Almost Normal Heegaard Splittings

by

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Abstract

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The study of three-manifolds via their Heegaard splittings was initiated by Poul Heegaard in 1898 in his thesis. Our approach to the subject is through *almost normal surfaces*, as introduced by Hyam Rubinstein [28] and *distance*, as introduced by John Hempel [12].

Among the results presented is a proof that every closed, orientable three-manifold has only finitely many Heegaard splittings with distance greater than 4, a new recognition algorithm for surface bundles over the circle, and a series of results which bound the distance of a splitting in terms of its structure as an almost normal surface.

Professor Andrew Casson
Dissertation Committee Co-Chair

Professor John Stallings
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To my parents,

for all their hard work.

INVENTION

I've done it, I've done it!
Guess what I've done!
Invented a light that plugs into the sun.
The sun is bright enough,
The bulb is strong enough,
But, oh, there's only one thing wrong...

The cord ain't long enough.

By Shel Silverstein

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Acknowledgements

This thesis would not exist without the kindly guidance of my advisor Professor Andrew Casson. Even after one of my many lengthy and often bewildering monologues on a problem he always gave me the benefit of the doubt, thinking carefully on what I had said. He would then tell me the error in my thinking and offer other profitable avenues to explore.

Of many people to whom credit is due I would like to especially thank Daniel Wilkerson, my early and ongoing collaborator, and Peter Brinkmann. Danny Calegari gave me valuable feed-back on an early version of Chapter 4 and maintained an interest in Chapter 6 even when my own attention waned. Without a sequence of illuminating conversations with Kevin Hartshorn and Dave Bachman the theorems of Chapters 3 and 7 would never have been conjectured, much less proved.

I am also grateful for the patient attention of Professors William Jaco and Cameron Gordon and their help in matters mathematical. Professor Morris Hirsch made possible my travel in this last year by granting me Graduate Student Researcher support. The mathematics departments of Yale University, Oklahoma State University, and the University of Texas at Austin generously provided office space and computer access while the students of those departments lent me their ears. I would also like to thank Karen Edwards, my fellow traveler of the past year.

The University of California at Berkeley mathematics department has been my home for the last nine years. Now that it is time to leave the nest I would like to express my appreciation to the faculty, students, and staff of that remarkable place. I never would have learned the beauty of mathematics without my teachers and fellow students there. Also, I would have been cold, unkempt, and hungry without my friends in the University Students Co-operative Association and my good friends Chris, Stephen, and, most of all, Diane.

Chapter 1

Introduction

This thesis studies three-manifolds from the combinatorial point of view. As such, our theorems discuss algorithmic issues and questions of finiteness. We make a special effort to understand, for a fixed three-manifold, the set of Heegaard splittings of that manifold.

In Chapter 2 we lay out the many of the necessary definitions. Almost all of the concepts that we require and which also may be found in the literature are contained within. Material newly introduced in this thesis is, for the most part, contained in the relevant chapter.

Chapter 3 picks up the theme of Heegaard splittings. We prove that a closed orientable three-manifold has only finitely many Heegaard splittings of distance 5 or higher. Along the way the elementary yet important concept of *blocks* is introduced.

Chapter 4 introduces the *tightening sequence*; a restricted version of Haken's normalization procedure. The somewhat technical distinction between these is explained in the introduction to Chapter 4. We should point out that the tightening procedure may be treated as a black box whose input is a reasonably behaved surface S and whose output is a compression body C with $\partial_+ C$ equal to S and $\partial_- C$ being normal. Furthermore, C satisfies several minimality properties used in later chapters.

Pursuing the combinatorial aspect, in Chapter 5 we study how the exchange annuli of a Haken sum lie inside the containing three-manifold. In particular, we make used of the concept of a *strongly irreducible surface homeomorphism*; we show that some fibre of a mapping torus with strongly irreducible monodromy must be a fundamental normal surface.

