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The pants complex has only one end

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1. De£nitions and statement of the main theorem

The purpose of this note is to prove the following theorem:

Theorem 4.1. *Let S be a closed, connected, orientable surface with genus $g(S) \geq 3$. Then the pants complex of S has only one end. In fact, there are constants $K = K(S)$ and $M = M(S)$ so that: if $R > M$, and P and Q are pants decompositions at distance greater than KR from a basepoint, then P and Q may be connected by a path which remains at least distance R from the basepoint.*

A *pants decomposition* of S consists of $3g(S) - 3$ disjoint essential non-parallel simple closed curves on S. Each component of the complement of the curves is a three-holed sphere; a *pants*. Then the *pants complex* $\mathscr{P}(S)$ is the metric graph whose vertices are pants decompositions of S, up to isotopy. Two vertices P, P' are connected by an edge of length one if P, P' differ by an *elementary move*. In an elementary move all curves of the pants are £xed except for one curve α. Remove α and let V be the component of the complement of the remaining curves which is not a pants. Then V contains α and is either a once-holed torus or a four-holed sphere. Now α is replaced by any curve β contained in V that intersects α minimally; in the torus case once, and in the sphere case twice. All edges of $\mathscr{P}(S)$ are assigned length 1. We let $d(\cdot,\cdot)$ be the distance function in $\mathscr{P}(S)$. The pants complex $\mathscr{P}(S)$ is known to be connected [HT80].

A path metric space (X, d) has *one end* if for any basepoint $O \in X$ and any radius R the complement of $B_R = B_R(O)$, the ball of radius R centered at O, has only one unbounded component. It is easy to see that the de£nition does not depend on the choice of the point O. Clearly having one end is a quasi-isometry invariant of path metric spaces. So, following Brock [Br03] (see also [Br02]) and Wolpert [Wo87], our theorem implies:

Corollary 1.1. *Fix S a closed, connected, orientable surface with genus three or higher. The Teichmüller space of S, equipped with the Weil-Peterson metric, has only one end.*

Finally, recall that the *curve complex* $\mathscr{C}(S)$ is the complex whose k-simplices consist of $k+1$ distinct isotopy classes of essential simple closed curves on S that have disjoint representatives on S. Or, in the case of a once-holed torus and four-holed sphere, $\mathscr{C}(S)$ is the Farey graph. From the metric point of view we will only be interested in the 1-skeleton of $\mathscr{C}(S)$. Each edge is assigned length 1. We let $d_S(\cdot,\cdot)$ denote the distance function in $\mathscr{C}(S)$.

2. The set of handle curves is connected

In this section we prove two combinatorial facts. First, the set of handle curves in the curve complex is connected and second, any pants decomposition is a bounded distance (in the pants complex) from a decomposition containing a handle curve.

Again assume S is a closed, connected, orientable surface with genus three or greater. We will call α a *handle curve* in S if α separates S into two surfaces: the once-holed torus $S(\alpha)$ and the rest of the surface.

We will need the following result. It was £rst proved by Farb and Ivanov [FI03] by different methods. Another proof has been given by McCarthy and Vautaw [MV03] by methods similar to ours. We include a proof for completeness.

Proposition 2.1. *If $g(S) \geq 3$, the subcomplex $\mathscr{H}(S) \subset \mathscr{C}(S)$ of handle curves is connected.*

Remark 2.2. Note that the hypothesis $g(S) \geq 3$ cannot be removed; it is easy to check that $\mathscr{H}(S)$, when S has genus 2, is an in£nite collection of points.

Remark 2.3. Note that Proposition 2.1 immediately implies that the set of separating curves in $\mathscr{C}(S)$ is also connected.

Remark 2.4. Our proof of Proposition 2.1 generalizes to the case $\partial S \neq \emptyset$. An interesting open question is the higher connectivity of $\mathscr{H}(S)$.

Before we begin the proof we will require a bit of terminology. Let $i(\cdot,\cdot)$ denote the geometric intersection number of two essential simple closed curves in S. Also, if δ is a separating curve in S we say that an arc β' is a *wave* for δ if $\beta' \cap \delta = \partial\beta'$ and β' is essential as a properly embedded arc in $S \setminus \delta$. We say that two waves β' and β'' for δ *link* if $\beta' \cap \beta'' = \emptyset$, both β' and β'' meet the same side of δ, and $\partial\beta'$ separates $\partial\beta''$ inside δ. Figure 1 shows a pair of linking waves.

