', \beta) \leq 2$ for every $\beta \in W^*$. This gives the desired diameter bound for $W \cup W^*$.

If, instead, ∂X fails to be wide in $N(\tau)$, then there is a properly embedded, essential arc $\gamma \subset X$ that is a subarc of a tie. Again, lift and extend to an essential arc $\gamma'' \subset$

S^X so that $i(\gamma'', \beta) = 0$ for any $\beta \in \mathcal{AC}^*$. This implies that $\operatorname{diam}_X(\mathcal{C}^*(\tau|X)) \le 2$. We also have $i(\gamma'', \alpha) \le 2$ for any $\alpha \in W$. Again, the diameter is bounded.

Now suppose that ∂X is wide. Thus, for every $b \in \mathcal{B}(\tau)$, the rectangle R_b meets ∂X in at most a pair of arcs. It follows that $N \cap X$, and thus N', is a union of at most $2|\mathcal{B}(\tau)|$ subrectangles of the form $R', R'' \subset R_b$. Suppose that $\alpha \in W$ and that $\beta \in W^*$. Then α and β each meet a subrectangle R' in at most two arcs. Thus α and β intersect in at most four points inside of R'. Thus $i(\alpha, \beta) \le 8|\mathcal{B}(\tau)|$. Since $|\mathcal{B}(\tau)| \le 6 \cdot \xi(S) = 18g - 18 + 6n$, Lemma 5.2 is proved. □

5.2. *Accessible intervals*

Suppose that $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence of birecurrent train tracks. Suppose that $X \subset S$ is an essential subsurface, yet not an annulus, with $\xi(X) < \xi(S)$. Define

$$m_X = \min\bigl\{i \in [0, N] \bigm| \operatorname{diam}_X\bigl(\mathcal{C}^*(\tau_i|X)\bigr) \ge 3\bigr\}$$

and

$$n_X = \max\bigl\{i \in [0, N] \bigm| \operatorname{diam}_X\bigl(\mathcal{C}(\tau_i|X)\bigr) \ge 3\bigr\}.$$

If either m_X or n_X is undefined or if $n_X < m_X$, then I_X, the *accessible interval*, is empty. Otherwise, $I_X = [m_X, n_X]$.

If $X \subset S$ is an annulus, then I_X is defined by replacing $\mathcal{C}$ by $\mathcal{A}$ and increasing the lower bound on diameter from 3 to 9. We may now state the structure theorem.

THEOREM 5.3

For any surface S with $\xi(S) \ge 1$, there is a constant $\mathsf{K}_0 = \mathsf{K}_0(S)$ with the following property. Suppose that $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence of birecurrent train tracks in S, and suppose that $X \subset S$ is an essential subsurface.

- *For every $[a, b] \subset [0, N]$, if $[a, b] \cap I_X = \emptyset$ and $\pi_X(\tau_b) \ne \emptyset$, then $d_X(\tau_a, \tau_b) \le \mathsf{K}_0$.*

Suppose that $i \in I_X$. If X is an annulus, then we have the following.

- *The core curve α is carried by and wide in τ_i.*
- *Both sides of α are combed in the induced track $\tau_i|X$.*
- *If $i + 1 \in I_X$, then $\tau_{i+1}|X$ is obtained by taking subtracks, slides, or at most a pair of splittings of $\tau_i|X$.*

If X is not an annulus, then we have the following.

- *When in efficient position, ∂X is wide with respect to τ_i.*
- *The track $\tau_i|X$ is birecurrent and fills X.*
- *If $i + 1 \in I_X$, then $\tau_{i+1}|X$ is either a subtrack, a slide, or a split of $\tau_i|X$.*

Proof
Fix an interval $[a,b] \subset [0,N]$. Note that $\tau_b \prec \tau_a$ and so $\tau_b^X \prec \tau_a^X$. Thus $\mathcal{AC}(\tau_b^X) \subset \mathcal{AC}(\tau_a^X)$, while $\mathcal{AC}^*(\tau_a^X) \subset \mathcal{AC}^*(\tau_b^X)$.

CLAIM
If $[a,b] \cap I_X = \emptyset$ *and if* $\pi_X(\tau_b) \neq \emptyset$, *then* $d_X(\tau_a, \tau_b) \leq \mathsf{K}_0$.

Proof
Fix, for the duration of the claim, a vertex cycle $\beta \in V(\tau_b)$ so that β cuts X. Since β is also carried by τ_a, there is, by Lemma 2.9, a vertex cycle $\alpha \in V(\tau_a)$ cutting X. Pick $\alpha' \in \pi_X(\alpha)$ and $\beta' \in \pi_X(\beta)$. Note that Lemma 2.8 implies that α' is wide in τ_a^X while β' is wide in τ_b^X. The proof divides into cases depending on the relative positions of a, b, m_X, and n_X.

Case I
Suppose that $n_X < a$ or n_X is undefined.

Note that $\beta' \prec \tau_a^X$. If X is an annulus, then, since $a \notin I_X$, the diameter of $\mathcal{A}(\tau_a|X)$ is at most 8; thus $d_X(\alpha, \beta) \leq 8$, and we are done.

Suppose that X is not an annulus. If β' is an arc, then Lemma 3.3 gives two cases: we may replace β' by γ, which is either a wide arc in τ_a^X or is an essential nonperipheral curve in $\tau_a|X$. (If β' is a curve, then let $\gamma = \beta'$.) In either case, Lemma 3.3 ensures that $i(\gamma, \beta') \leq 2$ and so $d_X(\gamma, \beta') \leq 4$. If γ is an arc, then both α' and γ are wide so Lemma 5.2 gives $d_X(\alpha, \beta) \leq \mathsf{K}_1 + 4$. If γ is a curve, pick any $\delta \in V(\tau_a|X)$. Then Lemma 5.2 implies that $d_X(\alpha', \delta) \leq \mathsf{K}_1$. Also, $a \notin I_X$ implies that $d_X(\delta, \gamma) \leq 2$. Thus $d_X(\alpha, \beta) \leq \mathsf{K}_1 + 6$.

Case II
Suppose that $b < m_X$ or m_X is undefined.

