AUTOMORPHISMS OF THE DISK COMPLEX

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Abstract. We show that the automorphism group of the disk complex is isomorphic to the handlebody group. Using this, we prove that the outer automorphism group of the handlebody group is trivial.

1. Introduction

We show that the automorphism group of the disk complex is isomorphic to the handlebody group. Using this, we prove that the outer automorphism group of the handlebody group is trivial. These results and many of the details of the proof are inspired by Ivanov's work [15] on the mapping class group and the curve complex. We begin with a review of the relevant story for surfaces.

1.1. **Surfaces.** Let $S = S_{g,n}$ be the compact connected orientable surface of genus g with n boundary components. We write $S = S_g$ when $n = 0$. Define $e(S) = -\chi(S) = 2g - 2 + n$ and define the *complexity* of S to be $\xi(S) = 3g - 3 + n$.

Suppose α is a simple closed curve, properly embedded in S. We say α is *inessential* if α cuts a disk off of S; otherwise α is *essential*. We say α is *peripheral* if it cuts an annulus off of S; otherwise α is *non-peripheral*. Note that essential curves may be peripheral.

Definition 1.2 (Harvey [10])**.** The *curve complex* $\mathcal{C}(S)$ is the simplicial complex with vertex set being ambient isotopy classes of essential, non-peripheral simple closed curves in S. The k–simplices are given by collections of $k + 1$ vertices having pairwise disjoint representatives.

In the usual abuse of notation, we typically do not distinguish a curve from its isotopy class. If $\mathcal{K}$ is a simplicial complex (as in [32, page 108]) then let $\operatorname{Aut}(\mathcal{K})$ be the group of simplicial automorphisms of $\mathcal{K}$.

Definition 1.3. The *mapping class group* $\mathcal{MCG}(S)$ is the group of homeomorphisms of S, up to isotopy. (Some authors refer to this as

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the *extended* mapping class group, as orientation reversing homeomorphisms are permitted.) There is a natural homomorphism $\mathcal{MCG}(S) \to \mathrm{Aut}(\mathcal{C}(S))$; we call any element of the image a *geometric automorphism*.

We now state a foundational result.

Theorem 1.4 (Ivanov's Theorem)**.** *If* $\xi(S) \geq 3$*, or if* $(g, n) = (0, 5)$*, then all elements of* $\mathrm{Aut}(\mathcal{C}(S_{g,n}))$ *are geometric.* □

This was first proved by Ivanov [15], for surfaces of genus at least two, and was extended to lower genus by Korkmaz and by Luo [19, 22].

One application of Theorem 1.4 is a version of a theorem of Royden [30]: for $g \geq 2$ any complex analytic isometry from a domain of Teichmüller space $\mathcal{T}(S_g)$, into $\mathcal{T}(S_g)$, is induced by a mapping class. Note that Royden's result was generalized by Earle and Kra to include punctured surfaces [8].

Work following Ivanov includes proofs that the mapping class group is the automorphism group of: the graph of ideal triangulations [20], the pants graph [24], the Hatcher-Thurston graph [12], the arc complex [13], the Torelli geometry [9], the complex of separating curves [5], the complex of non-separating curves [11], the arc and curve complex [21], and the duality graph [31]. These results align in a pleasing fashion with Ivanov's metaconjecture: "Every object naturally associated to a surface S with sufficiently rich structure has $\mathcal{MCG}(S)$ as its group of automorphisms." [16, Problem 6].

Many of these proofs follow a common plan, as do our results; this is laid out in Section 3. However, since we work with three-manifolds our cut-and-paste arguments have an extra dimension; in the same line, our theorems conclude with the mapping class group of a handlebody, not of a surface. This does not contradict Ivanov's metaconjecture; the disk complex is naturally associated to a handlebody, not to its boundary. Note that Aramayona and Souto [2] have a preprint showing that all automorphisms of the sphere complex are geometric.

Ivanov [15] uses Theorem 1.4 to obtain a new proof of *algebraic rigidity* of the mapping class group. This was first proven by Ivanov and by McCarthy [15, 25] using other methods.

Theorem 1.5. *Suppose that* $e(S) \geq 3$*. Then the outer automorphism group of the mapping class group* $\mathcal{MCG}(S)$ *is trivial.* □

To prove such a result, one fixes attention on a subgroup of the mapping class group, defines a suitable combinatorial complex and then establishes *combinatorial rigidity* (as in Theorem 1.4). Algebraic rigidity may then follow; such a plan is carried out in many of the papers referenced above.

1.6. **Handlebodies.** The rest of the paper is devoted to a parallel development of Section 1.1, in the handlebody setting. Let $V = V_{g,n}$ be the genus g handlebody with n *spots*: a regular neighborhood of a finite, polygonal, connected graph in $\mathbb{R}^3$ with n disjoint closed disks chosen on the boundary. See Figure 1.7 for a picture of $V_{2,2}$. We write $V = V_g$ when $n = 0$. Let $\partial_0 V$ denote the union of the spots. We will assume that all ambient isotopies of V fix $\partial_0 V$ setwise. Let $\partial_+ V$ be the closure of $\partial V - \partial_0 V$. So $\partial_+ V$ is homeomorphic to $S_{g,n}$. Define $e(V) = e(\partial_+ V)$ and $\xi(V) = \xi(\partial_+ V)$.

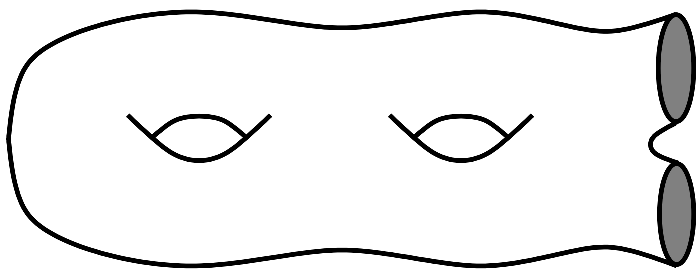

FIGURE 1.7. A genus two handlebody with two spots.

A properly embedded disk $D \subset V$, with $\partial D \subset \partial_+ V$, is *essential* or *non-peripheral* exactly as its boundary is in $\partial_+ V$.

Definition 1.8 (McCullough [26])**.** The *disk complex* $\mathcal{D}(V)$ is the simplicial complex with vertex set being ambient isotopy classes of essential, non-peripheral disks in V. The k–simplices are given by collections of $k + 1$ vertices having pairwise disjoint representatives.

Again, we do not notationally distinguish a disk from its isotopy class. Note that there is a natural simplicial injection $\mathcal{D}(V) \to \mathcal{C}(\partial_+ V)$ taking a disk to its boundary.

Definition 1.9. The *handlebody group* $\mathcal{H}(V)$ is the group of homeomorphisms of V, fixing the spots setwise, up to ambient isotopy. We call the image of the natural homomorphism $\mathcal{H}(V) \to \mathrm{Aut}(\mathcal{D}(V))$ the group of *geometric automorphisms*.

Note that there is also a natural monomorphism $\mathcal{H}(V) \to \mathcal{MCG}(\partial_+ V)$ which takes $f \in \mathcal{H}(V)$ to $f|\partial_+ V$.

Our main theorem is an analogue of Theorem 1.4.

Theorem 9.3. *If a handlebody* $V = V_{g,n}$ *satisfies* $e(V) \geq 3$ *then all elements of* $\mathrm{Aut}(\mathcal{D}(V))$ *are geometric.*

The plan of the proof is given in Section 3. The proof itself is completed in Section 9. Section 4 shows that Theorem 9.3 is sharp; all handlebodies V with $e(V) \leq 2$ exhibit some kind of exceptional behavior. Theorem 9.3 has the following consequence.

Theorem 9.4. *If a handlebody* $V = V_{g,n}$ *satisfies* $e(V) \geq 3$ *then the natural map* $\mathcal{H}(V) \to \mathrm{Aut}(\mathcal{D}(V))$ *is an isomorphism.*

In Section 10 we use Theorem 9.3 to prove an analogue of Theorem 1.5.

Theorem 10.1. *If* $e(V) \geq 3$ *then the outer automorphism group of the handlebody group is trivial.*

2. Background

The genus zero case of Theorem 1.4 is contained in the thesis of the first author [19, Theorem 1].

