THE DISJOINT ANNULUS PROPERTY

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Abstract. A Heegaard splitting of a closed, orientable three-manifold satisfies the *Disjoint Annulus Property* if each handlebody contains an essential annulus and these are disjoint. This paper proves that, for a fixed three-manifold, all but finitely many splittings have the disjoint annulus property. As a corollary, all but finitely many splittings have *distance* three or less, as defined by Hempel.

1. History and overview

Great effort has been spent on the classification problem for Heegaard splittings of three-manifolds. Haken's lemma [2], that all splittings of a reducible manifold are themselves reducible, could be considered one of the first results in this direction. *Weak reducibility* was introduced by Casson and Gordon [1] as a generalization of reducibility. They concluded that a weakly reducible splitting is either itself reducible or the manifold in question contains an incompressible surface. Thompson [16] later defined the *disjoint curve property* as a further generalization of weak reducibility. She deduced that all splittings of a toroidal three-manifold have the disjoint curve property.

Hempel [5] generalized these ideas to obtain the *distance* of a splitting, defined in terms of the curve complex. He then adapted an argument of Kobayashi [9] to produce examples of splittings of arbitrarily large distance. Hartshorn [3], also following the ideas of [9], proved that Hempel's distance is bounded by twice the genus of any incompressible surface embedded in the given manifold.

Here we introduce the *twin annulus property* (TAP) for Heegaard splittings as well as the weaker notion of the *disjoint annulus property* (DAP). As discussed below, a splitting has the disjoint annulus property if each handlebody contains an essential annulus and these are disjoint. The TAP is essentially found in [16] while the DAP arises naturally in Kobayashi's discussion of the *strong rectangle condition* in [8].

Next, we apply the ideas of strong irreducibility and normal surface theory as in [15] and [12]. This gives Theorem 6.3: every splitting in M, of sufficiently large genus, satisfies the DAP. We next need Jaco and Rubinstein's solution to the Waldhausen conjecture: If M is closed, orientable, and atoroidal then there are only finitely many strongly irreducible splittings, up to isotopy, in each genus. This yields:

Theorem 6.5. *In any closed orientable three-manifold there are only finitely many Heegaard splittings, up to isotopy, which do not satisfy the disjoint annulus property.*

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As a corollary there are only finitely many splittings, up to isotopy in the given manifold, which have distance greater than three in the sense of [5]. This result is not sharp, as shown by the Casson-Gordon-Parris examples:

Fact. There is a closed Haken three-manifold containing infinitely many, pairwise non-isotopic, distance two splittings. (See [3].)

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2. Heegaard splittings and the DAP

This section recalls standard notation and reviews several notions from the theory of Heegaard splttings. (For an excellent survey of the subject, see [13].) Also, the twin and disjoint annulus properties are introduced.

In this paper, M will be used to denote a closed, orientable three-manifold. A *handlebody* is a compact three-manifold which is homeomorphic to a closed regular neighborhood of a finite, connected graph embedded in $\mathbb{R}^3$. A *Heegaard splitting* (or simply a *splitting*), $H \subset M$, is a closed, embedded, orientable surface with complement a disjoint pair of handlebodies, V and W. The *genus* of the splitting is the genus of H, $g(H) = 1 - \chi(V)$. A properly embedded disk $D \subset V$ is *essential* if ∂D is essential inside $H = \partial V$. Recall that a splitting H is *reducible* if there are essential disks, $D \subset V$ and $E \subset W$, such that $\partial D = \partial E$.

A strict parallel can now be drawn, using annuli instead of disks.

Definition. A properly embedded annulus $A \subset V$ is *essential* if A is incompressible and not boundary parallel.

Thus $\partial A = \partial_+ A \cup \partial_- A$ is essential in H. An analogue of reducibility is the following:

Definition. A Heegaard splitting has the *twin annulus property* if there are essential annuli, $A \subset V$ and $B \subset W$, such that $\partial_+ A = \partial_- B$ while $\partial_- A \cap \partial_+ B = \emptyset$.

Remark 2.1. Suppose that $H \subset M$ has the TAP. Boundary-compressing the given annuli reveals that H also satisfies Thompson's *disjoint curve property*, as defined in [16]. Both notions are essentially equivalent to the splitting having distance two or less, in the sense of [5].

Returning to essential disks [1] gives the following fruitful notion:

Definition. A splitting $H \subset M$ is *weakly reducible* if there are essential disks, $D \subset V$ and $E \subset W$, such that $\partial D \cap \partial E = \emptyset$. If H is not weakly reducible then it is *strongly irreducible*.