If X is an annulus, then Lemma 5.1 gives wide duals $\alpha^* \in V^*(\tau_a|X)$ and $\beta^* \in V^*(\tau_b|X)$ so that $d_X(\alpha', \alpha^*), d_X(\beta', \beta^*) \leq 8$. It follows that the arc α^* also lies in $\mathcal{A}^*(\tau_b|X)$. Since $b \notin I_X$, we have $d_X(\alpha^*, \beta^*) \leq 8$. Thus $d_X(\alpha, \beta) \leq 24$, as desired.

If X is not an annulus, then by Lemma 5.2 there is a wide dual $\alpha^* \in W^*(\tau_a^X)$ such that $d_X(\alpha', \alpha^*) \leq \mathsf{K}_1$. Again, α^* is also an element of $\mathcal{AC}^*(\tau_b^X)$ but may not be wide there. If α^* is an arc, then Lemma 3.3 gives two cases: we may replace α^* by γ^* which is either a wide dual arc to τ_b^X or is an essential nonperipheral dual curve to τ_b^X. (If α^* is a curve, then let $\gamma^* = \alpha^*$.) So $i(\alpha^*, \gamma^*) \leq 2$ and thus $d_X(\alpha^*, \gamma^*) \leq 4$. If γ^* is a wide dual arc, then Lemma 5.2 implies that $d_X(\gamma^*, \beta') \leq \mathsf{K}_1$ and so

$d_X(\alpha, \beta) \le 2\mathsf{K}_1 + 4$. If γ^* is a dual curve, then, as $b \notin I_X$, any dual wide curve $\delta^* \in V^*(\tau_b|X)$ has $d_X(\gamma^*, \delta^*) \le 2$. Again, Lemma 5.2 implies that $d_X(\delta^*, \beta') \le \mathsf{K}_1$ and so $d_X(\alpha, \beta) \le 2\mathsf{K}_1 + 6$.

Case III
Suppose that $a \le n_X < c < m_X \le b$.

The first two cases bound $d_X(\tau_c, \tau_b)$ and $d_X(\tau_a, \tau_c)$; thus we are done by the triangle inequality.

Case IV
Suppose that $a \le n_X < m_X \le b$ and that $m_X = n_X + 1$.

Let $c = n_X$ and $d = m_X$. The first two cases bound $d_X(\tau_d, \tau_b)$ and $d_X(\tau_a, \tau_c)$. Since $V(\tau_c)$ and $V(\tau_d)$ have bounded intersection, $d_X(\tau_c, \tau_d)$ is also bounded and the claim is proved. □

Now fix $i \in I_X$.

CLAIM
If X is an annulus, then we have the following.

- *The core curve α is carried by and is wide in τ_i.*
- *Both sides of α are combed in the induced track $\tau_i|X$.*
- *If $i + 1 \in I_X$, then $\tau_{i+1}|X$ is obtained by taking subtracks, slides, or at most a pair of splittings of $\tau_i|X$.*

Proof
Since $i \in I_X$, both $\mathcal{A}(\tau_i|X)$ and $\mathcal{A}^*(\tau_i|X)$ have diameter at least 9. From Lemma 5.1, deduce that the inclusions $V \subset \mathcal{A}$ and $V^* \subset \mathcal{A}^*$ are strict. Thus by Lemma 3.5, the core curve α is both carried by and dual to $\tau_i|X$. The second statement now follows from Lemma 3.6. Thus at least one side of α is combed in τ_i^X. Projecting from S^X to S, we find that $\alpha \prec \tau_i$. If α is not wide in τ_i, then we deduce that neither side of α is combed in τ^X, a contradiction.

Suppose that τ_i slides to τ_{i+1}. Then, up to isotopy, τ_{i+1} slides to τ_i. Since slides do not kill essential arcs, it follows that $\tau_{i+1}|X$ is obtained from $\tau_i|X$ by an at most countable collection of slides.

Now suppose that τ_{i+1} is obtained by splitting τ_i along a large branch b. Thus τ_{i+1}^X is obtained from τ_i^X by splitting all of the countably many lifts of b. Every essential arc carried by $\tau_{i+1}|X$ is also carried by τ_i^X. Let $\tau' \subset \tau_i^X$ be the union of these essential routes. It follows that $\tau_{i+1}|X$ is obtained from τ' by splitting along

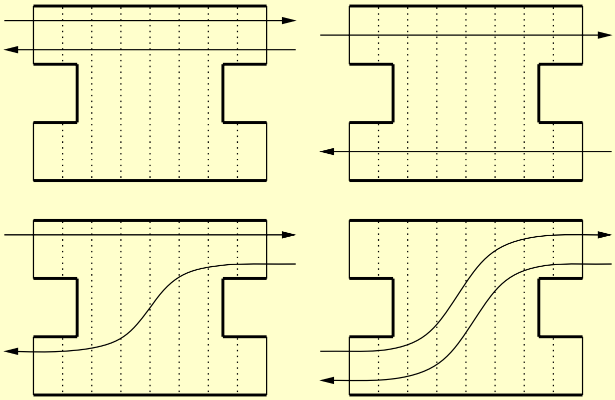

Figure 12. Four of the possible ways for an oriented, carried, wide curve α to meet a large rectangle R_b of $N(\tau)$. Note that when X is an annulus and α is the core curve, the upper left picture implies that neither side of α is combed in $\tau|X$. Splitting the upper right deletes zero or two components of $\tau_i|X - \alpha$. In the bottom row, only the left splitting is possible when $i, i+1 \in I_X$. On the bottom left, one component of $\tau_i|X - \alpha$ is deleted and $\tau_i|X$ is split once. On the bottom right $\tau_i|X$ is split twice.

lifts of b that are also large branches of τ'. Since both sides of α are combed in $\tau_i|X$, the same is true in τ' and so any component of $\tau' - \alpha$ is a tree without large branches. The track τ' therefore has only finitely many large branches, all contained in α. Since α is wide in τ_i, there are at most two preimages of the large branch b contained in $\alpha \subset S^X$. Thus $\tau_{i+1}|X$ is obtained from τ' by at most two splittings. This proves the claim. (See Figure 12 for pictures of how α may be carried by τ_i and how splitting b effects $\tau_i|X$.) □

CLAIM

Suppose that X is not an annulus.