Theorem 2.1. *If* $g = 0$ *and* $n \geq 5$ *then all elements of* $\mathrm{Aut}(\mathcal{C}(S_{0,n}))$ *are geometric.* □

Spotted balls are the simplest handlebodies. Accordingly:

Lemma 2.2. *The natural maps* $\mathcal{D}(V_{0,n}) \to \mathcal{C}(S_{0,n})$ *and* $\mathcal{H}(V_{0,n}) \to \mathcal{MCG}(S_{0,n})$ *are isomorphisms.*

Proof. The three-manifold $V_{0,n}$ is an n–spotted ball. Every simple closed curve in $\partial_+ V$ bounds a disk in V. This proves that $\mathcal{D}(V_{0,n}) \to \mathcal{C}(S_{0,n})$ is a surjection and thus, by the remark immediately after Definition 1.8, an isomorphism.

It follows from the Alexander trick [1] that the map of mapping class groups is an injection. Since $\mathcal{MCG}(S_{0,n})$ is generated by half twists and a reflection [3, Theorem 4.5] the inclusion is in fact surjective. □

The genus zero case of Theorem 9.3 is an immediate corollary. Higher genus requires further preparation.

Suppose that V is a handlebody. Two disks $D, E \in \mathcal{D}(V)$ are *topologically equivalent* if there is a mapping class $f \in \mathcal{H}(V)$ so that $f(D) = E$. The *topological type* of D is its equivalence class in $\mathcal{D}(V)$.

For any simplicial complex $\mathcal{K}$ and for any simplex $\sigma \in \mathcal{K}$ we define *link* of σ to be the subcomplex

$$\mathrm{link}(\sigma) = \{\tau \in \mathcal{K} \mid \sigma \cap \tau = \emptyset,\ \sigma \cup \tau \in \mathcal{K}\}.$$

So if $\mathbb{D}$ is a simplex of $\mathcal{D}(V)$ then $\mathrm{link}(\mathbb{D})$ is the subcomplex of $\mathcal{D}(V)$ spanned by disks E disjoint from all and distinct from all $D \in \mathbb{D}$.

If $X \subset Y$ is a properly embedded submanifold then we write $n(X)$ and $N(X)$ to denote open and closed regular neighborhoods of X in Y.

If X is codimension zero then the *frontier* of X in Y is the closure of $\partial X - \partial Y$.

A simplex $\mathbb{D} \in \mathcal{D}(V)$ is a *cut system* if $V - n(\mathbb{D})$ is a spotted ball. Thus the number of vertices in $\mathbb{D}$ is exactly the genus of V. Note that every disk of $\mathbb{D}$ yields two spots of $V - n(\mathbb{D})$.

Recall that for simple curves α, β, properly embedded in S, the *geometric intersection number* $i(\alpha, \beta)$ is the minimum possible intersection number between ambient isotopy representatives.

Two disks $D, E \in \mathcal{D}(V)$ are *dual* if $i(\partial D, \partial E) = 2$. Equivalently, all representatives of D meet E and, after a suitable ambient isotopy of D, the disks D and E intersect along a single arc. Equivalently, after a suitable ambient isotopy of D, a regular neighborhood of $D \cup E$ is a four-spotted ball with all spots essential in V. (Note that the spots of $V_{0,4}$ may be peripheral in V.) See Figure 2.3.

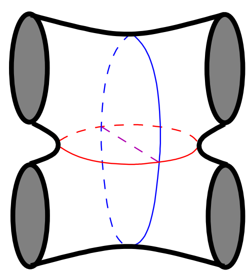

FIGURE 2.3. Every spot of the $V_{0,4}$ containing a pair of dual disks is essential in V. The spots of $V_{0,4}$ may be peripheral.

If $\mathbb{D} = \{D_i\}$ is a cut system then define $\mathrm{dual}_i(\mathbb{D})$ to be the subcomplex spanned by the disks $E \in \mathcal{D}(V)$ which are dual to D_i and disjoint from D_j for all $j \neq i$. Define $\mathrm{dual}(\mathbb{D})$ to be the subcomplex spanned by $\cup_i \mathrm{dual}_i(\mathbb{D})$.

3. The plan of the proof of Theorem 9.3

Let $V = V_{g,n}$ be a genus g handlebody with n spots. Suppose that $g \geq 1$ and $e(V) \geq 3$. Let ϕ be any automorphism of $\mathcal{D}(V)$; so ϕ preserves the combinatorics of $\mathcal{D}(V)$. It follows that any topological property of V admitting a combinatorial characterization is preserved by ϕ. This observation is the driving force behind our proof. Applying it, Lemma 5.1 proves that ϕ preserves the topological types of disks. In addition, ϕ sends cut systems to cut systems (Lemma 5.10). Next

Lemma 7.2 shows that ϕ preserves duality. Also, for any cut system $\mathbb{D} = \{D_i\}$, the complex $\mathrm{dual}_i(\mathbb{D})$ is connected (Lemma 7.3).

Fix now a cut system $\mathbb{D}$. Pick any geometric automorphism f_{cut} so that $f_{\mathrm{cut}}(\mathbb{D}) = \phi(\mathbb{D})$, vertex-wise; f_{cut} exists by Lemma 5.10. Define $\phi_{\mathrm{cut}} = f_{\mathrm{cut}}^{-1} \circ \phi$. Thus

$$\phi_{\mathrm{cut}}|\mathbb{D} = \mathrm{Id}\,.$$

Let $V' = V - n(\mathbb{D}) =\cong V_{0,2g+n}$ be the spotted ball obtained by cutting V along a regular neighborhood of $\mathbb{D}$. Let $D_i^{\pm}$ be the two spots of V' that are parallel to D_i in V. Now, since ϕ_{cut} preserves $\mathrm{link}(\mathbb{D}) \cong \mathcal{D}(V')$, by Theorem 2.1 and Lemma 2.2 there is a homeomorphism $f \colon V' \to V'$ so that the induced automorphism $f \in \mathrm{Aut}(\mathcal{D}(V'))$ satisfies $f = \phi_{\mathrm{cut}}|\,\mathrm{link}(\mathbb{D})$. Lemma 6.1 proves that for every i either f fixes the spots D_i^+ and D_i^- or f interchanges the spots. Thus f can be glued to give a homeomorphism $f_{\mathrm{link}} \colon V \to V$ as well as a geometric automorphism $f_{\mathrm{link}} \in \mathrm{Aut}(\mathcal{D}(V))$. Define $\phi_{\mathrm{link}} = f_{\mathrm{link}}^{-1} \circ \phi_{\mathrm{cut}}$. Thus

$$\phi_{\mathrm{link}}|\mathbb{D} \cup \mathrm{link}(\mathbb{D}) = \mathrm{Id}\,.$$

Recall that ϕ_{link} preserves duals by Lemma 7.2. For every $D_i \in \mathbb{D}$ pick some dual $E_i \in \mathrm{dual}_i(\mathbb{D})$. By Lemma 8.1 there is an integer $m_i \in \mathbb{Z}$ so that $T_i^{m_i}(E_i) = \phi_{\mathrm{link}}(E_i)$, where T_i is the Dehn twist about D_i. Define $f_{\mathrm{dual}} = \prod T_i^{m_i}$ and define $\phi_{\mathrm{dual}} = f_{\mathrm{dual}}^{-1} \circ \phi_{\mathrm{link}}$. Letting $\mathbb{E} = \{E_i\}$ we have

$$\phi_{\mathrm{dual}}|\mathbb{D} \cup \mathrm{link}(\mathbb{D}) \cup \mathbb{E} = \mathrm{Id}\,.$$

Recall that Lemma 7.3 proves that $\mathrm{dual}_i(\mathbb{D})$ is connected. Therefore, a *crawling* argument, given in Lemma 8.2, proves that

$$\phi_{\mathrm{dual}}|\mathbb{D} \cup \mathrm{link}(\mathbb{D}) \cup \mathrm{dual}(\mathbb{D}) = \mathrm{Id}\,.$$

Wajnryb [33] proves that the *cut system complex* is connected. Thus we may likewise crawl through $\mathcal{D}(V)$ and prove (Section 9) that

$$\phi_{\mathrm{dual}} = \mathrm{Id}$$

and so prove that

$$\phi = f_{\mathrm{cut}} \circ f_{\mathrm{link}} \circ f_{\mathrm{dual}}.$$

Thus ϕ is geometric.