The appendix then proves:

Lemma 8.3. *If H is weakly reducible then H satisfies the twin annulus property.*

We will also require the following lemma, essentially due to Kobayashi [8]:

Lemma 2.2. *If M is toroidal then every strongly irreducible splitting $H \subset M$ has the twin annulus property.*

There is an analogue of weak reducibility for annuli:

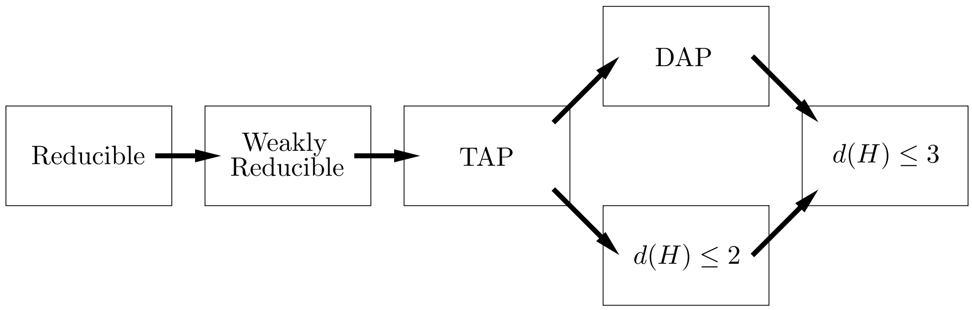


FIGURE 1. Implications

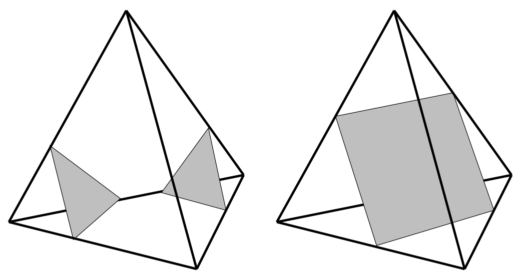


FIGURE 2. Normal disks

Definition. A Heegaard splitting has the *disjoint annulus property* if there are essential annuli, $A \subset V$ and $B \subset W$, such that $\partial A \cap \partial B = \emptyset$.

Note that if $H \subset M$ has the TAP then H also has the DAP, by pushing $\partial_+ A$, in H, to be disjoint from and parallel to $\partial_- B$.

Remark 2.3. If $H \subset M$ satisfies the DAP then H has distance three or less, in the sense of [5]. To see this, note that an essential annulus boundary-compresses to give a disjoint essential disk. See Figure 1 for a diagram of the relations between these various properties of Heegaard splittings.

3. Normal surface theory and blocks

This section presents the few necessary tools from normal surface theory. For a more complete treatment consult [7] or [4].

Fix a triangulation, T, of N, a compact three-manifold. Denote the i-skeleton of T by T^i. A surface S, properly embedded in N, is *normal* if S is transverse to the skelata of T and intersects each tetrahedron in a collection of *normal disks*. See Figure 2 for pictures of the two kinds of normal disks; the *normal triangle* and the *normal quadrilateral*.

A surface S, properly embedded in N, is *almost normal* if S is transverse to the skelata of T and intersects each tetrahedron but one in a collection of normal disks. In the remaining tetrahedron S yields a collection of normal disks and at most one *almost normal piece*. Two of the five kinds of almost normal pieces are shown in Figure 3.

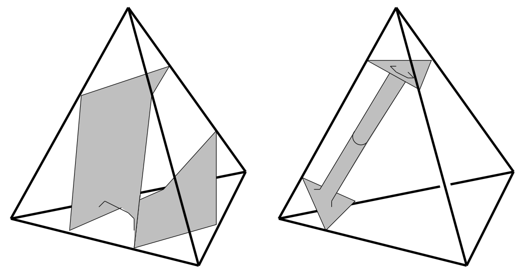


FIGURE 3. Almost normal pieces

Suppose now that $N = \tau$ is a single tetrahedron. Let $S \subset \tau$ be a disjoint collection of normal disks together with at most one almost normal piece.

Definition. A *block*, B, is the closure of a connected component of $\tau \smallsetminus S$. Suppose B is adjacent to exactly two normal disks, D and E. Suppose also that D is properly isotopic, in τ, to E relative to the vertices of τ. Then B is a *product block*. All other blocks are called *core blocks*.

See Figure 4 for pictures of the two product blocks and six of the many possible core blocks. In the figures I have shaded some of the faces — these are the faces which lie in the surface S.