- *When in efficient position, ∂X is wide with respect to τ_i.*
- *The track $\tau_i|X$ is birecurrent and fills X.*
- *If $i+1 \in I_X$, then $\tau_{i+1}|X$ is either a subtrack, a slide, or a split of $\tau_i|X$.*

Proof

Since $i \in I_X$, the second conclusion of Lemma 5.2 implies that ∂X is wide. The induced track $\tau_i|X$ carries a pair of curves at distance at least 3; thus $\tau_i|X$ fills X. Also, $\tau_i|X$ is recurrent by definition. For any branch $b' \in \tau_i|X \subset S^X$, let $b \subset \tau$ be the image in S. Since τ is transversely recurrent, there is a dual curve β meeting b. Lifting β to a curve or arc $\beta' \subset S^X$ gives a dual to $\tau|X$ meeting b'. Thus $\tau|X$ is transversely recurrent with respect to arcs and curves, as defined in Section 3.1.

Now, if τ_i slides to τ_{i+1}, then, as in the annulus case, $\tau_i|X$ slides to $\tau_{i+1}|X$. Suppose instead that τ_i splits to τ_{i+1} along the branch $b \in \mathcal{B}(\tau_i)$. Thus $\tau_i|X$ splits (or isotopes) to a track τ' so that $\tau_{i+1}|X$ is a subtrack. Let $R_b \subset N(\tau_i)$ be the rectangle corresponding to the branch b. Isotope ∂X into efficient position with respect to $N(\tau_i)$, and recall that X is to the left of ∂X. Note that, by an isotopy, we may arrange for all curves in $\mathcal{C}(\tau_i|X)$ to be disjoint from ∂X. Let $\beta \subset R_b$ be a *central splitting arc*: a carried arc completely contained in R_b.

If $\beta \cap X$ is empty, then $\tau_i|X$ is identical to $\tau_{i+1}|X$. If $\beta \subset X$, then $\tau_{i+1}|X$ is either a subtrack or a splitting of $\tau_i|X$, depending on how the carried curves of $\tau_i|X$ meet the lift of R_b. In all other cases, $\tau_{i+1}|X$ is a subtrack of $\tau_i|X$. (See Figure 12 for some of the ways carried subarcs of ∂X may meet R_b.) If $\partial X \cap R_b$ contains a tie, then $\tau_i|X$ is identical to $\tau_{i+1}|X$. This completes the proof of the claim. □

Thus Theorem 5.3 is proved. □

We now rephrase a result of Masur and Minsky [17] using the *refinement* procedure of Penner and Harer [21, p. 122].

THEOREM 5.4 ([17, Theorem 1.3])
For any surface S with $\xi(S) \geq 1$, there is a constant $Q = Q(S)$ with the following property. For any sliding and splitting sequence $\{\tau_i\}_{i=0}^N$ of birecurrent train tracks in S, the sequence $\{V(\tau_i)\}_{i=0}^N$ forms a Q-unparameterized quasi-geodesic in $\mathcal{C}(S)$.

Theorem 5.3 implies that the same result holds after subsurface projection.

THEOREM 5.5
For any surface S with $\xi(S) \geq 1$, there is a constant $Q = Q(S)$ with the following property. For any sliding and splitting sequence $\{\tau_i\}_{i=0}^N$ of birecurrent train tracks in S and for any essential subsurface $X \subset S$, if $\pi_X(\tau_N) \neq \emptyset$, then the sequence $\{\pi_X(\tau_i)\}_{i=0}^N$ is a Q-unparameterized quasi-geodesic in $\mathcal{C}(X)$.

Proof
By the first conclusion of Theorem 5.3, we may restrict attention to the subinterval $[p, q] = I_X \subset [0, N]$.

Fix any vertex $\alpha \in V(\tau_q|X)$. Note that α is carried by $\tau_i|X$ for all $i \leq q$. So define $\sigma_i \subset \tau_i|X$ to be the minimal pretrack carrying α. Since σ_i does not carry any peripheral curves, σ_i is a train track. Note that σ_i is recurrent by definition and transversely recurrent by Lemma 3.2. Applying Theorem 5.3, for all $i \in [p, q-1]$, the track σ_{i+1} is a slide, a split, or identical to the track σ_i.

Theorem 5.4 implies that the sequence $\{V(\sigma_i)\}$ is a Q-unparameterized quasi-geodesic in $\mathcal{C}(X)$. Note that $d_X(\sigma_i, \tau_i|X)$ is uniformly bounded because σ_i is a sub-track.

Since $\alpha \prec \tau_i|X$, the curve α is also carried by τ_i. By Lemma 2.9, there is a vertex cycle $\beta_i \prec \tau_i$ that cuts X. Since β_i is wide (Lemma 2.8), any element $\beta_i' \in \pi_X(\beta)$ is carried by and wide in τ_i^X. It follows that β_i' and the vertex cycles of $\tau_i|X$ have bounded intersection. Thus $d_X(\tau_i, \tau_i|X)$ is uniformly bounded, and we are done. □

6. Further applications of the structure theorem

We now turn to Theorems 6.1 and 6.2; both are slight generalizations of a result of Hamenstädt [10, Corollary 3]. Our proofs, however, rely on Theorem 5.3 and are quite different from the proof found in [10].

6.1. The marking and train-track graphs

Suppose that S is not an annulus. A finite subset $\mu \subset \mathcal{AC}(S)$ *fills* S if for all $\beta \in \mathcal{C}(S)$ there is a $\gamma \in \mu$ such that $i(\beta, \gamma) \neq 0$. If $\mu, \nu \subset \mathcal{AC}(S)$, then we define

$$i(\mu, \nu) = \sum_{\alpha \in \mu, \beta \in \nu} i(\alpha, \beta).$$

Also, let $i(\mu) = i(\mu, \mu)$ be the self-intersection number. A set μ is a *k-marking* if μ fills S and $i(\mu) \leq k$. Two sets μ, ν are *ℓ-close* if $i(\mu, \nu) \leq \ell$.

Define $k_0 = \max_\tau i(V(\tau))$, where τ ranges over tracks with vertex cycles $V(\tau)$ filling S. Define $\ell_0 = \max_{\tau,\sigma} i(V(\tau), V(\sigma))$, where σ ranges over tracks obtained from τ by a single splitting. Referring to [16] for the necessary definitions, we define $k_1 = \max_\mu i(\mu)$, where μ ranges over *complete clean markings* of S. Define $\ell_1 = \max_{\mu,\nu} i(\mu, \nu)$, where ν ranges over markings obtained from μ by a single *elementary move*. Define $\ell_2 = \max_\tau \min_\mu i(V(\tau), \mu)$.