Delving deeper into the particulars of triangulations, Chapter 6 uses practically every tool introduced thus far to study surface bundle structures. We show that if a closed

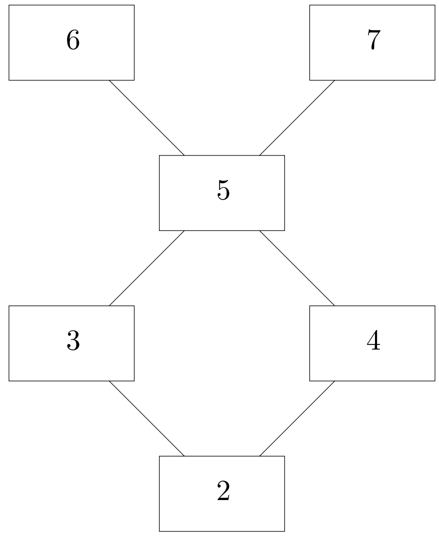


Figure 1.1: The dependencies of the chapters

orientable manifold has a given surface bundle structure $\mathcal{F}$ and a triangulation T then there is a *dicing* collection of normal and almost normal surfaces which are isotopic to fibres of $\mathcal{F}$. Combining this with work of Tollefson and Wang [36] procures a new algorithm which decides whether or not a closed, orientable, irreducible, atoroidal three-manifold is a surface bundle over the circle.

Finally, Chapter 7 again picks up the theme of Heegaard splittings. The goal of this chapter is to provide bounds on the *distance* of a Heegaard splitting provided data about its presentation as a normal or almost normal surface. A representative theorem: There is a constant $d_2 \in \mathbb{N}$ such that if H is an almost normal splitting which is least weight (in a certain sense) and has almost normal piece an annulus then the distance of H is at most d_2 times the number of tetrahedra in the triangulation of M. The theorems in this last chapter should be thought of as progress towards a more effective version of the main theorem of Chapter 3.

Figure 1.1 shows how the chapters depend upon each other. Chapter 7 relies mainly on the ideas of Chapter 4 and does not draw on the theorems in Chapter 3.

Chapter 2

Background Material

This chapter develops the basic tools we require. The expert reader and perhaps even the novice may safely skip most of this material, referring back to it as necessary. Unfortunately, there is no one reference which collects all of the required definitions in one place. The reader should, however, read Section 2.1 in order to familiarize themselves with our notation.

In addition to the underlying material we also reproduce a few of the main theorems in the field as a form of motivation. We assume a basic familiarity with common notions in three-manifold topology, all of which may be found in [13] or [14].

2.1 Notation

This section sets out the various notations in use throughout the rest of the thesis.

M^3 will always denote a three-manifold. The superscript will occasionally be dropped. We use $\eta_A(B)$ ($\overline{\eta}_A(B)$) to denote an open (closed) regular neighborhood of the submanifold B inside of the manifold A.

The notation $X \smallsetminus Y$ denotes the set obtained by intersecting X with the complement of Y. We use $A - B$ to mean $A \smallsetminus \eta_A(B)$; here $A - B$ is the space obtained by *cutting A along B*. Also, define $|A|$ to be the number of connected components of A as a topological space; occasionally we will abuse this notation and apply it to a finite set without topological structure. If $B \subset A$ then $\mathrm{fr}_A(B)$ will denote $\overline{B} \cap \overline{A \smallsetminus B}$; the *frontier of B inside of A*. The subscript is omitted when the containing space is clear from context.

If $F \subset M$ is a properly embedded, 2-sided surface — that is, $|\eta_M(F) - F| = 2$ —

then the transverse orientation is often made explicit in figures by drawing arrows pointing in the appropriate direction. The side of F pointed towards will be called the *tail* side, while the opposite side (pointed away from) will be called *heads*.

A *thickening* of F is a closed, regular, product neighborhood $\iota : F \times I \to M$. The side $\iota(F \times 1)$ is the *heads* side and correspondingly $\iota(F \times 0)$ is the *tails* side. Also, if $\mathcal{F}$ is a map and A is a subset of the domain then $\mathcal{F}|A$ is the restriction of $\mathcal{F}$ to A.

Finally, if α and β are two simple closed curves in a surface F then $i(\alpha, \beta)$ denotes the minimal geometric intersection number of α and β.

2.2 Heegaard splittings

A pleasant introduction to Heegaard splittings, with an extensive collection of references may be found in Scharlemann's survey article [29].

2.2.1 Compression bodies

Definition. A *compression body* V is a three-manifold which has a decomposition as

$$H \times I \bigcup (\cup D_i \times I) \bigcup (\cup B_j)$$

where H is a closed connected orientable surface, the $D_i \times I$ are thickened disks attached to the head of $H \times I$ along $\partial D_i \times I$, and the B_j are three-balls capping off some of the S^2 boundary components of $\partial V \setminus (H \times 0)$.

Note that a compression body need not be irreducible. (For a variant definition see [3].) It is standard to divide the boundary of V into two components, $\partial_+ V = H \times 0$ and $\partial_- V = \partial V \setminus \partial_+ V$, the latter of which may be empty. This gives:

Definition. If V is a compression body with $\partial_- V = \emptyset$ then V is a *handlebody.*

Note that the Euler characteristic of a handlebody, $\chi(V)$, equals $1 - g(H)$ where $g(H)$ is the genus of H.