Finally we de£ne *double surgery* as follows. Suppose we are given a linking pair of waves β' and β'' for an essential separating curve δ. Form the closed regular neighborhood $U = \text{neigh}(\delta \cup \beta' \cup \beta'')$. Let δ' be the component of ∂U which is not

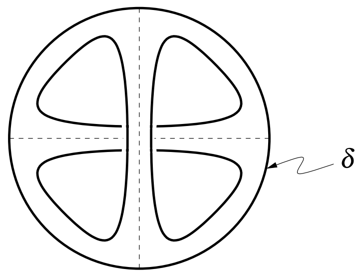

Figure 1: The dotted lines are β' and β''.

homotopic to δ. We say that δ' is obtained from δ via double surgery along β' and β''. Again, see Figure 1 for a picture of δ, δ', and U. Note that δ' is necessarily a separating curve and is disjoint from δ. Furthermore, the curves δ and δ' cobound a two-holed torus. We deduce that δ' is also essential as long as the component of $S \setminus \delta$ containing $\beta' \cup \beta''$ is not a handle.

We are now equipped to prove the proposition:

Proof of Proposition 2.1. Let $\alpha, \beta \in \mathscr{H}(S)$ be handle curves and S a closed orientable surface of genus at least three. Suppose that α and β are *tight*: α has been isotoped to make $|\alpha \cap \beta| = i(\alpha, \beta)$. If $i(\alpha, \beta) = 0$ then there is nothing to prove. If $i(\alpha, \beta) > 0$ we will £nd a curve $\gamma \in \mathscr{H}(S)$ with $i(\gamma, \alpha) = 0$ and $i(\gamma, \beta) < i(\alpha, \beta)$. By induction, γ will be connected to β in $\mathscr{H}(S)$, proving the proposition.

We £nd γ via the following inductive procedure. Recall that $S(\alpha)$ is the handle which α bounds. To begin, we de£ne $\delta_0 \subset S \setminus S(\alpha)$ to be a parallel copy of α, still intersecting β tightly. At stage k by induction we will be given an essential separating curve δ_k where

- $i(\alpha, \delta_k) = 0$,
- δ_k is tight with respect to β, and
- $i(\delta_k, \beta) < i(\delta_{k-1}, \beta)$, if $k > 0$.

Let T_k be the component of $S \setminus \delta_k$ which does not contain α. If T_k is a handle, then we take $\gamma = \delta_k$ and we are done with the inductive procedure. If $i(\delta_k, \beta) = 0$ then we may take γ to be any handle curve inside T_k. As this γ satis£es $i(\alpha, \gamma) = i(\beta, \gamma) = 0$ £nding γ would £nish the proposition. From now on we assume that T_k is not a handle and that $i(\delta_k, \beta) > 0$.

We now attempt to do a double surgery of δ_k into T_k. Either we will £nd γ directly or the curve resulting from double surgery, δ_{k+1}, will satisfy the induction hypothesis.

As the geometric intersection with β is always decreasing, this procedure will stop after £nitely many steps yielding the desired handle curve.

So all that remains is to do the double surgery. Suppose for the moment that $\beta', \beta'' \subset \beta \cap T_k$ are linking waves for δ_k. As described above we may form δ_{k+1} via a double surgery along β' and β''. Isotope δ_{k+1}, in the complement of δ_k, to be tight with respect to β. As noted in the de£nition of double surgery, δ_{k+1} is an essential separating curve which is disjoint from α. Finally note that $i(\delta_{k+1}, \beta) \leq i(\delta_k, \beta) - 4$. Thus all of the induction hypotheses are satis£ed.

Suppose now that we cannot £nd linking waves among the arcs $\beta \cap T_k$. Choose instead an *outermost* wave $\beta' \subset \beta \cap T_k$: that is, there exists an arc $\delta'_k \subset \delta_k$ such that $\delta'_k \cap \beta = \partial \delta'_k = \partial \beta'$. See Figure 2.

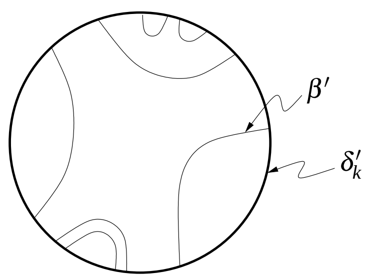

Figure 2: The arc β' is an outermost wave.

Here there are two remaining cases. If $\delta'_k \cup \beta'$ is a separating curve take $\delta_{k+1} = \delta'_k \cup \beta'$ and note that the induction hypotheses are easily veri£ed. The £nal possibility is that $\delta'_k \cup \beta'$ is not separating. See Figure 3.