Note that there are only finitely many tracks τ and finitely many complete clean markings μ, up to the action of $\mathcal{MCG}(S)$. Since $|\mathcal{B}(\tau)| \leq 6 \cdot \xi(S)$, the number of splittings of τ is also bounded. Lemma 2.4 of [16] bounds the number of elementary moves for μ. Thus the quantities k_0, k_1, ℓ_0, ℓ_1 are well defined. An upper bound for ℓ_2 can be obtained by surgering $V(\tau)$ to obtain a complete clean marking (see the discussion preceding [1, Lemma 6.1]). Now define $k = \max\{k_0, k_1\}$ and $\ell = \max\{\ell_0, \ell_1, \ell_2\}$. Define $\mathcal{M}(S)$ to be the *marking graph*: the vertices are k-markings and the edges are given by ℓ-closeness. (When S is an annulus, we take $\mathcal{M}(S) = \mathcal{A}(S)$. Recall that $\mathcal{A}(S)$ is quasi-isometric to $\mathcal{MCG}(S, \partial) \cong \mathbb{Z}$.)

That $\mathcal{M}(S)$ is connected now follows from the discussion at the beginning of [16, Section 6.4]. Accordingly, define $d_{\mathcal{M}(S)}(\mu, \nu)$ to be the length of the shortest edge-path between the markings μ and ν.

Since the above definitions are stated in terms of geometric intersection number, the mapping class group $\mathcal{MCG}(S)$ acts via isometry on $\mathcal{M}(S)$. Counting the appropriate set of ribbon graphs proves that the action has finitely many orbits of vertices and edges. The Alexander method (see [6, Section 2.4]) proves that vertex stabilizers are finite and hence the action is proper. It now follows from the Milnor–Švarc lemma (see [2, Proposition I.8.19]) that any Cayley graph for $\mathcal{MCG}(S)$ is quasi-isometric to $\mathcal{M}(S)$.

Define $\mathcal{T}(S)$, the *train-track graph*, as follows: vertices are isotopy classes of birecurrent train tracks $\tau \subset S$ such that $V(\tau)$ fills S. Connect vertices τ and σ by an edge exactly when σ is a slide or split of τ. Let $d_{\mathcal{T}(S)}(\tau, \upsilon)$ be the minimal number of edges in a path in $\mathcal{T}(S)$ connecting τ to υ, if such a path exists. Note that the map $\tau \mapsto V(\tau)$ from $\mathcal{T}(S)$ to $\mathcal{M}(S)$ sends edges to edges (or to vertices) and thus is distance nonincreasing. (For further discussion of graphs tightly related to $\mathcal{T}(S)$, see [10].)

We adopt the following notation. If $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence in $\mathcal{T}(S)$ and if $I = [p,q] \subset [0,N]$ is a subinterval, then $|I| = q - p$ and $d_{\mathcal{T}(S)}(I) = d_{\mathcal{T}(S)}(\tau_p, \tau_q)$. If $\tau, \sigma \in \mathcal{T}(S)$, then define

$$d_{\mathcal{M}(X)}(\tau,\sigma) = d_{\mathcal{M}(X)}\big(V(\tau|X), V(\sigma|X)\big).$$

Also take $d_{\mathcal{M}(X)}(I) = d_{\mathcal{M}(X)}(\tau_p, \tau_q)$.

THEOREM 6.1
For any surface S with $\xi(S) \geq 1$, there is a constant $Q = Q(S)$ with the following property. Suppose that $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence in $\mathcal{T}(S)$. Then the sequence $\{V(\tau_i)\}_{i=0}^N$, as parameterized by splittings, is a Q-quasi-geodesic in the marking graph.

Before proving this, we give our final generalization of [10, Corollary 3], which follows from Theorem 6.1.

THEOREM 6.2
For any surface S with $\xi(S) \geq 1$, there is a constant $Q = Q(S)$ with the following property. If $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence in $\mathcal{T}(S)$, injective on slide subsequences, then $\{\tau_i\}$ is a Q-quasi-geodesic.

Notice that here, unlike Theorem 6.1, the parameterization is by index. In the proof we use the following.

LEMMA 6.3
There is a constant $A = A(S)$ such that if $\{\tau_i\}_{i=1}^N$ is an injective sliding sequence in $\mathcal{T}(S)$, then $N + 1 \leq A$. □

Proof of Theorem 6.2
Let $\{\tau_i\}$ be the given sliding and splitting sequence in $\mathcal{T}(S)$. Let $I = [p, q] \subset [0, N]$ be a subinterval. Note that $d_{\mathcal{T}(S)}(I) \leq |I|$ because $\{\tau_i\}$ is an edge-path in $\mathcal{T}(S)$.

Define $\mathcal{S}(I)$ to be the set of indices $r \in I$, where $r + 1 \in I$ and τ_{r+1} is a splitting of τ_r. Thus $|I| \leq_A |\mathcal{S}(I)|$, where A is the constant of Lemma 6.3. Now, Theorem 6.1 implies that $|\mathcal{S}(I)| \leq_Q d_{\mathcal{M}(S)}(I)$. Finally, since $\tau \mapsto V(\tau)$ is distance nonincreasing, we have $d_{\mathcal{M}(S)}(I) \leq d_{\mathcal{T}(S)}(I)$. Deduce that $|I| \leq_Q d_{\mathcal{M}(S)}(I)$, for a somewhat larger value of Q. □

Remark 6.4
Note that we have not used the connectedness of $\mathcal{T}(S)$, an issue that appears to be difficult to approach combinatorially. For a proof of connectedness, see [10, Corollary 3.7].

6.2. Hyperbolicity and the distance estimate
To prove Theorem 6.1, we will need the following.

THEOREM 6.5 ([15, Theorem 1.1])
For every connected compact orientable surface X, there is a constant δ_X such that $\mathcal{C}(X)$ is δ_X-hyperbolic. □

An important consequence of the Morse lemma (see [2, Theorem III.H.1.7]) is a reverse triangle inequality.