4. Small handlebodies

This section covers the handlebodies V with $e(V) \le 2$. We start with genus zero. If $n \le 3$ then $\mathcal{D}(V_{0,n})$ is empty. By Lemma 2.2 the mapping class groups of V and $\partial_+ V$ are equal. Thus

$$\mathcal{H}(V_0),\ \mathcal{H}(V_{0,1}) \cong \mathbb{Z}/2\mathbb{Z}$$

while

$$\mathcal{H}(V_{0,2}) \cong K_4 \quad \text{and} \quad \mathcal{H}(\mathrm{V}_{0,3}) \cong \mathbb{Z}/2\mathbb{Z} \times \Sigma_3.$$

Here K_4 is the Klein four-group and Σ_3 is the symmetric group on three objects [29, Appendix A].

In genus zero with $n = 4$ the disk complex $\mathcal{D}$ is a countable collection of vertices with no higher dimensional simplices. Thus $\mathrm{Aut}(\mathcal{D}) = \Sigma_\infty$ is uncountable. However, there are only countably many geometric automorphisms. In fact, by Lemma 2.2, the mapping class group $\mathcal{H}(V_{0,4})$ is isomorphic to $K_4 \rtimes \mathrm{PGL}(2, \mathbb{Z})$ [29, Appendix A].

For genus one, if $n = 0$ or 1 then $\mathcal{D}$ is a single point and $\mathrm{Aut}(\mathcal{D})$ is trivial. On the other hand

$$\mathcal{H}(V_1),\ \mathcal{H}(V_{1,1}) \cong \mathbb{Z} \rtimes K_4.$$

For $V = V_{1,2}$ matters are more subtle. The subcomplex $\mathrm{NonSep}(V) \subset \mathcal{D}(V)$, spanned by non-separating disks, is a copy of the Bass-Serre tree for the meridian curve in $S_{1,1} = \partial_+ V_{1,1}$ [18]. Thus $\mathrm{NonSep}(V)$ is a copy of T_∞: the regular tree with countably infinite valance. Now, if $E \in \mathcal{D}(V)$ is separating then there is a unique disk D disjoint from E; also, D is necessarily non-separating. It follows that $\mathcal{D}(V)$ is a copy of the tree $\mathrm{NonSep}(V)$ with countably many leaves attached to every vertex. Thus $\mathrm{Aut}(\mathcal{D})$ contains a copy of $\mathrm{Aut}(T_\infty)$ as well as countably many copies of Σ_∞; so $\mathrm{Aut}(\mathcal{D})$ is uncountable. As usual $\mathcal{H}(V)$ is countable; thus $\mathrm{Aut}(\mathcal{D})$ contains non-geometric elements. However, Luo's treatment of $\mathcal{C}(S)$ [22] suggests the following problem.

Problem 4.1. Suppose that $V = V_{1,2}$. Let $\mathcal{G}$ be the subgroup of $\mathrm{Aut}(\mathcal{D}(V))$ consisting of automorphisms preserving duality: if $\phi \in \mathcal{G}$ and D, E are dual then so are $\phi(D), \phi(E)$. Is every element of $\mathcal{G}$ geometric?

Note that this approach of recording duality is precisely correct for the four-spotted ball; the complex where simplices record duality in $V_{0,4}$ is the *Farey tessellation*, $\mathcal{F}$, and every element of $\mathrm{Aut}(\mathcal{F})$ is geometric. See [22, Section 3.2].

The last exceptional case is $V = V_2$. Let $\mathrm{NonSep}(V)$ be the subcomplex of $\mathcal{D}(V)$ spanned by non-separating disks. Then $\mathrm{NonSep}(V)$ is an increasing union, as follows. Let $\mathcal{N}_0$ be a single triangle and form $\mathcal{N}_{i+1}$ by attaching (to every free edge of $\mathcal{N}_i$) a countable collection of triangles. The complex $\mathrm{NonSep}(V)$ is the increasing union of the $\mathcal{N}_i$. A careful discussion of $\mathrm{NonSep}(V)$ is given by Cho and McCullough [6, Section 4]

We obtain $\mathcal{D}(V)$ by attaching a countable collection of triangles to every edge of $\mathrm{NonSep}(V)$. To see this note that every separating disk E divides V into two copies of $V_{1,1}$. These copies of $V_{1,1}$ have meridian disks, say D and D'. Thus $\operatorname{link}(E)$ is an edge and the triangle $\{E, D, D'\}$ has two free edges in $\mathcal{D}(V)$, as indicated. Finally, there is a countable collection of separating disks lying in $V - (D \cup D')$, again as indicated.

It follows that $\operatorname{Aut}(\mathcal{D}(V_2))$ is uncountable. Again, as in Problem 4.1, we may ask: are all "duality-respecting" elements $f \in \operatorname{Aut}(\mathcal{D}(V_2))$ geometric? We end this section with another open problem.

Problem 4.2. Suppose that V is a handlebody with $e(V)$ and genus both sufficiently large. Show that $\operatorname{Aut}(\operatorname{NonSep}(V)) = \mathcal{H}(V)$.

A solution to Problem 4.2 may lead to a simplified proof of Theorem 10.1.

5. Topological types

The goal of this section is the following.

Lemma 5.1. *For any handlebody V and for any $\phi \in \operatorname{Aut}(\mathcal{D}(V))$ the automorphism ϕ preserves topological types of disks.*

Note that V_1, $V_{1,1}$ and $V_{0,4}$ are the only handlebodies where $\mathcal{D}(V)$ has dimension zero. (When $\mathcal{D}(V)$ is empty its dimension is -1.) Furthermore V_1 and $V_{1,1}$ are the only handlebodies where $\mathcal{D}(V)$ is a single point.

Definition 5.2. A simplicial complex $\mathcal{K}$ is *flag* when, in dimensions two and higher, a simplex is present if its faces are. (Equivalently, minimal non-faces have dimension one.) Note that the disk complex $\mathcal{D}(V)$ is flag.

We call $V_{0,3}$, the three-spotted ball, a *solid pair of pants*. Thus $\xi(V) = 3g - 3 + n$ is the number of disks in a *pants decomposition* of V; this is also the number of vertices of any maximal simplex in $\mathcal{D}(V)$. Note that $e(V) = 2g - 2 + n$ is the number of solid pants in a pants decomposition. We will call $V_{1,1}$ a *solid handle*. Suppose now that $E \subset V$ is a separating disk with $V - n(E) = X \cup Y$. If X or Y is a solid pants then we call E a *pants disk*. If X or Y is a solid handle then we call E a *handle disk*.

Definition 5.3. If $\mathcal{K}$ and $\mathcal{L}$ are non-empty simplicial complexes with disjoint vertex sets then their *join* is the complex

$$\mathcal{K} * \mathcal{L} = \mathcal{K} \cup \{\sigma \cup \tau \mid \sigma \in \mathcal{K},\ \tau \in \mathcal{L}\} \cup \mathcal{L}.$$

For example, if $E \subset V$ is an essential separating disk, yet not a pants disk, then $\operatorname{link}(E) \subset \mathcal{D}(V)$ is a join.

A flag complex decomposes uniquely as a join; here is a version of that fact.

Lemma 5.4. *Suppose that $\mathcal{K}$, $\mathcal{L}$ are simplicial complexes, both flag, non-empty, and not joins. Then $\mathcal{K}$ and $\mathcal{L}$ can be recovered from $\mathcal{K} * \mathcal{L}$.*

Proof. Let $\mathcal{J} = \mathcal{K} * \mathcal{L}$ be given. Note that $\mathcal{J}$ is flag. Let $\mathcal{G}$ be the graph on the vertex set $\mathcal{J}^{(0)}$ of $\mathcal{J}$, defined as follows. For any distinct $u, v \in \mathcal{J}^{(0)}$ we have $\{u, v\} \in G$ if and only if $\{u, v\} \notin \mathcal{J}$.

Let u be a vertex of $\mathcal{K}$. Let $\mathcal{G}_u \subset \mathcal{G}$ be the connected component containing u. By induction on distance from u in $\mathcal{G}_u$ deduce $\mathcal{G}_u^{(0)} \subset \mathcal{K}^{(0)}$. Define $\mathcal{K}_u$ to be the subcomplex of $\mathcal{J}$ spanned by the vertices of $\mathcal{G}_u$. It follows that $\mathcal{K}_u$ is a subcomplex of $\mathcal{K}$.

Suppose, for a contradiction, that $\mathcal{K}_u \neq \mathcal{K}$. Let $\mathcal{K}^u$ be the subcomplex of $\mathcal{K}$ spanned by vertices not in $\mathcal{K}_u$. It follows that, in $\mathcal{J}$ and thus in $\mathcal{K}$, every vertex of $\mathcal{K}_u$ is connected to every vertex of $\mathcal{K}^u$. Since $\mathcal{K}$ is flag deduce that $\mathcal{K} = \mathcal{K}_u * \mathcal{K}^u$, a contradiction. □

The main hypothesis of Lemma 5.4 holds in our setting.