4. Blocked and shrunken submanifolds

This section deals with the submanifolds of a triangulated three-manifold naturally contained in unions of blocks. Let (M, T) be a closed, orientable, triangulated three-manifold.

Definition. A three-dimensional submanifold $X \subset (M, T)$ is *blocked* if ∂X is an almost normal surface and X is a union of blocks.

As a specific example, if $H \subset (M, T)$ is connected, separating, normal surface then the closure of a component of $M \smallsetminus H$ is a blocked submanifold.

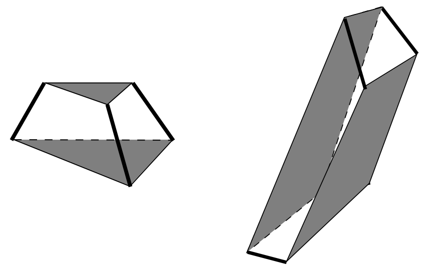
Remark 4.1. Note that a blocked submanifold contains at most $6|T^3| + 2$ core blocks, where $|T^3|$ is the number of tetrahedra in T. To see this, note that a tetrahedron contains at most five parallel families of normal disks. Thus tetrahedra not meeting an almost normal piece contribute at most six core blocks. The almost normal piece (if it exists) gives the final pair of core blocks.

Definition. Let $X \subset (M, T)$ be blocked. A three-dimensional submanifold $Y \subset X$ is *shrunken* if there is a union of blocks $\overline{Y} \subset X$, with

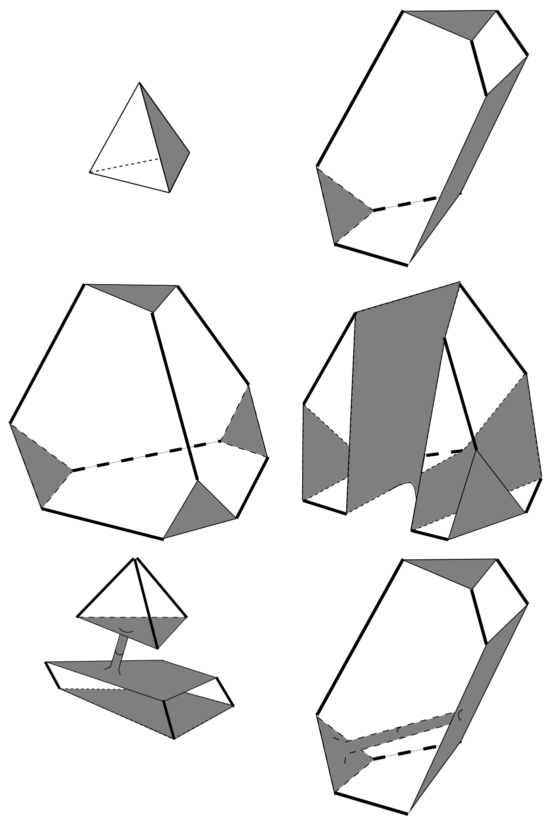
$$Y = \overline{Y} \smallsetminus \eta(\mathrm{fr}(\overline{Y}) \smallsetminus \partial X).$$

Here $\eta(\cdot)$ denotes a regular open neighborhood taken inside of M. Thus the shrunken Y is obtained by removing a neighborhood of $\overline{Y}$'s frontier, taken inside of X. (That is, by *shrinking* $\overline{Y}$.) Note that Y uniquely determines $\overline{Y}$ as well as the reverse.

Let $\partial_h Y = Y \cap \partial X$ denote the *horizontal boundary* of Y. Also, the *vertical boundary* of Y, $\partial_v Y$, equals the closure of $\partial Y \smallsetminus \partial X$. This terminology is prompted by the fact that if $\overline{Y}$ is a union of product blocks then Y forms an I-bundle over a surface. Below, $|\cdot|$ denotes the number of components of a topological space.



The product blocks



A few core blocks

FIGURE 4. Some of the possible blocks

Definition. If Y is a shrunken submanifold then define the *complexity* of Y:

$$c(Y) = \chi(Y) + |\partial_v Y|.$$

This in an exact parallel of the *complexity* of a compact surface G, $c(G) = \chi(G) + |\partial G|$. If Y is in fact an I-bundle then $c(Y)$ agrees with the complexity of Y's base surface. A more delicate measure of complexity, $c'(\cdot)$, is also required:

Definition. Let Y be a shrunken submanifold which is an I-bundle. Then $\widehat{Y}$, the *large part* of Y, equals the union of all components $Z \subset Y$ which have $c(Z) \leq 0$. Now define $c'(Y) = c(\widehat{Y})$.