LEMMA 6.6
For every δ and Q, there is a constant $\mathsf{R}_0 = \mathsf{R}_0(\delta, Q)$ with the following property. For any δ-hyperbolic space $\mathcal{X}$, for any Q-unparameterized quasi-geodesic $f : [m, n] \to \mathcal{X}$, and for any $a, b, c \in [m, n]$, if $a \leq b \leq c$, then

$$d_{\mathcal{X}}(\alpha, \beta) + d_{\mathcal{X}}(\beta, \gamma) \leq d_{\mathcal{X}}(\alpha, \gamma) + \mathsf{R}_0,$$

where $\alpha, \beta, \gamma = f(a), f(b), f(c)$. □

We now take $\mathsf{R}_0 = \mathsf{R}_0(\delta, Q(S))$, where $\delta = \max\{\delta_X \mid X \subset S\}$, as provided by Theorem 6.5, and where $Q(S)$ is the constant of Theorem 5.5. The next central result needed is the *distance estimate* for $\mathcal{M}(S)$. Let $[\cdot]_C$ be the *cutoff function*

$$[x]_C = \begin{Bmatrix} 0, & \text{if } x < C \\ x, & \text{if } x \geq C \end{Bmatrix}.$$

We may now state the distance estimate.

THEOREM 6.7 ([16, Theorem 6.12])
For any surface S, there is a constant $\mathsf{C}(S)$ such that for every $C \geq \mathsf{C}(S)$ there is an $\mathsf{E} \geq 1$ and such that for all $\mu, \nu \in \mathcal{M}(S)$,

$$d_{\mathcal{M}(S)}(\mu, \nu) =_{\mathsf{E}} \sum [d_X(\mu, \nu)]_C,$$

where the sum ranges over essential subsurfaces $X \subset S$. □

6.3. Marking distance
Suppose that $\{\tau_i\}_{i=0}^N \subset \mathcal{T}(S)$ is a sliding and splitting sequence. Let $V_i = V(\tau_i)$ be the set of vertex cycles of τ_i. As $i(V_i) \leq k_0$ and $i(V_i, V_{i+1}) \leq \ell_0$, the map $i \mapsto V_i$ gives rise to an edge-path in $\mathcal{M}(S)$.

Suppose that $[p,q] \subset [0, N]$. Let $\mathcal{S}_X(p,q)$ be the set of indices $r \in [p, q-1]$ so that $\tau_{r+1}|X$ is a splitting of $\tau_r|X$. (When X is an annulus, $\tau_{r+1}|X$ may also differ from $\tau_r|X$ by a pair of splits.)

Remark 6.8
We do not place indices r onto S_X, where $\tau_{r+1}|X$ is a subtrack of $\tau_r|X$; the number of such indices is bounded by a constant depending only on X.

Recall that we omit the subscript from $\mathcal{S}_X$ when $X = S$. As a piece of notation, set $I_S = [0, N]$. When $X \subset S$ is essential, take $V(\tau|X)$ to be the vertex cycles of the induced track. Recall that $I_X \subset I_S$, defined in Section 5.2, is the accessible interval for $X \subset S$. If $J = [m,n] \subset [0, N]$ is an interval, then define $\mathcal{S}_X(J) = \mathcal{S}_X(m,n)$, $d_X(J) = d_X(\tau_m, \tau_n)$, and so on. Here is the crucial tool for the proof of Theorem 6.1.

PROPOSITION 6.9
Suppose that $X \subset S$ is an essential subsurface and that $J_X \subset I_X$ is a subinterval. There is a constant $A = A(X)$, independent of the sequence $\{\tau_i\}$, such that $|\mathcal{S}_X(J_X)| \leq_A d_{\mathcal{M}(X)}(J_X)$.

Proof of Theorem 6.1
Suppose that $\{\tau_i\}_{i=0}^N$ is a sliding and splitting sequence. Define $V_i = V(\tau_i)$. By our definition of $\mathcal{M}(S)$, and since slides do not effect $P(\tau)$ (see [21, Proposition 2.2.2]), the map $i \mapsto V_i$ gives an edge-path in $\mathcal{M}(S)$. Thus $d_{\mathcal{M}}(V_i, V_j) \leq |\mathcal{S}(i,j)|$. The oppo-

site inequality, with multiplicative and additive error at most $A(S)$, follows from Proposition 6.9. □

The remainder of this paper gives the proof of Proposition 6.9. We begin with a sketch.

We induct on the complexity of the subsurface $X \subset S$. Partition the interval J_X into two kinds of subintervals. The first are *inductive* intervals. They arise from proper subsurfaces $Y \subset X$ for which the projection $d_Y(J_X)$ is above a given threshold. The second are *straight* intervals: those for which the diameter of the projection to all proper subsurfaces is below another threshold.

Lemma 6.12 uses the structure theorem (Theorem 5.3) to show that intervals disjoint from the inductive intervals are straight. The main technical step of the proof of Proposition 6.9 is Lemma 6.13: the number of splittings in a straight interval $I \subset J_X$ is bounded by the diameter $d_X(I)$, measured in the curve complex of X. This again uses our structure theorem and also the distance estimate (Theorem 6.7), the latter saying that if all strict subsurface projections are small, then marking distance is quasi-equal to distance in $\mathcal{C}(X)$.

We then divide the straight intervals into the *long* and the *short*: those which are longer than a threshold defined by the reverse triangle inequality (Lemma 6.6) and those which are shorter. Lemma 6.18 bounds the total number of splittings in long straight intervals by $d_X(J_X)$. The sum in short straight intervals is bounded by the number of inductive intervals (Lemma 6.19). In Lemma 6.22, we apply the inductive hypothesis provided by Proposition 6.9 to prove that the number of splittings in all inductive intervals is bounded by a sum of marking distances. The distance estimate, in turn, implies that the sum of marking distances is bounded by a sum of subsurface projections.

Adding these estimates, on the number of splittings in straight intervals and in inductive intervals, produces the desired bound of Proposition 6.9.