Lemma 5.5. *For any handlebody V the complex $\mathcal{D}(V)$ is not a join.*

Proof. When $e(V) \leq 2$ this can be checked case-by-case, following Section 4. The remaining handlebodies all admit disks D, E that *fill*: every disk F meets at least one of D or E. It follows that any edge-path in the one-skeleton $\mathcal{D}^{(1)}(V)$ connecting D to E has length at least three. However, the diameter of the one-skeleton of a join is either one or two. □

Here is our first combinatorial characterization of a family of disks.

Lemma 5.6. *A disk $E \in \mathcal{D}(V)$ is a separating disk yet not a pants disk if and only if* $\operatorname{link}(E)$ *is a join. Furthermore,* $\operatorname{link}(E)$ *is uniquely realized as a join, up to permuting the factors.*

Proof. For the forward direction of the first claim, suppose that $V - n(E) = X \cup Y$, where neither X nor Y is a solid pants. Since E is essential and non-peripheral $\mathcal{D}(X)$ and $\mathcal{D}(Y)$ are both non-empty. Thus $\operatorname{link}(E) = \mathcal{D}(X) * \mathcal{D}(Y)$ is a join.

On the other hand, if E is non-separating then $\operatorname{link}(E)$ is isomorphic to $\mathcal{D}(V_{g-1,n+2})$. If E is a pants disk then $\operatorname{link}(E) \cong \mathcal{D}(V_{g,n-1})$. Neither of these is a join by Lemma 5.5.

We now prove the second claim. The complexes $\mathcal{D}(X)$ and $\mathcal{D}(Y)$ are flag, non-empty, and, by Lemma 5.5, not joins. It follows from Lemma 5.4 that we may recover $\mathcal{D}(X)$ and $\mathcal{D}(Y)$ from $\operatorname{link}(E)$. □

The next three lemmas give combinatorial characterizations of handle disks, non-separating disks, and pants disks. Recall that a *cone* is the join of a point (the *apex*) with some non-empty simplicial complex (the *base*).

Lemma 5.7. *Suppose that $V \neq V_{1,2}$. Then $E \in \mathcal{D}(V)$ is a handle disk if and only if* $\operatorname{link}(E)$ *is a cone.*

Proof. Suppose that E cuts off a solid handle X with meridian D. Let Y be the other component of $V - n(E)$. Since $V \neq V_{1,2}$ we have that $\mathcal{D}(Y)$ is non-empty; in particular E is not a pants disk. Thus $\operatorname{link}(E) = \mathcal{D}(X) * \mathcal{D}(Y)$. As $\mathcal{D}(X) = \{D\}$ we are done with the forward direction.

Now suppose that $\operatorname{link}(E)$ is a cone with apex D. Since a cone is the join of the apex with the base, by Lemma 5.6 the disk E is separating but not a pants disk. Let $V - n(E) = X \cup Y$. Thus $\operatorname{link}(E) = \mathcal{D}(X) * \mathcal{D}(Y)$. However, by Lemma 5.6 the decomposition of $\operatorname{link}(E)$ is unique; breaking symmetry we may assume that $\mathcal{D}(X) = \{D\}$. Thus X is a solid handle and we are done. □

Lemma 5.8. *Suppose that $V \neq V_{1,2}$. Then $D \in \mathcal{D}(V)$ is non-separating if and only if there is an $E \in$* $\operatorname{link}(D)$ *so that* $\operatorname{link}(E)$ *is a cone with apex D.* □

Thus being a separating disk is a combinatorial property; we use this fact to characterize pants disks.

Lemma 5.9. *Suppose that $V \neq V_{1,2}$. Then $E \in \mathcal{D}(V)$ is a pants disk if and only if E is separating and* $\operatorname{link}(E)$ *is not a join.* □

We now combinatorially characterize cut systems.

Lemma 5.10. *Suppose that $e(V) \geq 3$. A simplex $\mathbb{D} \in \mathcal{D}(V)$ is a cut system if and only if the following properties hold.*

- *For every pair of disks $D, E \in \operatorname{link}(\mathbb{D})$ the complex $\operatorname{link}(E) \cap \operatorname{link}(\mathbb{D})$ is not a cone with apex D.*
- *For every proper subset $\sigma \subsetneq \mathbb{D}$ there is a pair of disks $D, E \in \operatorname{link}(\sigma)$ so that the complex $\operatorname{link}(E) \cap \operatorname{link}(\sigma)$ is a cone with apex D.*

Proof. The forward direction follows from Lemma 5.8 and the definition of a cut system. (When V is a spotted ball the only cut system is the empty set; the empty set has no proper subsets.)

Now for the backwards direction. From the first property and by Lemma 5.8 deduce that $V' = V - n(\mathbb{D})$ is a collection of spotted balls. If V' has at least two components then there is a proper subset $\sigma \subset \mathbb{D}$

which is a cut system for V. Thus $V - n(\sigma)$ is a spotted ball and this contradicts the second property. $\square$

The next lemma is the handlebody version of [19, Lemma 4.5] and [22, Lemma 2.1].

Lemma 5.11. *When $\xi(V) \geq 2$, the genus and number of spots of V are combinatorially characterized. In general, if V, W are handlebodies with $\mathcal{D}(V) \cong \mathcal{D}(W)$ then either*

- *$V \cong W$ or*
- *$V, W \in \{V_1, V_{1,1}\}$ or*
- *$V, W \in \{V_0, V_{0,1}, V_{0,2}, V_{0,3}\}$.*

Proof. Fix a handlebody $V = V_{g,n}$. When $e(V) \leq 2$ the lemma can be checked case-by-case, following Section 4. If $e(V) \geq 3$ then by Lemma 5.10 cut systems in $\mathcal{D}(V)$ are combinatorially characterized, and thus so is g, the genus of V. Since $\xi(V)$ is the number of vertices in any maximal simplex, we may compute n, the number of spots of V. Thus $\mathcal{D}(V) \cong \mathcal{D}(W)$ implies that $V \cong W$. $\square$

We are now equipped to prove the main lemma.

Proof of Lemma 5.1. Let $V = V_{g,n}$ and choose $\phi \in \mathrm{Aut}(\mathcal{D}(V))$. When $e(V) \leq 2$, Lemma 5.1 can be checked case-by-case, following Section 4. So suppose that $e(V) \geq 3$.

By Lemmas 5.8 and 5.9 the automorphism ϕ preserves the set of non-separating disks and also the set of pants disks.

Suppose that $E \in \mathcal{D}(V)$ is a separating disk yet not a pants disk. Writing $V - n(E) = X \cup Y$ we have $\mathrm{link}(E) = \mathcal{D}(X) * \mathcal{D}(Y)$. By Lemma 5.6 this join is realized uniquely and so we can recover $\mathcal{D}(X)$ and $\mathcal{D}(Y)$. By Lemma 5.11 we may recover the genus and number of spots of X and Y. Thus ϕ preserves the topological type of E. $\square$

6. Regluing

Suppose that $\mathbb{D} = \{D_i\}$ is a cut system for V.

Lemma 6.1. *Suppose that $\phi \in \mathrm{Aut}(\mathcal{D}(V))$ fixes $\mathbb{D}$, vertex-wise. Then there is an element $f \in \mathcal{H}(V)$ such that f and ϕ agree on $\mathbb{D} \cup \mathrm{link}(\mathbb{D})$.*

Proof. Let $V' = V - n(\mathbb{D})$ and let $D_i^{\pm}$ be the two spots of V' that are parallel to D_i in V. By Lemma 2.2 there is a homeomorphism f of V' so that the induced geometric automorphism equals $\phi | \mathrm{link}(\mathbb{D})$. We wish to glue f to get a homeomorphism of V: it suffices to show that for every i the homeomorphism f preserves the spots $D_i^{\pm}$.

Let $\mathrm{handle}_i(\mathbb{D})$ be the set of disks $E \in \mathcal{D}(V)$ so that

- $\operatorname{link}(E)$ is a cone with apex D_i and
- $E \in \operatorname{link}(\mathbb{D})$.

That is, $\operatorname{handle}_i(\mathbb{D})$ is the set of disks in the complement of $\mathbb{D}$ that cut off a solid handle, with meridian D_i.