We are now equipped to define:

Definition. A *Heegaard splitting* of a closed, orientable three-manifold M is a closed, connected, embedded, orientable surface $H \subset M$ such that $M - H$ is a disjoint union of two handlebodies.

For the most part we will choose notation so that $M - H = V \amalg W$. Note that there is a natural identification of the three surfaces H, ∂V, and ∂W because $\eta(H)$ is a product neighborhood.

The *genus* of the splitting is simply the genus $g(H)$ of the splitting surface. The terms "genus of the splitting" and "genus of the handlebody" are used interchangeably in the sequel. It has long been known that every closed three-manifold has a Heegaard splitting — this follows quickly from Moise's Theorem:

Theorem 2.2.1 (Moise [25]). *Every closed three-manifold may be triangulated.*

One then deduces the existence of a Heegaard splitting by taking V to be a closed regular neighborhood of the one-skeleton of a given triangulation and taking $H = \partial V$. See [29]. It follows that M must have a Heegaard splitting of minimal genus. These *minimal splittings* are of special algebraic and topological interest.

2.2.2 Essential disks and irreducibility

This section discusses the various notions of reducibility possible for a Heegaard splitting. This material is standard, except for the definition of *filling* given below.

Recall that a closed three-manifold M is *reducible* if there is a two-sphere in M which does not bound a three-ball. Otherwise M is *irreducible.* Also, M is *toroidal* if M contains an incompressible torus. If M is not toroidal, it is called *atoroidal.* Finally, a closed, orientable, irreducible three-manifold is *Haken* if it contains a two-sided incompressible surface.

Definition. A disk D properly embedded in a manifold M is *essential* if ∂D does not bound a disk in ∂M. If D is properly embedded and not essential then it is *trivial.*

If M is given, together with a Heegaard splitting H, then we set $\mathcal{D}_V = \{$proper isotopy classes of essential disks in $V\}$. $\mathcal{D}_W$ is defined similarly.

Definition. A Heegaard splitting $H \subset M$ is *reducible* if there are disks $A \in \mathcal{D}_V$ and $B \in \mathcal{D}_W$ such that $\partial A = \partial B$. If no such pair exists then H is *irreducible.*

This is a natural definition, given the following lemma of Haken:

Lemma 2.2.2 (Haken [6]). *If M is reducible then every Heegaard splitting of M is reducible.*

Remark 2.2.3. In some sense this is a model theorem for much of the work discussed in this section — a bit of topological information about M yields significant restrictions on the set of all Heegaard splittings of M.

There is another standard definition to be made:

Definition. A Heegaard splitting $H \subset M$ is *stabilized* if there are disks $A \in \mathcal{D}_V$ and $B \in \mathcal{D}_W$ such that $|\partial A \cap \partial B| = 1$.

Note that if M is irreducible then a Heegaard splitting is reducible if and only if it is stabilized. This is a nontrivial result, following from Waldhausen's complete classification of splittings of the three-sphere [37].

Most modern treatments of Heegaard splittings have as their starting point the following definition due to Casson and Gordon [3]:

Definition. A Heegaard splitting $H \subset M$ is *weakly reducible* if there are disks $A \in \mathcal{D}_V$ and $B \in \mathcal{D}_W$ such that $\partial A \cap \partial B = \emptyset$. If no such pair exists then H is *strongly irreducible.*

Strongly irreducible splittings are a natural starting point for the study of non-Haken three-manifolds. This is because of:

Theorem 2.2.4 (Casson, Gordon [3]). *If M is non-Haken then every weakly reducible splitting of M is reducible.*

It is a somewhat surprising fact, shown by combining the work of Casson and Gordon [2] and Parris [27], that there are certain Haken manifolds each admitting infinitely many pairwise non-isotopic strongly irreducible Heegaard splittings. See [23] or [31].

We end this section with Thompson's disjoint curve property:

Definition. A Heegaard splitting $H \subset M$ has the *disjoint curve property* if there are disks $A \in \mathcal{D}_V$ and $B \in \mathcal{D}_W$ and an essential simple closed curve $\gamma \subset H$ such that

$$\partial A \cap \gamma = \gamma \cap \partial B = \emptyset.$$

If no such pair exists then the Heegaard splitting H is *filling.*

The notion of a filling splitting is essentially due to Hempel [12] but this terminology is new. For further discussion of filling splittings see Chapter 3.