6.4. *Inductive and straight intervals*

We fix two thresholds $\mathsf{T}_0, \mathsf{T}_1$ so that

$$\max\{6\mathsf{N}_1 + 2\mathsf{N}_2 + 2\mathsf{K}_0(X) + 2, 2\mathsf{R}_0, M_2(X), \mathsf{C}(X)\} \le \mathsf{T}_0(X),$$
$$\max\{\mathsf{T}_0(X) + 2\mathsf{R}_0, \mathsf{B}_0\mathsf{N}_2\} \le \mathsf{T}_1(X).$$

Here N_1 is an upper bound for $d_Y(\alpha, \beta)$, where $Y \subset S$ is any essential subsurface, τ is a track, and α and β are wide with respect to τ. The constant B_0 is an upper bound for the number of branches in any induced track. The constant N_2 is an upper bound for the distance (in any subsurface projection) between the vertices of

τ (or $\tau|X$) and the vertices of a single splitting or subtrack of τ. Also, $M_2(X)$ is the constant provided by [16, Lemma 6.1].

Recall that the interval $J_X \subset I_X$ is given in Proposition 6.9.

Definition 6.10
Suppose that $Y \subset X$ is an essential subsurface with $\xi(Y) < \xi(X)$. If $d_Y(J_X) \geq \mathsf{T}_0(X)$, then we call Y an *inductive* subsurface of X, and we take $J_Y = I_Y \cap J_X$ as the associated *inductive subinterval* of J_X. If $d_Y(J_X) < \mathsf{T}_0(X)$, then we set $J_Y = \emptyset$.

Suppose that I is a subinterval of J_X. Define $\operatorname{diam}_Y(I)$ to be the diameter, in $\mathcal{AC}(Y)$, of the union $\bigcup_{i \in I} \pi_Y(\tau_i)$.

Definition 6.11
A subinterval $I \subset J_X$ is a *straight subinterval* for X if for all essential subsurfaces $Y \subset X$, with $\xi(Y) < \xi(X)$, we have $\operatorname{diam}_Y(I) \leq \mathsf{T}_1(X)$.

LEMMA 6.12
If $I \subset J_X$ is disjoint from all inductive subintervals of J_X, then I is straight for X.

Proof
Fix an essential $Y \subset X$ with $\xi(Y) < \xi(X)$. It suffices to show, for every subinterval $J \subset I$, that $d_Y(J) \leq \mathsf{T}_1(X)$.

If $J \cap I_Y = \emptyset$, then Theorem 5.3 implies that $d_Y(J) \leq \mathsf{K}_0$. Suppose that J meets I_Y; thus $J_Y = \emptyset$ by hypothesis, and so Y is not inductive. It follows that $d_Y(I) < \mathsf{T}_0(X)$. By Lemma 6.6, we have $d_Y(J) < \mathsf{T}_0(X) + 2\mathsf{R}_0$. □

LEMMA 6.13
There is a constant $A = A(X)$, independent of $\{\tau_i\}$, such that if $I \subset J_X$ is straight, then $|\mathcal{S}_X(I)| \leq_A d_X(I)$.

Proof
If X is an annulus, then by Theorem 5.3, for every $r \in I$, the core curve $\alpha \subset X$ is carried by and wide in τ_r. It follows that the number of switches in $\alpha \subset \tau_r|X$ is bounded by some constant $K = K(S)$. Let $q = \max I$, and pick any $\beta \in V(\tau_q|X)$. As in the proof of Theorem 5.5, let $\sigma_r \subset \tau_r|X$ be the minimal subtrack carrying β. Thus σ_r has either exactly four branches and two switches, or is an embedded arc. It follows that every $K^2/4$ consecutive splitting in $S_X(I)$ induces at least one splitting in the sequence of tracks $\{\sigma_r\}$. Therefore, the singleton sets $V(\sigma_r)$ form a quasi-geodesic in $\mathcal{A}(X)$. Since $V(\sigma_r) \subset V(\tau_r|X)$, the proof is complete when X is an annulus.

We assume for the rest of the proof that X is not an annulus. The map $i \mapsto V(\tau_i|X)$, taking tracks to their vertex cycles, is generally not injective (see, e.g., [21, Proposition 2.2.2]). However, we do have the following.

CLAIM 6.14
There is a constant $\mathsf{N}_0 = \mathsf{N}_0(X)$, *independent of* $\{\tau_i\}$, *such that if* $V(\tau_r|X) = V(\tau_s|X)$, *then* $|\mathcal{S}_X(r,s)| \le \mathsf{N}_0$.

Proof
Let $\mu = V(\tau_r|X)$. Our hypothesis on $\tau_s|X$ and induction proves that $V(\tau_t|X) = \mu$ for all $t \in [r,s]$. Recurrence and uniqueness of carrying (see [20, Proposition 3.7.3]) implies that $\tau_{t+1}|X$ is a split or a slide of $\tau_t|X$, and not a subtrack, for all $t \in [r, s-1]$.

If $t \in [r,s]$ and $b \in \mathcal{B}(\tau_t|X)$, then define $w_\mu(b) = \sum_{\alpha\in\mu} w_\alpha(b)$. Let

$$M(t) = \big(w_\mu(b) : b \text{ is a large branch of } \tau_t|X\big)$$

be the sequence of given numbers arranged in nonincreasing order. Note that if $\tau_{t+1}|X$ is a slide of $\tau_t|X$, then $M(t+1) = M(t)$. However, if $t \in \mathcal{S}_X(r,s)$, then the recurrence of $\tau_t|X$ implies that $M(t+1) < M(t)$, in lexicographic order. Since there are only finitely many possibilities for an induced track $\tau|X$, up to the action of $\mathcal{MCG}(X)$, the claim follows. □

Notice that if $V(\tau_{i+1}|X) \ne V(\tau_i|X)$, then $V(\tau_{i+1}|X) \ne V(\tau_j|X)$ for $j \le i$. This is because $P(\tau_{k+1}|X) \subset P(\tau_k|X)$ for all k. Using $C = 1 + \max\{\mathsf{C}(X), \mathsf{T}_1(X)\}$ as the cutoff in Theorem 6.7 gives some constant of quasi-equality, say, E. Define $\mathsf{R}_1 = \mathsf{E} + 1$.