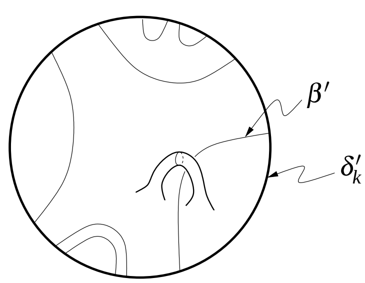

Figure 3: The curve $\beta' \cup \delta'_k$ is nonseparating.

Since $\delta'_k \cup \beta'$ is not separating there exists a properly embedded essential arc $\beta'' \subset T_k$ such that $\beta'' \cap \beta = \emptyset$ and $|\beta'' \cap \delta'_k| = 1$. Then β' and β'' link. Do a double surgery along these waves to obtain δ_{k+1}. Isotope δ_{k+1}, in the complement of δ_k, to be tight with respect to β. Again, all of the induction hypotheses are easily veri£ed, as we have $i(\delta_{k+1}, \beta) \leq i(\delta_k, \beta) - 2$. This completes the second induction step and hence completes the proof of Proposition 2.1. □

We also require

Lemma 2.5. *There is a constant $M = M(S)$ such that the pants decompositions containing a handle curve are M-dense in the space of all pants decompositions*

Proof. The mapping class group acts co-compactly on the space of pants decompositions. □

3. Subsurface projections and distances

Here we set forth three lemmas studying the pants complex. The £rst is a slight re£nement of an idea of Masur and Minsky [MM00], the second gives a condition for a pants decomposition to lie outside of a large ball about the origin in $\mathscr{P}(S)$, and the third provides us with useful paths lying outside of such a ball.

Fix attention on a subsurface $W \subset S$ which has ∂W essential in S and which is not an annulus or a pants (a three-holed sphere). Suppose that γ is an essential simple closed curve in S which is not isotopic to a boundary component of ∂W. Suppose further that γ either lies in the interior of W or has non-zero geometric intersection with ∂W. Isotope γ to be tight with respect to ∂W.

We brie¤y de£ne the *subsurface projection* $\pi_W(\gamma)$ (see [MM00] for a more through discussion). If $\gamma \subset W$ then set $\pi_W(\gamma) = \gamma$. If not, then for every arc $\alpha \subset \gamma \cap W$ take every curve of $\partial(\text{neigh}(\alpha \cup \partial W))$ which is not isotopic into ∂W. Let $\pi_W(\gamma)$ be this set of curves and note that $\pi_W(\gamma) \subset \mathscr{C}(W)$ has diameter at most 2 ([MM00] Lemma 2.3).

Similarly, given a pants decomposition P we may project each curve of P into W. We denote the resulting image $\pi_W(P) \subset \mathscr{C}(W)$. This again has diameter at most 2. By $d_W(P, P')$ we mean the distance in the curve complex of W between the sets $\pi_W(P)$ and $\pi_W(P')$.

Let $[x]_C$ be the function on $\mathbb{N}$ giving zero if $x < C$ and giving x if $x \geq C$. We will need the following result from [MM00] (see Theorem 6.12 and Section 8 of that paper):

Lemma 3.1. *There is a constant $C_0 = C_0(S) \geq 1$ such that for any $D \geq C_0$ there are constants $\lambda_1 = \lambda_1(D) \geq D$ and $\varepsilon_1 = \varepsilon_1(D) \geq 0$ with the following property: for any pants decompositions P and P' we have*

$$\frac{1}{\lambda_1} \sum_V [d_V(P, P')]_D - \varepsilon_1 \leq d(P, P') \leq \lambda_1 \sum_V [d_V(P, P')]_D + \varepsilon_1, \tag{3.1}$$

where the sums range over subsurfaces $V \subset S$ with essential boundary and where V is neither an annulus nor a pants. □

We have a £nal de£nition: Choose a basepoint $O \in \mathscr{P}(S)$. Fix $R > 0$. A curve $\alpha'' \subset S$ is *R-distant* from the basepoint O if, for any pants decomposition P containing α'', we have $d(P,O) > R$.

Fix now $C > \max\{2, C_0\}$. Let $\lambda_1 = \lambda_1(C) \geq C$ and $\varepsilon_1 = \varepsilon_1(C) \geq 0$ as in Lemma 3.1.