If $Y = \widehat{Y}$ then we say that Y is *large*. Again, this parallels the notation for surfaces: A compact surface G is large if no component of G has positive complexity. Note that, if Y is a shrunken submanifold which is an I-bundle, then $c'(Y) \leq c(Y)$.

We end this section with a discussion of how to obtain new shrunken submanifolds from old. Let Y be a shrunken submanifold of the blocked X. Let $B \subset X$ be a block which meets the interior of Y. Then $Y \smallsetminus \eta(B)$ is again a shrunken submanifold inside of X. The following lemma records how the complexity of $Y \smallsetminus \eta(B)$ differs from that of Y:

Lemma 4.2. *There is a positive integer a_1 independent of (M, T), X, Y, and B such that*

$$c(Y \smallsetminus \eta(B)) \leq c(Y) + a_1.$$

Proof. There are only finitely many combinatorial types of block. Hence $\chi(Y)$ and $|\partial_v Y|$ can only increase by a bounded amount. □

Remark 4.3. In fact, the change in complexity can be controlled by twice the number of vertical faces of the block B. By listing all possible blocks one finds an easy upper bound, $a_1 \leq 16$. However, we will not prove this as it is not required in the sequel.

5. Shelling high genus splittings

This section deals with the main objects of interest: almost normal Heegaard splittings. Here [15] serves as a reference. A pair of blocked handlebodies and a sequence of shrunken submanifolds will now be derived from each such splitting.

Fix (M, T) a closed, orientable, triangulated three-manifold. Suppose that $H \subset M$ is an almost normal Heegaard splitting. Then H divides M into two blocked handlebodies, V and W. Label the blocks of V and W inductively: In the first step all core blocks are labeled zero. During the $(i+1)^{\text{th}}$ step every unlabeled product block receives the label i if and only if it is adjacent, across H, to a block labeled $i - 1$.

Definition. Suppose V and W are labeled as above. Obtain V_i by shrinking $\overline{V_i}$, the union of all blocks inside of V with label at least i. The *shelling* of V is the sequence of submanifolds, $\{V_0, V_1, V_2, \ldots V_n\}$. The shelling of W is defined symmetrically.

If i is greater than zero V_i is a I-bundle. Thus, when $i > 0$, the vertical boundary of V_i is a collection of annuli properly embedded in the handlebody V.

Several remarks are in order.

Remark 5.1. As M is orientable, all I-bundles considered are orientable.

Remark 5.2. If $i < j$ then $V_i \supset V_j$. Thus $c'(V_i) \leq c'(V_j)$. This follows directly from Lemma 8.5 applied to the base surfaces of V_i and V_j.

Remark 5.3. If Y and Z are large, connected I-bundles with a component of $\partial_h Y$ contained in $\partial_h Z$ then

$$c(Z) \leq \frac{1}{2} c(Y).$$

Again, this follows from Lemma 8.5 applied to the relevant components of $\partial_h Y$ and $\partial_h Z$.

Remark 5.4. As V_0 equals V so $c(V_0) = 1 - g(H)$, where $g(H)$ is the genus of H. By Remark 4.1 and Lemma 4.2 we have $c(V_1) \leq 1 - g(H) + a_1 \cdot (6|T^3| + 2)$. Here a_1 is the constant provided by Lemma 4.2 and $|T^3|$ is the number of tetrahedra in the triangulation T.

Remark 5.5. Suppose that $i > 0$. If a tetrahedron τ meets the almost normal piece then τ contains at most six families of parallel product blocks. If not, then τ contains at most five such families. Each family contains at most two blocks labeled i. Thus, when $i > 0$ the I-bundle V_{i+1} is obtained by removing at most $10 \cdot |T^3| + 2$ blocks from V_i. By Lemma 4.2 we have

$$c(V_{i+1}) \leq c(V_i) + 12a_1|T^3|.$$

6. High genus implies the DAP

Enough tools are now at hand to prove the main technical proposition:

Proposition 6.1. If (M, T) is a closed, orientable, triangulated three-manifold and $H \subset M$ is an almost normal Heegaard splitting of genus $g(H) > 496(a_1|T^3|)^2$ then H satisfies the disjoint annulus property.

Proof. As H is almost normal the handlebodies V and W are blocked submanifolds of M. We will first find an essential annulus $A \subset V$.