Lemma 2.2.5 (Thompson [34], Kobayashi [22]). *If M is toroidal then every Heegaard splitting of M has the disjoint curve property.*

We should also point out:

Theorem 2.2.6 (Hempel [12]). *If M is a Seifert fibred space then every Heegaard splitting of M has the disjoint curve property.*

This follows fairly directly from the remarkable classification theorem of Moriah and Schultens [26].

Remark 2.2.7. It is now a straight-forward observation that if M admits a filling splitting then M is irreducible, atoroidal, and not a Seifert fibred space.

2.3 Distance

This section presents Hempel's notion of the *distance* of a Heegaard splitting (see [12]) and the *translation distance* of a surface homeomorphism (see [24].)

2.3.1 The curve complex

Let H be a closed orientable surface and let $\Delta^0(H)$ be the set of isotopy classes of essential simple closed curves in H. Following Harvey [8] we define the graph $\Delta^1(H)$, the one-skeleton of the *curve complex on H*, which has $\Delta^0(H)$ as its vertex set and has an edge between isotopy classes $[\alpha], [\beta] \in \Delta^0(H)$ if and only if there are $\alpha' \in [\alpha]$ and $\beta' \in [\beta]$ such that $\alpha' \cap \beta' = \emptyset$. We bestow a metric on $\Delta^1(H)$ by assuming that every edge is isometric to the unit interval in $\mathbb{R}^1$.

For the most part the distinction between a curve and its isotopy class will be suppressed.

Definition. The *distance* $d_H(\alpha, \beta)$ between vertices $\alpha, \beta \in \Delta^0(H)$ is the length of the shortest path in $\Delta^1(H)$ connecting α to β.

Lemma 2.3.1 (Hempel [12]). *If H is a closed orientable surface and α, β are vertices of $\Delta^0(H)$ then*

$$d_H(\alpha, \beta) \leq 2\log_2(i(\alpha, \beta)) + 2.$$

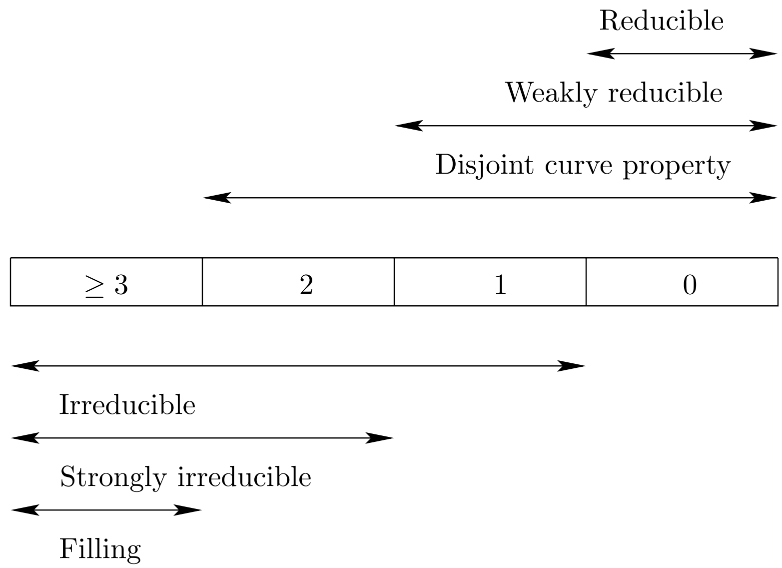


Figure 2.1: The set of Heegaard splittings of M.

2.3.2 Distances of Heegaard splittings

Definition. If $\mathcal{V}$ and $\mathcal{W}$ are subsets of $\Delta^0(H)$ then set

$$d_H(\mathcal{V}, \mathcal{W}) = \min\{d_H(\alpha, \beta) \ | \ \alpha \in \mathcal{V}, \beta \in \mathcal{W}\}.$$

Now take $H \subset M$ a Heegaard splitting and set $\mathcal{V} = \{$essential simple closed curves in H which bound disks in $V\}$. Define $\mathcal{W}$ similarly.

Definition. The *distance* of a Heegaard splitting $H \subset M$ is $d(H) = d_H(\mathcal{V}, \mathcal{W})$.

This definition of Hempel's [12] is a far-reaching extension of the notions introduced in Section 2.2. It is straight-forward to verify:

- H is reducible or stabilized $\iff d(H) = 0$.
- H is weakly reducible $\iff d(H) \leq 1$.
- H has the disjoint curve property $\iff d(H) \leq 2$.