Lemma 3.2. *Fix $R > 1$. Fix a handle curve α and some curve $\alpha'' \subset S(\alpha)$ satisfying $d_{S(\alpha)}(O, \alpha'') > \lambda_1(R + \varepsilon_1) + 2$. Then α'' is R-distant from O.*

Proof. Since $\lambda_1 > C$ and $R > 1$ we have $d_{S(\alpha)}(O, \alpha'') \geq C$. Fix any pants decomposition P containing α''. As $\pi_{S(\alpha)}(P)$ has diameter at most two we have

$$d_{S(\alpha)}(P,O) \geq d_{S(\alpha)}(\alpha'', O) - 2.$$

So, by the left inequality of Equation 3.1 we have

$$d(P,O) \geq \frac{1}{\lambda_1}[d_{S(\alpha)}(P,O)]_C - \varepsilon_1 = \frac{1}{\lambda_1} d_{S(\alpha)}(P,O) - \varepsilon_1 \geq \frac{1}{\lambda_1}(d_{S(\alpha)}(\alpha'', O) - 2) - \varepsilon_1 > R.$$

□

As the Farey graph for $S(\alpha)$ has in£nite diameter, and as the diameter of $\pi_{S(\alpha)}(O)$ is bounded, such curves α'' exist in abundance. We now turn to the existence of paths lying outside of the R-ball about the basepoint. As a bit of notation let $B_R = B_R(O)$ be the ball of radius R centered at the basepoint O.

Note that it follows from Equation 3.1 that projections of size exactly C or $C+1$ cannot account for the entire pants distance between P and P'. Namely there are constants $\lambda_2 = \lambda_2(C) > 1$ and $\varepsilon_2 = \varepsilon_2(C) > 0$ such that

$$\sum_V [d_V(P,P')]_{C+2} \leq \sum_V [d_V(P,P')]_C \leq \lambda_2 \sum_V [d_V(P,P')]_{C+2} + \varepsilon_2. \tag{3.2}$$

Choose $K = K(C) > 0$ so that for all $R \geq 1$,

$$\frac{1}{2\lambda_2\lambda_1}\left((K-1)R - \varepsilon_1 - \lambda_1\varepsilon_2 - 2\lambda_2\lambda_1 - \lambda_2\lambda_1^2(R+\varepsilon_1)\right) > \lambda_1(R+\varepsilon_1) \tag{3.3}$$

Lemma 3.3. *Suppose P_0 is a pants decomposition of S such that $P_0 \notin B_{(K-1)R}$ and P_0 contains a curve α which bounds a handle $S(\alpha)$. Then there is a pants decomposition P_1 and a path $\{P_{i/n}\}_{i=0}^n$ so that*

- $P_{0/n} = P_0$, $P_{n/n} = P_1$,
- $P_{i/n}$ *differs from* $P_{(i+1)/n}$ *by a single elementary move,*
- *for all* i, $P_0|(S \setminus S(\alpha)) = P_{i/n}|(S \setminus S(\alpha))$,
- *for all* i, $P_{i/n} \notin B_R$
- *The curve* $\alpha'' = P_1 \cap S(\alpha)$ *is R-distant from O.*

Proof. Let $\alpha' \in P_{0/n} = P_0$ be the curve contained in $S(\alpha)$. Consider a geodesic segment in the Farey graph connecting α' to $\beta \in \pi_{S(\alpha)}(O)$, where β is chosen as close as possible to α'. Extend this segment through α' to a geodesic ray L in the direction opposite β. The ray L meets the segment only at α'. Move along L distance more than $\lambda_1(R+\varepsilon_1)+2$ from α' to a point α''. Let $P_{i/n}$ be the path obtained by making elementary moves along the curves in L and £xing the pants in $S \setminus S(\alpha)$. This path has all of the desired properties except perhaps the fourth. (The £fth follows from Lemma 3.2). It remains to show that $P_{i/n} \notin B(R)$.

There are two cases. Suppose £rst that $d_{S(\alpha)}(\beta, \alpha') > \lambda_1(R+\varepsilon_1)+2$. Then by Lemma 3.2 for any i the curve $P_{i/n} \cap S(\alpha)$ is R-distant. So $P_{i/n} \notin B(R)$ and Lemma 3.3 holds in this case.

Next suppose that $d_{S(\alpha)}(\beta, \alpha') \leq \lambda_1(R+\varepsilon_1)+2$. Then by Equation 3.1 and the right hand side of Equation 3.2

$$(K-1)R \leq d(P_0, O) \leq \lambda_1 \sum_V [d_V(P_0, O)]_C + \varepsilon_1 \leq$$

$$\leq \lambda_2\lambda_1 \sum_V [d_V(P_0, 0)]_{C+2} + \lambda_1\varepsilon_2 + \varepsilon_1 \leq$$

$$\leq \lambda_2\lambda_1 \sum_{V \neq S(\alpha)} [d_V(P_0, O)]_{C+2} + 2\lambda_2\lambda_1 + \lambda_2\lambda_1^2(R+\varepsilon_1) + \lambda_1\varepsilon_2 + \varepsilon_1.$$