Suppose that $[p,q] = I$, the straight subinterval of J_X given by Lemma 6.13. We define a function $\rho\colon [0,M] \to I$ as follows. Let $\rho(0) = p$, and let $\rho(n+1)$ be the smallest element in $[\rho(n), q]$ with $d_{\mathcal{M}(X)}(\tau_{\rho(n)}, \tau_{\rho(n+1)}) = \mathsf{R}_1$. (If $\rho(n+1)$ is undefined, then take $M = n+1$ and $\rho(M) = q$.) Let $B(\mu)$ be the ball of radius R_1 about the marking $\mu \in \mathcal{M}(X)$. Define

$$\mathsf{V} = \max\big\{|B(\mu)| : \mu \in \mathcal{M}(X)\big\}.$$

Deduce from Claim 6.14 that, for all $n \in [0, M-1]$,

$$\big|\mathcal{S}_X\big(\rho(n), \rho(n+1)\big)\big| \le \mathsf{N}_0\mathsf{V}.$$

Thus

$$|\mathcal{S}_X(I)| \le \mathsf{N}_0\mathsf{V} \cdot M.$$

So to prove Lemma 6.13, it suffices to bound M from above in terms of $d_X(I)$.

CLAIM 6.15

Fix $n \in [0, M-2]$. *Let* $\tau, \sigma = \tau_{\rho(n)}, \tau_{\rho(n+1)}$. *Then* $d_X(\tau, \sigma) \geq \mathsf{R}_0 + 1$.

Proof

We use Theorem 6.7. Note that $d_{\mathcal{M}(X)}(\tau, \sigma) = \mathsf{R}_1$. Since R_1 is greater than the additive error, there is at least one nonvanishing term in the sum $\sum_{Y \subset X} [d_Y(\tau, \sigma)]_C$.

However, since $[\rho(n), \rho(n+1)] \subset [p, q]$ and since $[p, q] = I$ is straight, we have $d_Y(\tau, \sigma) \leq \mathsf{T}_1(X)$ for all $Y \subset X$ with $\xi(Y) < \xi(X)$. Thus $d_X(\tau, \sigma)$ is the only term of the sum greater than the cutoff C. Since $C > \mathsf{T}_1(X) \geq \mathsf{R}_0$, we have $d_X(\tau, \sigma) \geq \mathsf{R}_0 + 1$, and the claim is proved. □

Thus we have

$$\begin{aligned} d_X(I) &\geq -(M-1) \cdot \mathsf{R}_0 + \sum_{n=0}^{M-1} d_X(\tau_{\rho(n)}, \tau_{\rho(n+1)}) \\ &\geq M - 1 + d_X(\tau_{\rho(M-1)}, \tau_{\rho(M)}) \\ &\geq M - 1, \end{aligned}$$

where the first and second lines follow from Lemma 6.6 and Claim 6.15, respectively. This completes the proof of Lemma 6.13. □

LEMMA 6.16

There is a constant $A = A(X)$ *with the following property. Suppose that* $J_Y \subset J_X$ *is an inductive interval. Suppose that* $I \subset J_Y$ *is a straight subinterval for* X. *Then* $|\mathcal{S}_X(I)| \leq A$.

Proof

Let $[p, q] = I$. Applying Theorem 5.3, as $p \in J_Y \subset I_Y$, the multicurve ∂Y is wide with respect to τ_p. It follows that ∂Y is also wide with respect to τ_p^X. Note that the curves of $V(\tau_p | X)$ are also wide with respect to τ_p^X. Repeating this discussion for q, and then applying Lemma 5.2 and the triangle inequality gives a uniform bound for $d_X(\tau_p, \tau_q)$. The lemma now follows from Lemma 6.13. □

6.5. *Long and short intervals*

Definition 6.17

A straight subinterval I for X is *short* if $d_X(I) \leq 4\mathsf{R}_0$. Otherwise I is *long*.

By Lemma 6.13, if I is a short straight interval, then $|\mathcal{S}_X(I)|$ is uniformly bounded by a constant depending only on X.

Let Ind be the set of inductive subsurfaces $Y \subset X$. Define $\mathsf{Ind}' = \mathsf{Ind} \cup \{X\}$. Note that, by Lemma 6.12, every maximal subinterval of $J_X - \bigcup_{Y \in \mathsf{Ind}} J_Y$ is straight. We partition these maximal subintervals into the sets Long and Short as the given interval is long or short, respectively.

LEMMA 6.18
There is a constant $A = A(X)$, independent of $\{\tau_i\}$, such that

$$\sum_{I \in \mathsf{Long}} |\mathcal{S}_X(I)| \le_A d_X(J_X).$$

Proof
From Lemma 6.13, we deduce that

$$\sum_{I \in \mathsf{Long}} |\mathcal{S}_X(I)| \le_A |\mathsf{Long}| + \sum_{I \in \mathsf{Long}} d_X(I),$$

where the first term on the right-hand side arises from addition of additive errors. By the definition of a long straight interval and from Lemma 6.6, we deduce that

$$4\mathsf{R}_0|\mathsf{Long}| \le \sum_{I \in \mathsf{Long}} d_X(I) \le d_X(J_X) + 2\mathsf{R}_0|\mathsf{Long}|.$$

Thus $2\mathsf{R}_0|\mathsf{Long}| \le d_X(J_X)$. These inequalities combine to prove the lemma, for a somewhat larger value of $A = A(X)$. □

LEMMA 6.19
There is a constant $A = A(X)$, independent of $\{\tau_i\}$, such that

$$\sum_{I \in \mathsf{Short}} |\mathcal{S}_X(I)| \le_A |\mathsf{Ind}'|.$$

Proof
By Lemma 6.13, the number of splittings in any short straight interval is a priori bounded (depending only on X). Since $|\mathsf{Short}| \le |\mathsf{Ind}'|$, the lemma follows. □

LEMMA 6.20
If $Z \in \mathsf{Ind}$, then

$$\operatorname{card}\{Y \in \mathsf{Ind} \mid Z \subset Y, \xi(Z) < \xi(Y)\} \le 2(\xi(X) - \xi(Z) - 1).$$

This follows from and is strictly weaker than [16, Theorem 4.7, Lemma 6.1]. We give a proof, using our structure theorem, to extract the necessary lower bound for $\mathsf{T}_0(X)$.