Let $\operatorname{pants}_i(\mathbb{D})$ be the set of disks $E \in \mathcal{D}(V')$ so that one component of $V' - n(E)$ is a solid pants meeting the spots $D_i^{\pm}$.

By Lemma 5.7, for all i the set $\operatorname{handle}_i(\mathbb{D})$ is combinatorially characterized and so preserved by ϕ_{cut}. It follows, for all i, that the homeomorphism $f \in \operatorname{Homeo}(V')$ preserves the set $\operatorname{pants}_i(\mathbb{D})$. Now fix i for the rest of the proof. Suppose that $f(D_i^+), f(D_i^-) = A, B$ where A, B are spots of V'. Let $E \in \operatorname{pants}_i(\mathbb{D})$ be any pants disk. Then $f(E)$ is a pants disk cutting off A and B. It follows that the spots A, B (in some order) equal the spots $D_i^{\pm}$ as desired. □

7. Duality

Recall that two disks $D, E \in \mathcal{D}(V)$ are *dual* if $i(\partial D, \partial E) = 2$ (see Figure 2.3). A *pentagon* $P \subset \mathcal{D}(V_{0,5})$ is a collection of five disks $P = \{E_i\}_{i=0}^4$ so that E_i and E_{i+1} are disjoint, for all i (modulo five). We say that the disks E_i, E_{i+2} are *non-adjacent* in P, for all i (modulo five).

Lemma 7.1 (pentagon lemma)**.** *Suppose that $V = V_{0,5}$. Two disks $D, E \in \mathcal{D}(V)$ are dual if and only if there is a pentagon P so that $D, E \in P$ and D, E are non-adjacent in P.*

Proof. Recall that $\mathcal{D}(V_{0,5}) \cong \mathcal{C}(S_{0,5})$, by Lemma 2.2. The *pentagon lemma* for $S_{0,5}$ (see [19, Theorem 3.2] or [22, Lemma 4.2]) implies that there is only one pentagon in $\mathcal{D}(V_{0,5})$, up to the action of the handlebody group. □

Lemma 7.2. *Suppose that $V = V_{g,n}$ has $e(V) \geq 3$. Two disks $D, E \in \mathcal{D}(V)$ are dual if and only if there is a simplex $\sigma \in \mathcal{D}(V)$ with*

- *$\operatorname{link}(\sigma) \cong \mathcal{D}(V_{0,5})$,*
- *D, E are non-adjacent in some pentagon of $\operatorname{link}(\sigma)$.*

It follows that every $\phi \in \operatorname{Aut}(\mathcal{D}(V))$ preserves duality. A handlebody $W \subset V$ is *cleanly embedded* if $\partial_+ W \subset \partial_+ V$ and every spot of W either

- is essential and non-peripheral in V or
- is a spot of V.

Proof of Lemma 7.2. Suppose that D, E are dual. Let X be the four-spotted ball containing them. Isotope X to be cleanly embedded. Let $\mathbb{E}$ be a pants decomposition of $V' = V - n(X)$. Now, there is at least one solid pants P in $V' - n(\mathbb{E})$ which has a spot, say F, which is parallel to a spot, F', of X. If not then $e(V) \leq 2$, a contradiction.

Let Z be the twice-spotted ball with spots F and F'. Let $Y = X \cup Z \cup P$. Note that Y is a five-spotted ball containing D and E, our original disks. Isotope Y to be cleanly embedded. Let $\mathbb{E}'$ be any pants decomposition of $V - n(Y)$. Add to $\mathbb{E}'$ any spots of Y which are non-peripheral in V. This then is the desired simplex $\sigma \in \mathcal{D}(V)$. Since D and E are dual the pentagon lemma implies that there is a pentagon in $\mathcal{D}(Y)$ making D, E non-adjacent.

The backwards direction follows from Lemma 5.11 and from the pentagon lemma. $\square$

We now discuss the dual complex. Fix a cut system $\mathbb{D} = \{D_i\}$. Recall that $\mathrm{dual}_i(\mathbb{D})$ is the subcomplex of $\mathcal{D}(V)$ spanned by the disks $E \in \mathcal{D}(V)$ which are dual to D_i and disjoint from D_j for all $j \neq i$.

Define U_i to be the spotted solid torus obtained by cutting V along all disks of $\mathbb{D}$ *except* D_i. Note that U_i has exactly $e(V)$–many spots, and this is at least three. Also, D_i is a meridian disk for U_i. Note that $\mathrm{dual}_i(\mathbb{D}) \subset \mathcal{D}(U_i)$. A disk $E \in \mathrm{dual}_i(\mathbb{D})$ is a *simple dual* if E is a pants disk in U_i.

Let $\mathcal{A}_i(\mathbb{D})$ be the complex where vertices are isotopy classes of arcs $\alpha \subset \partial_+ U_i$ so that

- α meets ∂D_i exactly once, transversely, and
- $\partial\alpha$ meets distinct spots of U_i.

A collection of vertices spans a simplex if they can be realized disjointly.

If an arc $\alpha \in \mathcal{A}_i(\mathbb{D})$ meets spots $A, B \in \partial_0 U_i$ then the frontier of $N(A \cup \alpha \cup B)$ is a simple dual, E_α.

Lemma 7.3. *If* $e(V) \geq 3$ *then the complex* $\mathrm{dual}_i(\mathbb{D})$ *is connected.*

It suffices to check this for $i = 1$. To simplify notation we write $D = D_1$, $U = U_1$, $\mathrm{dual}(D) = \mathrm{dual}_i(\mathbb{D})$ and $\mathcal{A}(D) = \mathcal{A}_1(\mathbb{D})$. Before proving Lemma 7.3 we require a sequence of claims.

Claim. For any pair of arcs $\alpha, \gamma \in \mathcal{A}(D)$ there is a sequence $\{\alpha_k\}_{k=0}^N \subset \mathcal{A}(D)$ so that:

- the arcs α_k, α_{k+1} are disjoint, for all $k < N$,
- $\alpha_0 = \alpha$ and $\alpha_N = \gamma$, and
- there is at most one spot in common between the endpoints of α_k and α_{k+1}, for all $k < N$.

Proof. Fix, for the remainder of the proof, an arc $\beta \in \mathcal{A}(D)$ so that α and β are disjoint and so that the endpoints of α and β share at most one spot. This is possible as U has at least three spots. Define the complexity of γ to be $c(\gamma) = i(\alpha, \gamma) + i(\beta, \gamma)$. Notice if $c(\gamma) = 0$ then we are done: one of the sequences

$$\{\alpha, \gamma\} \quad \text{or} \quad \{\alpha, \beta, \gamma\}$$

has the desired properties.

Now induct on $c(\gamma)$. Suppose, breaking symmetry, that α meets a spot, say $A \in \partial_0 U$, so that $\gamma \cap A = \emptyset$. If $i(\alpha, \gamma) = 0$ then the sequence $\{\alpha, \gamma\}$ has the desired properties. If not, then let x be the point of $\alpha \cap \gamma$ that is closest, along α, to the endpoint $\alpha \cap A$. Let $\alpha' \subset \alpha$ be the subarc connecting x and $\alpha \cap A$. Let N be a regular neighborhood, taken in $\partial_+ U$, of $\gamma \cup \alpha'$. The frontier of N, in $\partial_+ U$, is a union of three arcs: one arc properly isotopic to γ and two more arcs γ', γ''.

The arcs γ' and γ'' are disjoint from γ and satisfy $c(\gamma') + c(\gamma'') \leq c(\gamma) - 1$. Also, since γ' and γ'' each have one endpoint on the spot A the arcs γ' and γ'' have exactly one spot in common with γ. Now, if $\alpha' \cap \partial D = \emptyset$ then one of γ', γ'' meets ∂D once and the other is disjoint. On the other hand, if $\alpha' \cap \partial D \neq \emptyset$ then α' meets ∂D once. Thus one of γ', γ'' meets ∂D once and the other meets ∂D twice. In either case we are done. □

Recall that if $\alpha \in \mathcal{A}(D)$ is an arc then E_α is the associated simple dual.

Claim. If $\alpha, \beta \in \mathcal{A}(D)$ are disjoint arcs, with at most one spot in common between their endpoints, then there is an edge-path in $\mathrm{dual}(D)$ of length at most four between E_α and E_β.