As in Section 5 obtain shellings of V and W, $\{V_i\}$ and $\{W_i\}$. Now, $c(V_0) = c(V) \leq -496(a_1|T^3|)^2$. The I-bundle V_1 contains at most $(6|T^3|+2)a_1$ components. It follows that some component, $Y_1 \subset V_1$, has

$$c(Y_1) \leq \frac{c(V_1)}{(6|T^3| + 2)a_1} \leq -62a_1|T^3| + 1.$$

Define $Y_i = (\widehat{V_i \cap Y_1})$ to be the large part of $V_i \cap Y_1$. Let $k > 1$ be the smallest integer such that $c(Y_k) > c(Y_1)$. The I-bundle Y_k is nonempty, as $c(Y_k) \leq -50a_1|T^3|+1$. Push each component of $\partial_v Y_k$ slightly into Y_k. Denote the resulting collection of annuli by $\mathcal{A} = \{A_j\}_{j=1}^m$.

Claim. Some annulus $A \in \mathcal{A}$ is essential in V.

Proceed by via contradiction: Suppose every $A_j \in \mathcal{A}$ is inessential. Deduce from Lemmata 8.6 and 8.7 that each A_j separates Y_1 into two pieces, one of which has complexity equal to 2. Then the entire collection $\mathcal{A}$ cuts Y_1 into $m + 1$ pieces. Of these exactly one has complexity not equal to 2. We deduce that Y_k is connected and has $c(Y_k) = c(Y_1)$. This contradicts our choice of k and proves the claim.

We must now find a disjoint essential annulus $B \subset W$. As the horizontal boundary of Y_k is contained in $\partial_h V_k$ and Y_k is large we have $\partial_h Y_k \subset \partial_h \widehat{W_{k-1}}$. It follows that $c(\widehat{W_{k-1}}) \leq \frac{1}{2}c(Y_k)$. Thus $c(\widehat{W_{k-1}}) \leq -25a_1|T^3| + \frac{1}{2}$ and $c(\widehat{W_{k+1}}) \leq -a_1|T^3| < 0$. Deduce that $\widehat{W_{k+1}}$ contains a properly embedded, nonseparating, vertical annulus, B. By Lemma 8.6 the annulus B must be essential in W.

Finally, note that $B \subset W_{k+1}$ while A is properly isotopic into the vertical boundary of $Y_k \subset V_k$. Thus A and B, after a small proper isotopy of A, are disjoint. We conclude that H satisfies the disjoint annulus property. □

Proposition 6.1 is cast into the realm of general splittings using:

Theorem 6.2 (Rubinstein [12], Stocking [15])**.** *If $H \subset (M, T)$ is a strongly irreducible Heegaard splitting then H is isotopic to an almost normal surface.*

We now deduce:

Theorem 6.3. *Fix M, a closed, orientable three-manifold. There is a constant $a_2(M)$ such that if $H \subset M$ is a Heegaard splitting, with $g(H) > a_2(M)$, then H satisfies the disjoint annulus property.*

Proof. Let T be a minimal triangulation of M and take $a_2(M) = 496(a_1|T^3|)^2$. If H is strongly irreducible then, by Theorem 6.2, we may isotope H to be almost normal with respect to T. Thus H has the DAP by Proposition 6.1. On the other hand, if H is weakly reducible then H satisfies the DAP by Lemma 8.3. □

Finally, we require Jaco and Rubinstein's solution to the Waldhausen Conjecture:

Theorem 6.4 (Jaco, Rubinstein [6])**.** *Fix M, a closed, orientable, atoroidal three-manifold. Then, for every $g \in \mathbb{N}$, there are only finitely many isotopy classes of strongly irreducible Heegaard splittings of genus g.*

From this follows:

Theorem 6.5. *In any closed orientable three-manifold there are only finitely many Heegaard splittings, up to isotopy, which do not satisfy the disjoint annulus property.*

Proof. We may restrict attention to strongly irreducible splittings with genus greater than one by applying Lemma 8.3 and the classification of splittings of lens spaces.

If M is toroidal then, by Lemma 2.2, every strongly irreducible splitting $H \subset M$ has the twin annulus property. Hence all splittings have the disjoint annulus property.

If M is atoroidal then, by Theorem 6.4, there are only finitely many strongly irreducible splittings (up to isotopy) with genus below $a_2(M)$. All splittings with higher genus have the DAP by Theorem 6.3. □

7. Conjecture

Our results could be strengthened in several ways.