Proof of Lemma 6.20

Suppose that $U \in \mathsf{Ind}$ contains Z. Suppose that $J_Z = [p,q]$ and that $J_X = [m,n]$. Thus ∂Z is wide with respect to τ_p. So $d_U(\tau_p, \partial Z) \le \mathsf{N}_1$, by the definition of N_1, and the same holds at the index q. Thus $d_U(\tau_p, \tau_q) \le 2\mathsf{N}_1$. The subsurface U *precedes* or *succeeds* Z if $d_U(\tau_m, \tau_p)$ or $d_U(\tau_q, \tau_n)$, respectively, is greater than or equal to $2\mathsf{N}_1 + \mathsf{N}_2 + \mathsf{K}_0(X) + 1$. Note that U must precede or succeed Z (or both) as otherwise $d_U(\tau_m, \tau_n) < 6\mathsf{N}_1 + 2\mathsf{N}_2 + 2\mathsf{K}_0(X) + 2 \le \mathsf{T}_0(X)$, a contradiction.

It now suffices to consider subsurfaces U and V that both succeed and both contain Z. If $\max J_U \le \max J_V$, then $U \subset V$, for, if not, ∂V cuts U while missing Z. Since ∂V is wide at the index $r = \max J_V$, we deduce that

$$\begin{aligned} d_U(\tau_q, \tau_n) &\le d_U(\tau_q, \partial Z) + d_U(\partial Z, \partial V) + d_U(\partial V, \tau_r) \\ &\quad + d_U(\tau_r, \tau_{r+1}) + d_U(\tau_{r+1}, \tau_n) \\ &\le 2\mathsf{N}_1 + \mathsf{N}_2 + \mathsf{K}_0(X) + 1, \end{aligned}$$

and this is a contradiction. Thus the surfaces in Ind that strictly contain Z, and succeed Z, are nested. □

Definition 6.21

Assign an index $r \in \mathcal{S}_X(J_X)$ to a subsurface $Y \subset X$ if $Y \in \mathsf{Ind}'$, if $r \in J_Y$, if $\tau_{r+1}|Y$ is a splitting of $\tau_r|Y$, and if there is no subsurface $Z \subset Y, \xi(Z) < \xi(Y)$ with those three properties.

LEMMA 6.22

There is a constant $A = A(X)$, independent of $\{\tau_i\}$, such that the number of splittings contained in inductive intervals is quasi-bounded by $|\mathsf{Ind}| + \sum_{Y \in \mathsf{Ind}} d_Y(J_X)$.

Proof

Fix $Y \in \mathsf{Ind}$. Consider an index $r \in J_Y$ that is assigned to X. Let $I \subset J_Y$ be the maximal interval containing r so that all indices in $\mathcal{S}_X(I)$ are assigned to X. We now show that I is straight. Let Z be any essential subsurface of X with $\xi(Z) < \xi(X)$, and let $[r,s] = J \subset I$ be any subinterval. If $J \cap I_Z = \emptyset$, then Theorem 5.3 implies that $d_Z(J) \le \mathsf{K}_0$. If J meets I_Z, then, as no splittings of J are assigned to Z, we deduce that $\tau_s|Z$ is obtained from $\tau_r|Z$ by sliding and taking subtracks only. Thus $d_Z(J) \le \mathsf{B}_0\mathsf{N}_2 \le \mathsf{T}_1(X)$, as desired.

By Lemma 6.16, we find that $|\mathcal{S}_X(I)|$ is bounded. It follows that the number of splittings in the inductive intervals is quasi-bounded by $\sum_{Y \in \mathsf{Ind}} |\mathcal{S}_Y(J_Y)|$.

By induction, Proposition 6.9 gives

$$|\mathcal{S}_Y(J_Y)| \le_A d_{\mathcal{M}(Y)}(J_Y).$$

Taking a cutoff of $C = 1 + \max\{\mathsf{C}(Y), \mathsf{T}_0(X) + 2\mathsf{R}_0\}$ and applying the distance estimate (Theorem 6.7), we have a quasi-inequality

$$d_{\mathcal{M}(Y)}(J_Y) \preceq_{\mathsf{E}} \sum_{Z \subset Y} [d_Z(J_Y)]_C.$$

Since $d_Z(J_Y) \le d_Z(J_X) + 2\mathsf{R}_0$ for all $Z \subset Y$, it follows that nonzero terms in the sum only arise for subsurfaces in $\mathsf{Ind}'(Y) = \{Z \in \mathsf{Ind}' \mid Z \subset Y\}$. Since $2\mathsf{R}_0 \le \mathsf{T}_0(X)$, we have $[d_Z(J_Y)]_C \le 2 \cdot d_Z(J_X)$. Making $A = A(X)$ larger if necessary, we have

$$|\mathcal{S}_Y(J_Y)| \preceq_A \sum_{Z \in \mathsf{Ind}'(Y)} d_Z(J_X).$$

Thus

$$\begin{aligned} \sum_{Y \in \mathsf{Ind}} |\mathcal{S}_Y(J_Y)| &\preceq_A |\mathsf{Ind}| + \sum_{Y \in \mathsf{Ind}} \sum_{Z \in \mathsf{Ind}'(Y)} d_Z(J_X) \\ &\preceq_A |\mathsf{Ind}| + \sum_{Y \in \mathsf{Ind}} d_Y(J_X), \end{aligned}$$

where the final quasi-inequality follows from Lemma 6.20, taking A larger as necessary. Note that the term $|\mathsf{Ind}|$ on the middle line arises by adding additive errors. This proves Lemma 6.22. □

Since every index in $\mathcal{S}_X(J_X)$ is either in a long or short straight interval or in an inductive interval, from Lemmas 6.18, 6.19, and 6.22, and by increasing A slightly, we have

$$|\mathcal{S}_X(J_X)| \preceq_A d_X(J_X) + |\mathsf{Ind}'| + \sum_{Y \in \mathsf{Ind}} d_Y(J_X).$$

Note that $|\mathsf{Ind}'| \preceq_A d_{\mathcal{M}(X)}(J_X)$; this follows from the hierarchy machine (in particular, [16, Lemma 6.2, Theorem 6.10]) and because $\mathsf{T}_0(X) \ge M_2(X)$, the constant in [16, Lemma 6.1]. Finally,

$$\sum_{Y \in \mathsf{Ind}'} d_Y(J_X) \preceq_A d_{\mathcal{M}(X)}(J_X)$$

follows from the distance estimate (Theorem 6.7) and because $\mathsf{T}_0(X) \ge \mathsf{C}(X)$. This completes the proof of Proposition 6.9. □

Acknowledgment. We thank Yair Minsky for enlightening conversations.

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