Proof. If α and β share no spots then $\{E_\alpha, E_\beta\}$ is a path of length one. Suppose that α and β share a single spot. Let A, B, C be the three spots that α and β meet, with both meeting C. Let α', β' be the subarcs of α, β connecting C to ∂D. There are two cases: either α' and β' are incident on the same side of ∂D or are incident on opposite sides.

Suppose that α' and β' are incident on the same side of ∂D, as shown on the left side of Figure 7.4. Then α' and β', together with subarcs of ∂C and ∂D bound a disk $\Delta \subset \partial U$. Note that Δ may contain spots, but it meets $A \cup B \cup C$ only along the subarc in ∂C. It follows that the disk F, defined to be the frontier of

$$N((A \cup B \cup C) \cup (\alpha \cup \beta) \cup \Delta),$$

is dual to D. The disk F is also essential as it separates at least three spots from a solid handle. So $\{E_\alpha, F, E_\beta\}$ is the desired path.

Suppose that α' and β' are incident on opposite sides of ∂D, as shown on the right side of Figure 7.4. Let $d \subset \partial D$ be either component of $\partial D - (\alpha \cup \beta)$. Let $\alpha'' = \overline{\alpha - \alpha'}$ and define β'' similarly. Define $\gamma \in \mathcal{A}(D)$

by forming the arc $\alpha'' \cup d \cup \beta''$ and using a proper isotopy of $\partial_+ U$ to make γ transverse to ∂D. Now apply the previous paragraph to the pairs $\{\alpha, \gamma\}$ and $\{\gamma, \beta\}$ to obtain the desired path of length four. $\square$

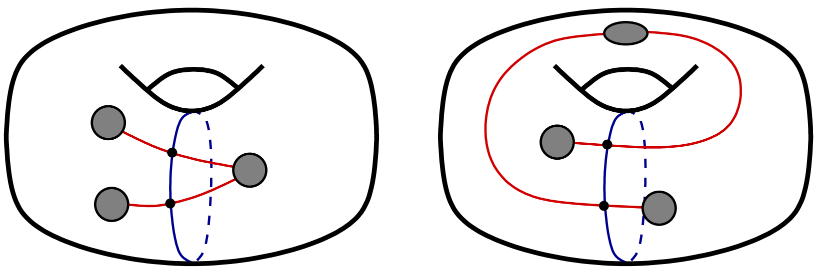

FIGURE 7.4. The two ways that disjoint dual arcs, sharing a spot, can meet the meridian disk D.

Claim. For every dual $E \in \mathrm{dual}(D)$ there is a simple dual connected to E by an edge-path, in $\mathrm{dual}(D)$, of length at most two.

Proof. The dual disk E is either separating or non-separating as shown on the left and right sides of Figure 7.5, respectively. In either case, the graph $\partial E \cup \partial D$ cuts ∂U into a pair of disks B, C and an annulus A. Each of B, C contain at least one spot.

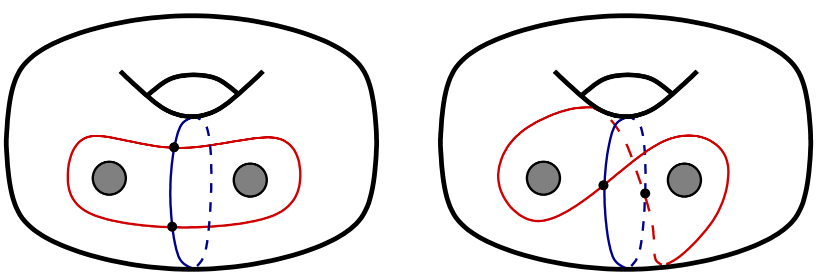

FIGURE 7.5. The two ways that a dual disk E can meet the meridian disk D.

When E is separating the disks B, C are adjacent along an subarc of ∂D. Connect a spot in B to a spot in C by an arc α that meets ∂D once and that is disjoint from ∂E. Thus E_α is disjoint from E.

Suppose E is non-separating, as shown on the right side of Figure 7.5. Then the two disks B, C meet only at the points of $\partial D \cap \partial E$. Now, if the annulus A contains a spot then we may connect a spot in B to a spot in A by an arc α meeting ∂D once and ∂E not at all. In this case we are done as in the previous paragraph.

If A contains no spots then, breaking symmetry, we may assume that B contains at least two spots while C contains at least one. Let $\delta \subset B$ be a simple arc meeting each of $\partial_0 U$ and E in a single endpoint. Let $B' \subset \partial_0 U$ be the spot meeting δ. Let N be a regular neighborhood of $B' \cup \delta \cup E$. Then the frontier of N contains two disks. One of these is isotopic to E while the other, say E', is non-separating, dual to D, and divides the spots as described in the previous paragraph. □

Thus equipped we can prove connectivity of $\mathrm{dual}(D)$.

Proof of Lemma 7.3. The first two claims imply that the set of simple duals in $\mathrm{dual}(D)$ is contained in a connected set. The third claim shows that every vertex in $\mathrm{dual}(D)$ is distance at most two from the set of simple duals. Thus $\mathrm{dual}(D)$ is connected. □

8. Crawling through the complex of duals

Let $\mathbb{D} = \{D_i\}$ be a cut system for V.

Lemma 8.1. *Suppose that $\phi_{\mathrm{link}} \in \mathrm{Aut}(\mathcal{D}(V))$ fixes $\mathbb{D}$ and $\mathrm{link}(\mathbb{D})$, vertex-wise. For any $E \in \mathrm{dual}_i(\mathbb{D})$ the disks E and $\phi(E)$ differ by some power of T_i, the Dehn twist about D_i.*

Proof. As usual, it suffices to prove this for $D = D_1$. Let $U = U_1$, the result of cutting V along all disks of $\mathbb{D}$ except D_1.

Let $X \subset U$ be the four-spotted ball filled by D and the dual disk E. Isotope X to be cleanly embedded. Let $\mathbb{F}$ be the components of $\partial_0 X$ which are not spots of U. Note that ϕ_{link} fixes D as well as every disk of $\mathbb{F}$. This, together with Lemma 7.2, implies that ϕ_{link} preserves the set of disks that are contained in X and dual to D.

Since $\mathcal{D}(X)$ equipped with the duality relation is a copy of $\mathcal{F}$, the Farey graph, it follows that E and $F = \phi_{\mathrm{link}}(E)$ differ by some number of half twists about D. If E and F differ by an odd number of half twists then E and F have different topological types in U, contradicting Lemma 5.1 applied to $\phi_{\mathrm{link}}|\mathcal{D}(U)$. Thus E and F differ by an even number of half twists, as desired. □

Let $\mathbb{E} = \{E_i\}$ be a collection of duals for the cut system $\mathbb{D}$: that is, $E_i \in \mathrm{dual}_i(\mathbb{D})$.

Lemma 8.2. *Suppose that $\phi_{\mathrm{dual}} \in \mathrm{Aut}(\mathcal{D}(V))$ fixes $\mathbb{D}$, $\mathrm{link}(\mathbb{D})$, and $\mathbb{E}$, vertex-wise. Then ϕ_{dual} fixes every vertex of $\mathrm{dual}_i(\mathbb{D})$, for all i.*

Proof. As usual, it suffices to prove this for $D = D_1$. Let $\text{dual}(D) = \text{dual}_1(\mathbb{D})$, let $E = E_1$, and let $U = U_1$. We *crawl* through $\text{dual}(D)$, as follows.

Suppose that $F, G \in \text{dual}(D)$ are adjacent vertices and suppose that $\phi_{\text{dual}}(F) = F$. By Lemma 8.1, the disks G and $G' = \phi_{\text{dual}}(G)$ differ by some number of Dehn twists about D. Also, as ϕ_{dual} is a simplical automorphism the disks F and G' are disjoint. Let X be the four-spotted ball filled by D and F. If G and G' are not equal then $G \cap X$ and $G' \cap X$ are also not equal and in fact differ by some non-zero number of twists; thus one of $G \cap X$ or $G' \cap X$ must cross F, a contradiction.

By hypothesis $\phi_{\text{dual}}(E) = E$. Suppose that G is any vertex of $\text{dual}(D)$. Since $\text{dual}(D)$ is connected (Lemma 7.3) there is a path $\mathcal{P} \subset \text{dual}(D)$ connecting E to G. Induction along $\mathcal{P}$ completes the proof. $\square$

9. Crawling through the disk complex

Before continuing we will need the following complex.

Definition 9.1 (Wajnryb [33])**.** The *cut system graph* $\mathcal{CG}(V)$ is the graph with vertex set being isotopy classes of unordered cut systems in V. Edges are given by pairs of cut systems with $g-1$ disks in common and the remaining pair of disks disjoint.