Question. Suppose that H is a Heegaard splitting of M. Is there a integer $a_3(M)$ such that $g(H) > a_3(M)$ implies that H satisfies the twin annulus property?

To my great embarrassment an affirmative answer is incorrectly claimed in my thesis [14]. However, the question seems to be difficult. Gaining this point would partially close the gap between our results and the Casson-Gordon-Parris examples mentioned in Section 1. Even more ambitious is the following:

Question. Do high distance Heegaard splittings always have minimal genus?

We end with a question first asked by Eric Sedgwick:

Question. Does there exist a non-Haken manifold with infinitely many non-isotopic, strongly irreducible Heegaard splittings?

8. Appendix

8.1. **Curves and Annuli.** Here the focus lies in proving that reducible and weakly reducible splittings have the twin annulus property. Recall that a simple closed curve γ, in the boundary of a handlebody, is *disk-busting* if γ meets the boundary of every essential disk.

Lemma 8.1. *Suppose $H \subset M$ is a Heegaard splitting. Then there is a curve $\gamma \subset H$ which is disk-busting for both handlebodies, V and W.*

We mimic an argument of [5] and [9]. As a reference for $\mathcal{PMF}(H)$, the space of projectively measured singular foliations on H, refer to [11].

Proof. If the genus of H, $g(H)$, equals one then any essential curve which does not bound a disk suffices. For the rest of the proof assume that $g(H) \geq 2$.

Let $\mathcal{V}$ be the set of isotopy classes of curves which bound disks in V. Let $\mathcal{V}'$ be the closure of $\mathcal{V}$ as a subset of $\mathcal{PMF}(H)$. We rely on the following theorem of Masur [10]: $\mathcal{V}'$ is nowhere dense in $\mathcal{PMF}(H)$. Thus there is an open set $U \subset \mathcal{PMF}(H)$ in the complement of $\mathcal{V}' \cup \mathcal{W}'$.

Let λ be a minimal, uniquely ergodic foliation lying inside of U. That is, λ contains no closed leaves and admits a projectively unique transverse measure. (Minimal, uniquely ergodic foliations are dense; see [11].) Let $\beta_i \subset H$ be a sequence of simple closed curves converging to λ in $\mathcal{PMF}(H)$.

Claim. There is a $K \in \mathbb{N}$ such that if $i > K$ then β_i is disk-busting for V.

As the same holds for W this will provide the conclusion of the lemma.

Suppose that infinitely many of the β_i fail to be disk-busting for V. Reindex so that all β_i fail to be disk-busting. Choose a sequence $\alpha_i \in \mathcal{V}$ such that $\alpha_i \cap \beta_i = \emptyset$.

As $\mathcal{PMF}(H)$ is compact pass to a subsequence (and reindex) so that the α_i converge to a projectively measured singular foliation, μ. From the continuity of intersection number as a function $\mathcal{MF}(H) \times \mathcal{MF}(H) \to \mathbb{R}$ (see 1.11 of [11]) we deduce

$$i(\lambda, \mu) = 0.$$

Now, as λ is minimal it follows that μ is Whitehead equivalent to λ as topological foliations (1.12 of [11]). But λ is also uniquely ergodic. Thus the measure on μ is projectively equivalent to that of λ. So λ is the same point as μ in $\mathcal{PMF}(H)$. That is, λ lies in $\mathcal{V}'$, a contradiction. □

We are now equipped to prove that reducible and weakly reducible splittings satisfy the TAP.

Lemma 8.2. *If $H \subset M$ is a reducible Heegaard splitting with genus $g(H) > 1$ then H satisfies the twin annulus property.*

Proof. Let $D \subset V$ and $E \subset W$ be essential disks with $\partial D = \partial E$. As $g(H) > 1$ we may assume that $D \cup E$ is separating. Pick one component of $M \smallsetminus (D \cup E)$. Let M' be the closed manifold obtained by gluing shut the boundary S^2 along the two disks D and E. Let H' be the resulting splitting of M'. Let $C \subset H'$ be the image of D.

Choose $\beta \subset H'$ an essential simple closed curve which meets C in a single arc. If $g(H') = 1$ then we require that the geometric intersection between β and the meridional disk of V' (and W') is at least two points. If $g(H') > 1$ then by Lemma 8.1 we may require that β is disk-busting for both V' and W'.

Let N be a regular neighborhood of $C \cup \beta$, taken in M'. Then $A' = \partial N \cap V'$ and $B' = \partial N \cap V'$ are incompressible in V' and W', as β is disk-busting. Also, A' and B' are boundary parallel to exactly one side, by our choice of β.