Let V be any subsurface disjoint from $S(\alpha)$. Since $P_{i/n}$ is constant in V, the projection $\pi_V(P_{i/n})$ is constant. Now let V be a subsurface that intersects $S(\alpha)$ or strictly contains $S(\alpha)$. Since $\alpha \in P_{i/n}$, it follows that $\pi_V(P_{i/n})$ contains $\pi_V(\alpha)$. Since each $\pi_V(P_{i/n})$ has diameter at most 2, $d_V(P_{i/n}, O) \geq d_V(P_0, O) - 2$. Thus for any subsurface V not isotopic to $S(\alpha)$, as $C > 2$, we have $[d_V(P_{i/n}, O)]_C \geq \frac{1}{2}[d_V(P_0, O)]_{C+2}$. Thus for all i,

$$\sum_{V \neq S(\alpha)} [d_V(P_{i/n}, O)]_C \geq \frac{1}{2} \sum_{V \neq S(\alpha)} [d_V(P_0, 0)]_{C+2} \geq$$

$$\geq \frac{1}{2\lambda_2\lambda_1}\left((K-1)R - \varepsilon_1 - \lambda_1\varepsilon_2 - 2\lambda_2\lambda_1 - \lambda_2\lambda_1^2(R+\varepsilon_1)\right) > \lambda_1(R+\varepsilon_1),$$

the last inequality following from Equation 3.3. So, by the left-hand side of Equation 3.1, for all i we have $d(P_{i/n}, O) > R$. □

4. Proof of the theorem

Recall the statement:

Theorem 4.1. *Let S be a closed, connected, orientable surface with genus $g(S) \geq 3$. Then the pants complex of S has only one end. In fact, there are constants $K = K(S)$ and $M = M(S)$ so that, if $R > M$, any pants decompositions P and Q, at distance greater than KR from a basepoint, can be connected by a path which remains at least distance R from the basepoint.*

Proof. We take M as de£ned in Section 2 and K as de£ned in Section 3.

Using Lemma 2.5 move P and Q a distance at most $M < R$ to obtain pants decompositions P_0 and Q_0. The lemma gives handle curves $\alpha_P \in P_0$, $\alpha_Q \in Q_0$. Also, $P_0, Q_0 \notin B_{(K-1)R}$.

Apply Lemma 3.3 twice in order to connect P_0 and Q_0 to pants decompositions P_1 and Q_1 satisfying all of the conclusions of the lemma. Let $\alpha''_P \in P_1$ and $\alpha''_Q \in Q_1$ be the R-distant curves lying in the handles $S(\alpha_P)$ and $S(\alpha_Q)$ respectively.

We must now construct a path from P_1 to Q_1. Consider £rst the case where $\alpha_P \neq \alpha_Q$.

Applying Proposition 2.1 we connect α_P to α_Q by a path of handle curves in $\mathscr{H}(S)$. Label these $\{\alpha_i\}_{i=1}^n$ where $\alpha_1 = \alpha_P$, $\alpha_n = \alpha_Q$, and $n > 1$. Note that in this step the hypothesis $g(S) > 2$ is used. Choose, for $i \in \{2, 3, \ldots, n-1\}$, any R-distant curve $\alpha''_i \subset S(\alpha_i)$. This requires Lemma 3.2. Again, for $i \in \{2, 3, \ldots, n-1\}$, extend the pair α_i, α''_i to a pants decomposition P_i. Finally set $P_n = Q_1$.

We connect P_i to P_{i+1} by a path where every pants decomposition in the £rst part of the path contains α_i and α''_i and every pants decomposition in the rest of the path contains α_{i+1} and α''_{i+1}. (This is possible because $\mathscr{P}(S \smallsetminus S(\alpha_i)) \cong \mathscr{P}(S \smallsetminus S(\alpha_{i+1}))$ are connected and because $S(\alpha_i)$ is disjoint from $S(\alpha_{i+1})$.) By Lemma 3.2, this path lies outside of the ball of radius R and we are done.

In the case which remains $\alpha_P = \alpha_Q$. Here there is no need for Proposition 2.1. Instead we choose any handle curve β which is disjoint from α_P. Note that β exists as $g(S) > 2$. Using Lemma 3.2 choose a R-distant $\beta'' \subset S(\beta)$ and extend this to a pants decomposition P_2. Set $P_3 = Q_1$. Connect P_1 to P_2 to P_3 as in the previous paragraph. This completes the proof. □

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