Theorem 9.2 (Wajnryb [33])**.** *The cut system graph $\mathcal{CG}(V)$ is connected.* $\square$

For the remainder of this section suppose $\Phi = \phi_{\text{dual}}$ is an automorphism of $\mathcal{D}(V)$ and $\mathbb{D}$ is a cut system so that Φ fixes $\mathbb{D}$, $\text{link}(\mathbb{D})$ and $\text{dual}(\mathbb{D})$, vertex-wise.

For the crawling step, suppose that $\mathbb{E}$ and $\mathbb{F}$ are cut systems that are adjacent in $\mathcal{CG}(V)$. Suppose that Φ fixes $\mathbb{E}$, $\text{link}(\mathbb{E})$ and $\text{dual}(\mathbb{E})$ vertex-wise. Let $\mathbb{G}$ be a pants decomposition obtained by adding the new disk of $\mathbb{F}$ to $\mathbb{E}$ and then adding non-separating disks until we have $3g-3+n$ disks. Since $\mathbb{F}, \mathbb{G} \subset \mathbb{E} \cup \text{link}(\mathbb{E})$ it follows that Φ fixes $\mathbb{F}$ and $\mathbb{G}$, vertex-wise.

Define X_i to be the non-pants component of

$$V - n(\mathbb{G} - \{G_i\}).$$

Note that $X_i \cong V_{0,4}$, as all disks in $\mathbb{G}$ are non-separating. Choose $\mathbb{H}, \mathbb{I} = \{H_i\}, \{I_i\}$, collections of disks, so that H_i, I_i are contained in X_i and G_i, H_i, I_i are pairwise dual in X_i. This is possible because $\mathcal{D}(V_{0,4})$, equipped with the duality relation, is a copy of the Farey graph. Note that $\mathbb{G} \cup \mathbb{H} \cup \mathbb{I}$ is contained in $\mathbb{E} \cup \text{link}(\mathbb{E}) \cup \text{dual}(\mathbb{E})$. Thus Φ fixes those disks as well.

Since Φ fixes $\mathbb{F}$ it follows that Φ fixes $\operatorname{link}(\mathbb{F})$, set-wise. By Theorem 2.1 the automorphism $f = \Phi|\operatorname{link}(\mathbb{F})$ is geometric. Let f also denote the given homeomorphism of the spotted ball $V' = V - n(\mathbb{F})$. Let $\mathbb{G}' = \mathbb{G} - \mathbb{F}$ and $\mathbb{H}', \mathbb{I}'$ be the disks of $\mathbb{H}, \mathbb{I}$ contained in V'. Thus f fixes all disks of $\mathbb{G}', \mathbb{H}', \mathbb{I}'$. Let $\{P_k\}$ enumerate the solid pants of $V - n(\mathbb{G})$. It follows that f permutes the solid pants $\{P_k\}$.

If f nontrivially permutes $\{P_k\}$ then, since each G_i is fixed, we find that adjacent solid pants are interchanged. This implies that $V' = P_1 \cup P_2$, contradicting our assumption that $e(V) \geq 3$.

So f fixes every P_k. Since all disks in $\mathbb{G}'$ are fixed, f is either orientation reversing, isotopic to a half twist, or isotopic to the identity on each of the P_k. Let $G_i \in \mathbb{G}'$ be any disk meeting P_k. Then $f|P_k$ cannot be orientation reversing because the triple G_i, H_i, I_i determines an orientation on X_i and hence on P_k. If $f|P_k$ is a half twist then P_k meets two spots of V'. Thus G_i meets two solid pants P_k, P_ℓ so that $X_i = P_k \cup P_\ell$. Now, as $e(V') \geq 3$, the solid pants P_ℓ meets at most one spot of V'. Thus $f|P_\ell$ is isotopic to the identity. So if $f|P_k$ is a half twist then $f(H_i) \neq H_i$, a contradiction. Deduce that f, when restricted to any solid pants, is isotopic to the identity. It follows that f is isotopic to a product of powers of Dehn twists about the disks of $\mathbb{G}'$. As f fixes the disks of $\mathbb{H}'$ these powers are trivial and f is isotopic to the identity on V', as desired. It follows that $\Phi|\operatorname{link}(\mathbb{F}) = \operatorname{Id}$.

As Φ fixes duals to $\mathbb{F}$ by Lemma 8.2 the automorphism Φ restricts to give the identity on $\operatorname{dual}(\mathbb{F})$. This completes the crawling step. Since every non-separating disk lies in some cut system and every separating disk lies in the link of some cut system, deduce that Φ is the identity map. This completes the proof of the main theorem.

Theorem 9.3. *If a handlebody $V = V_{g,n}$ satisfies $e(V) \geq 3$ then all elements of $\operatorname{Aut}(\mathcal{D}(V))$ are geometric.* □

Note that Theorem 9.3 is sharp, as is its consequence Theorem 9.4: when $e(V) \leq 2$ the conclusions are false. See Section 4.

Theorem 9.4. *If a handlebody $V = V_{g,n}$ satisfies $e(V) \geq 3$ then the natural map $\mathcal{H}(V) \to \operatorname{Aut}(\mathcal{D}(V))$ is an isomorphism.*

Proof. Theorem 9.3 shows that the natural map is surjective. Suppose that the mapping class f lies in the kernel. As in the discussion of crawling through $\mathcal{CG}(V)$ given above, let $\mathbb{G} = \{G_i\}$ be a pants decomposition of V so that all of the G_i are non-separating. Let $\{P_k\}$ enumerate the solid pants of this decomposition. Let $X_i = P_j \cup P_k$ be the four-spotted ball containing G_i in its interior. Let $\mathbb{H}, \mathbb{I} = \{H_i\}, \{I_i\}$ be collections of disks so that H_i, I_i are contained in X_i and G_i, H_i, I_i

are pairwise dual in X_i. All of these disks are fixed by f. It follows that f is isotopic to the identity. □

10. Automorphisms of the handlebody group

As with Theorem 1.5, combinatorial rigidity of the disk complex leads to algebraic rigidity of the handlebody group.

Theorem 10.1. *If $e(V) \geq 3$ then the outer automorphism group of the handlebody group is trivial.*

Since $\mathcal{H} = \mathcal{H}(V)$ is centerless this may be restated as $\mathrm{Aut}(\mathcal{H}) \cong \mathcal{H}$. If $g = 0$ then Theorem 10.1 follows from Lemma 2.2 and the first author's thesis [19, Theorem 3]. For the rest of this section we restrict to the case $g \geq 1$ and we assume that $e(V) \geq 3$.

The idea of the proof is to turn an element $\phi \in \mathrm{Aut}(\mathcal{H})$ into an automorphism of the disk complex $\mathcal{D}(V)$. We do this, following [14], by giving an algebraic characterization of Dehn twists inside of $\mathcal{H}$. We then apply Theorem 9.3 to ϕ to find the corresponding geometric automorphism. Using Lemma 10.3 and following [15, Section 4] then gives the desired result.

The following lemmas follow from the identical statements for the mapping class group of a surface [17].

Lemma 10.2. *Suppose D and E are essential disks. The twists T_D, T_E commute if and only if D and E can be made disjoint via ambient isotopy.* □

Lemma 10.3. *For any twist T_D and for any homeomorphism h we have $hT_Dh^{-1} = T_{h(D)}$.* □

Lemma 10.4. *For any pair of disks D and E and any non-zero m and n, if $T_D^m = T_E^n$ then $D = E$ and $m = n$.* □

A finite index subgroup $\Gamma < \mathcal{H}$ is *pure* if every reducible class in Γ fixes every component of every reducing set. For example, the kernel of $\mathcal{H} \to \mathrm{Aut}(H_1(\partial_+ V, \mathbb{Z}/3\mathbb{Z}))$ is pure [14, Theorem 1.2]. An element of a group is *primitive* if it is not a proper power in the group. We write $C(G)$ for the center of a group G. We write $C_G(g)$ for the centralizer of g in G.

Suppose that $\Gamma < \mathcal{H}$ is pure and finite index. Suppose that D is an essential, non-peripheral disk. Let $T = T_D$ be the Dehn twist about D. It follows that there is a positive power n so that T^n lies in Γ. Following Ivanov [14] observe that $C(C_\Gamma(T^n))$ is generated by a power of T when D is not a handle disk. When D is a handle disk (and recalling that $e(V) \geq 3$) the group $C(C_\Gamma(T^n))$ has rank two; the other generator is a power of T_E, where E is the meridian of the solid handle determined by D. Equipped with this we may algebraically characterize Dehn twists on non-handle disks.