Let A and B be the preimages of A' and B' inside V and W respectively. These annuli are essential and, as $\partial A = \partial B$, it follows that H has the twin annulus property. □

Lemma 8.3. *If $H \subset M$ is a weakly reducible Heegaard splitting with genus $g(H) > 1$ then H satisfies the twin annulus property.*

Remark 8.4. In the proof we rely on Lemma 8.2 to deal with the reducible case.

Proof. Choose maximal families of disjoint, essential, nonparallel disks $\mathcal{D} \subset V$ and $\mathcal{E} \subset W$ such that $\partial\mathcal{D} \cap \partial\mathcal{E} = \emptyset$. As H is connected there is a component of $H \setminus (\mathcal{D} \cup \mathcal{E})$ whose closure meets both $\mathcal{D}$ and $\mathcal{E}$. Denote the closure of this component by G.

We claim that $|\partial G| = 2$ and the interior of G is disjoint from $\mathcal{D} \cup \mathcal{E}$. To see this suppose that ∂G meets distinct disks $D, D' \in \mathcal{D}$. (The case where interior(G) meets a disk is similar.) Let α be any embedded simple arc in G connecting D to D'. Let N be a closed regular neighborhood of $\alpha \cup D \cup D'$ taken in V. Then N is a solid pair of pants with frontier properly isotopic to $D \cup D' \cup D''$. But D'' is essential and may be added to $\mathcal{D}$. This contradicts the maximality of $\mathcal{D}$.

If G is an annulus then H is reducible. Lemma 8.2 then implies that H satisfies the TAP. Suppose now that G is not an annulus and suppose that $D \in \mathcal{D}$ and $E \in \mathcal{E}$ meet ∂G.

Let α be a nonseparating simple closed curve in G. Let β and γ be disjoint simple arcs in G connecting ∂D and ∂E, respectively, to opposite sides of α. Construct a pair of annuli as follows: Let $N(D)$ be a regular neighborhood, taken in V, of $\alpha \cup \beta \cup D$. Then $N(D)$ is a solid torus with frontier properly isotopic to $D \cup A$. Here A is an annulus properly embedded in V. Similarly construct $N(E) \subset W$ with frontier properly isotopic to $E \cup B$.

Now, if A is nonseparating in V then it is not boundary parallel. If A is separating then it separates ∂D from ∂E in H and again is not boundary parallel. The same holds for B.

Finally, if A compresses in V then there is a compression which is disjoint from D. It follows that α must bound a disk in V. But this implies that $\mathcal{D}$ was not maximal, a contradiction. Deduce that A and B are essential annuli which, after proper isotopy, share $\alpha \subset H$ as a common boundary component. □

8.2. **Surfaces.** Recall that the *complexity* of a compact surface G is defined to be the quantity $c(G) = \chi(G) + |\partial G|$. This is the same as the Euler characteristic of the surface obtained by capping off all boundary components of G by disks. The lemma below essentially states that Euler characteristic increases when a surface is compressed.

Lemma 8.5. *Suppose that G and F are compact, connected surfaces with $G \subset F$. Then $c(F) \leq c(G)$.*

Proof. If $G = F$ then the statement is trivial. Assume then that $\partial G \neq \emptyset$ and that G has been isotoped so that $\partial G \cap \partial F = \emptyset$.

Let $G' = \bigcup_{i=1}^n G_i = F \setminus \text{interior}(G)$, where each G_i is connected. It follows that $n \leq |\partial G|$ and that $|\partial G'| = |\partial G| + |\partial F|$. As $c(S) \leq 2$ for any compact, connected surface S we have:

$$c(G') = \sum c(G_i) \leq 2n \leq 2|\partial G|.$$

Omitting the middle yields:

$$\chi(G') + |\partial G'| \leq 2|\partial G|.$$

Thus:

$$\chi(G') + |\partial F| \leq |\partial G|.$$

Now add $\chi(G)$ to both sides to obtain the desired inequality. □

8.3. ***I*-Bundles.** As a matter of terminology, a subset of an I-bundle is called *vertical* if it is a union of fibres. The next pair of lemmata supply us with many essential annuli inside of handlebodies.

Lemma 8.6. *Let Y be an I-bundle embedded in a handlebody V such that $\partial_h Y \subset \partial V$. Let A be a properly embedded vertical annulus of Y. Suppose that A is non-separating inside of Y. Then A is an essential annulus in V.*

Proof. As A is nonseparating in Y, A is nonseparating in V. Thus A is not boundary parallel. It is left to show that A is incompressible.

Pick B, a vertical annulus or Mobius band in Y, such that $\alpha = A \cap B$ is a single fibre. For a contradiction suppose that $D \subset V$ is a compressing disk for A. As ∂D is isotopic to the core curve of A, $|\alpha \cap \partial D|$ is odd.

However, by general position, $D \cap B$ is a compact one-manifold with boundary $\alpha \cap \partial D$. But compact one-manifolds have an even number of points on their boundary. Thus A is incompressible. □

Lemma 8.7. *Let Y be an I-bundle embedded in a handlebody V such that $\partial_h Y \subset \partial V$. Let A be a properly embedded vertical annulus of Y. Suppose that A separates Y into components Z_0 and Z_1 with $c(Z_0), c(Z_1) < 2$. Then A is an essential annulus in V.*

Proof. As $c(Z_0) < 2$ there is a vertical rectangle B properly embedded in Z_0 such that $\partial_v B \subset A$ and B does not separate Z_0.

Suppose, for a contradiction, that $D \subset V$ is a compressing disk for A. Assume that a collar of ∂D, taken inside of D, lies in Z_0. (The other case is handled similarly.) Among all compressing disks for A, with collar in Z_0, choose D to minimize the quantity $|D \cap B| + |\partial D \cap B|$. The next two paragraphs show that $D \cap B$ consists of a single arc.

Suppose γ is an innermost simple closed curve of $D \cap B$ inside of D. Thus γ bounds disks $E \subset D$ and $C \subset B$. The two-sphere $E \cup C$ bounds a ball in V, as handlebodies are irreducible. An ambient isotopy of E across this ball pushes D to a disk D'. The disk D' compresses A, has a collar in Z_0, has $|\partial D' \cap B| = |\partial D \cap B|$, and has $|D' \cap B| < |D \cap B|$. However, this contradicts the minimality assumption. Thus $D \cap B$ contains no simple closed curves.

Suppose now that D intersects one component of $\partial_v B$ more than once. Recall that ∂D is a core curve in A. Thus there are arcs $\alpha \subset \partial D$, $\beta \subset \partial_v B$ such that $\alpha \cap \partial_v B = \partial\beta$ and $\alpha \cup \beta$ bounds a disk $A' \subset A$. An ambient isotopy of α across A' pushes D to D'. Then D' compresses A, has a collar in Z_0, has $|\partial D' \cap B| < |\partial D \cap B|$, and has $|D' \cap B| \leq |D \cap B|$. Again, this is a contradiction. Thus $D \cap B$ contains a single arc, γ.

The arc γ cuts D into a pair of disks, E' and E''. Likewise, $\partial_v B$ cuts A into a pair of vertical rectangles, A' and A''. Relabeling if necessary we have interior$(A') \cap E'' = \emptyset$. Thus $A' \cup B$ is a vertical annulus or Mobius band which is nonseparating in Y, because B was nonseparating in Z_0.

Now, E' is a compressing disk for $A' \cup B$. So if $A' \cup B$ is an annulus we have a contradiction of Lemma 8.6. If $A' \cup B$ is a Mobius band then E' shows that $A' \cup B$ is two-sided. This contradicts the fact that V is orientable. It follows that A is incompressible.

Finally, suppose A is boundary parallel in V. Let $C \subset \partial V$ be the annulus cobounded by the two components of ∂A. Let R be the solid torus cobounded by $A \cup C$. Suppose that $Z_0 \subset R$. (The other possibility is symmetric.) Thus $\partial_h Z_0 \subset C$.

If $c(Z_0) < 1$ then $\partial_h Z_0$ is a nonplanar surface. But $\partial_h Z_0 \subset C$ which is planar. This is a contradiction.

Suppose instead that $c(Z_0) = 1$. Thus Z_0 contains a vertical Mobius band B. So $\partial_h B \subset C$. If $\partial_h B$ bounds a disk, D, in C then $B \cup D$ is an $\mathbb{RP}^2$ embedded in V, which is impossible. Thus $\partial_h B$ is a core curve for C. Now choose E, a meridional disk for the solid torus R, such that $\partial E \cap C = \alpha$ is a single arc. Then $|\alpha \cap B|$ is even, as $E \cap B$ is a one-manifold. However, $|\alpha \cap B|$ must be odd, as $\partial_h B$ is a core curve for C.

This last contradiction establishes that A is not boundary parallel. Thus A is essential. □

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