Lemma 10.5. *Suppose $\Gamma < \mathcal{H}$ is pure and finite index. Then $\{f_i\}_{i \in I} \subset \mathcal{H}$ is a collection of Dehn twists along a pants decomposition of non-handle disks in V if and only if the following conditions hold.*

- *The subgroup $A = \langle f_i \mid i \in I \rangle$ is free Abelian of rank $\xi(V)$.*
- *Each f_i is primitive in $C_{\mathcal{H}}(A)$.*
- *For all $i \in I$ and all $n \in \mathbb{N} - \{0\}$, if $f_i^n \in \Gamma$ then $C(C_\Gamma(f_i^n)) \cong \mathbb{Z}$.*

Proof. The forwards direction is identical to the forwards direction of [14, Theorems 2.1, 2.2]. The backwards direction is similar in spirit to the backwards direction of [14, Theorem 2.1] but some details differ. Accordingly we sketch the backwards direction.

The mapping class f_i cannot be periodic or pseudo-Anosov as that would contradict the first property. Following [4], let $\Theta \subset S = \partial_+ V$ be the *canonical reduction system* for the Abelian group A. Let $\{X_j\}$ be the components of $S - n(\Theta)$ and let $\{Y_k\}$ be the collection of annuli $N(\Theta)$. By [4, Lemma 3.1(2)] the number of annuli in $\{Y_k\}$ plus the number of non-pants in $\{X_j\}$ equals $\xi(V)$. It follows that every non-pants X_j has complexity one (so is homeomorphic to $S_{0,4}$ or $S_{1,1}$).

Fix a power n (independent of i) to ensure that $f_i^n \in \Gamma$. For each X_j of complexity one there is some f_i^n so that $f_i^n|X_j$ is pseudo-Anosov. Suppose that $f = f_1^n$, $X = X_1$ has complexity one, and $f|X$ is pseudo-Anosov. Let $\lambda^\pm$ be the stable and unstable laminations of $f|X$. For every i, the mapping $f_i^n|X$ is either the identity or pseudo-Anosov. Note that in the latter case the stable and unstable laminations of $f_i^n|X$ agree with $\lambda^\pm$: otherwise a ping-pong argument gives a rank two free group in A, a contradiction. Thus, there are powers $k_i \neq 0$ so that for each i either $f_i^{nk_i}|X$ is the identity or identical to $f^k|X$. (Here $k = k_1$.) For each i where $f_i^{nk_i}|X = f^k|X$ we temporarily replace $f_i^{nk_i}$ by $f_i^{nk_i} f^{-k}$.

Continuing in this manner we obtain a collection of at least $|\Theta|$–many elements in $A \cap \Gamma$ that are supported inside of the union of annuli $\{Y_k\}$. Let $B < A \cap \Gamma$ be the free Abelian group containing them; note that B has rank at least $|\Theta|$. Since B is pure, it follows that all elements of B are compositions of powers of Dehn twists along disjoint curves. It follows that B has rank exactly $|\Theta|$. Oertel [28, Theorem 1.11] and McCullough [27, Theorem 1] prove that every curve in Θ either bounds a disk or cobounds an annulus with some other curve of Θ. However, each annulus reduces the possible rank of B by one; it follows that every curve in Θ bounds a disk.

Let γ be any essential non-peripheral component of ∂X. It follows that f^k commutes with T_γ, that T_γ lies in $\mathcal{H}$ by the above paragraph, and that T_γ to some power lies in $C(C_\Gamma(f^k))$. Deduce that $C(C_\Gamma(f^k))$ has rank at least two, contradicting the third property. It follows that every component X_j is a pants and that $|\Theta| = \xi(V)$. Thus every f_i^n is a composition of powers of disjoint twists. Again, by the third property each f_i is some power of a single twist. By the second property (following [14]) f_i is in fact a twist about a disk D_i. Finally, if D_i is a handle disk then $C(C_\Gamma(f_i^n))$ is Abelian of rank two, contradicting the third property. □

Corollary 10.6. *With the notation of Lemma 10.5: all the disks are non-separating if and only if all of the f_i are conjugate.* □

It remains to give an algebraic characterization of Dehn twists about handle disks. Let $\xi = \xi(V)$.

Lemma 10.7. *An element $g \in \mathcal{H}$ is a Dehn twist on a handle disk G if and only if there is a collection $\mathbb{D} = \{D_i\}_{i=1}^{\xi-1}$ of non-handle disks and another pair of non-separating disks E and F with the following properties. (Below $T_i = T_{D_i}$.)*

- *$\mathbb{D} \cup \{E\}$ and $\mathbb{D} \cup \{F\}$ are pants decompositions.*
- *g commutes with T_i for all i.*
- *No pair from $\{T_E, T_F, g\}$ commutes.*
- *For any pure $\Gamma < \mathcal{H}$ there are powers m, n so that g^n and T_1^m generate $C(C_\Gamma(g^n))$.*
- *The lantern relation* [7, page 333] *is satisfied:*

$$T_E T_F \cdot g = T_1^2 T_2 T_3.$$

In the previous line, when $V = V_{1,3}$, the twist T_3 is omitted.

Proof. We first sketch the forward direction. Suppose that $g = T_G$ is a twist on a handle disk. Let $X \cup Y = V - n(G)$, where X is a solid handle with meridian disk D_1; so $e(Y) \geq 2$. It follows that Y has a solid pants decomposition $\{D_i\}_{i=2}^{\xi-1}$ where none of the D_i are handle disks in V. Let $\mathbb{D} = \{D_1\} \cup \{D_i\}$. Let Z be the non-pants component of $V - n(\mathbb{D})$. Thus $Z \cong V_{0,4}$ is a four-spotted ball that contains G. Choose disks E and F in Z that, together with G, form a Farey triangle of the correct orientation. Note that E and F are necessarily non-separating. Also, T_E, T_F, T_G satisfy a lantern relation.

Now we sketch the backwards direction. Suppose that $\mathbb{D}$, E, F and g are given, with the properties listed. Define Z to be the non-pants component of $V - n(\mathbb{D})$. Again Z must be a four-spotted ball (and not a solid handle) as the twists T_E and T_F do not commute. As in

Lemma 10.5 the algebraic hypotheses imply that g is a product of powers of at most two disjoint Dehn twists. These twists are along a disk G inside of Z and a disk D' outside of Z. Let $T' = T_{D'}$. Since $\mathbb{D}$ gives a pants decomposition of the complement of Z, and since g commutes with all of the T_i, deduce that T' agrees with one of the T_i. Since T_1^m lies in $C(C_\Gamma(g^n))$ it follows that $D' = D_1$. Since the center of the centralizer has rank two deduce that G is a handle disk and D_1 is the non-separating meridian of the corresponding solid handle. Thus there are integers r, s so that

$$g = T_G^r T_1^s.$$

We now use the hypothesis of the lantern relation.

$$T_1^2 T_2 T_3 = T_E T_F \cdot g = T_E T_F T_G^r T_1^s$$

so

$$T_E T_F = T_G^{-r} T_1^{2-s} T_2 T_3.$$

Recall that there is a natural map $\mathcal{H}(V) \to \mathcal{MCG}(\partial_+ V)$ taking a twist on a disk to a twist on its boundary. Thus, the above equality gives a relation among Dehn twists in $\mathcal{MCG}(\partial_+ V)$. According to Margalit [23] the right-hand side is a *multi-twist* and so, by Theorem 1 from [23], we find $r = 1$ and $s = 0$. Thus $g = T_G$ is a Dehn twist on a handle disk, as desired. $\square$

The proof of Theorem 10.1 now follows, essentially line-by-line, the proof of either [15, Theorem 2] or [19, Theorem 3]. $\square$

To extend our algebraic characterization of twists inside of $\mathcal{H}(V)$ (Lemma 10.5, Corollary 10.6 and Lemma 10.7) to a characterization of powers of twists inside of finite index pure subgroups $\Gamma < \mathcal{H}(V)$ appears to be a delicate matter. Following Ivanov [15], resolving this would answer the following.

Problem 10.8. Show that the abstract commensurator of $\mathcal{H}(V)$ is isomorphic to $\mathcal{H}(V)$. Show that $\mathcal{H}(V)$ is not arithmetic.

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