Remark 2.3.2. We may rephrase Remark 2.2.7 as follows: If M is reducible, toroidal, or Seifert fibred then all splittings of M have distance less than or equal to 2.

Remark 2.3.3. Hartshorn, in his thesis [7], gives a result similar in spirit. Let $H_i \subset M$ be a sequence of Heegaard splittings produced via the Casson-Gordon-Parris method. Hartshorn demonstrates that all of these, except perhaps H_0, have distance less than or equal to two.

We end this portion by giving a rough diagram of the "set of all Heegaard splittings" in Figure 2.1. Note that the above remarks are far from the end of the story:

Theorem 2.3.4 (Hartshorn [7]). *If M is a Haken manifold containing an incompressible surface of genus g then every Heegaard splitting $H \subset M$ has $d(H) \leq 2g$.*

Theorem 2.3.5 (Hempel [12], Feng Luo). *For every $n \in \mathbb{N}$ there is a closed orientable manifold M and a Heegaard splitting $H \subset M$ with $d(H) \geq n$.*

2.3.3 Surface bundles and translation distance

Here we collect a few necessary definitions regarding surface bundles over the circle. This material is standard, except for the definition of *strongly irreducible* given below.

Definition. If H is a closed orientable surface then the *mapping class group* of H is the group of isotopy classes of homeomorphisms $h : H \to H$.

We typically will not distinguish between a homeomorphism and the element of the mapping class group which it represents. Note that the mapping class group of H acts on the complex of curves in a natural way: $[h] \cdot [\alpha] = [h(\alpha)]$.

Definition. If h is an element of the mapping class group then the *translation distance of* h is

$$d(h) = \min \left\{ d_H(\gamma, h(\gamma)) \;\mid\; \gamma \in \Delta^0(H) \right\}.$$

Definition. A mapping class element h is *strongly irreducible* if $d(h) \geq 2$. Equivalently, h is strongly irreducible if $\gamma \cap h(\gamma) \neq \emptyset$ for every essential simple closed curve in H.

This simple definition does not, to my knowledge, appear in the literature.

Definition. Suppose that $h : H \to H$ is a homeomorphism. The *mapping torus of* h is the space $H \times I/\sim$ where the equivalence relation is given by $(x, 1) \sim (h(x), 0)$.

Definition. Suppose that M is a three-manifold. A *surface bundle structure* on M is a pair, (h, ϕ), where h is an element of the mapping class group of H and ϕ is a homeomorphism between the mapping torus of h and M.

The mapping h is the *monodromy* and the genus of H is the *genus* of the bundle structure on M. A surface bundle is occasionally presented in the form of a foliation, $\mathcal{F}$, in which all leaves are fibres of the bundle.

For further discussion of strongly irreducible bundle structures on a fixed three-manifold see the last section of Chapter 5.

2.4 Position with respect to a product structure

This section defines *thin* and *bridge* position of one-manifolds properly embedded in a three-manifold. Let C be a three-dimensional submanifold of M, called the *core* of M. Let F be a compact surface and let $K \subset M$ be a properly embedded one-manifold.

Definition. A *product structure* $\mathcal{F}$ on (M, C) is a homeomorphism $\mathcal{F} : F \times I \to (M - C)$.

We refer to $F_r = \mathcal{F}(F \times r)$ as a *level* of the product structure. A fruitful example to have in mind is when M is given together with a Heegaard splitting H. C is then the union of the cores of the two handlebodies V and W and we take $H = F_{1/2} = \mathcal{F}(F \times 1/2)$.

Let $\mathcal{F}$ be a product structure on (M, C). Following [33] we adapt Gabai's notion of *thin position* (see [4]) to our situation. In order to set up the machinery of thin position we must discuss how the given one-manifold meets levels of a product structure.

Definition. The *weight*, $w_K(S)$, of a surface $S \subset M$ is the number of points in $S \cap K$.

Definition. A properly embedded one-manifold K and a product structure $\mathcal{F}$ are *transverse* if the following conditions hold:

1. K is transverse to ∂C.

2. All but finitely many levels of $\mathcal{F}$ are transverse to the one-manifold K.

3. Each nontransverse level S_i, $i \in \{1, \ldots, n\}$, has exactly one singular intersection with K. Furthermore, every such intersection is either a local maximum or a local minimum. "Cubic" intersections are not allowed.

All nontransverse intersections of K with a level of $\mathcal{F}$ look like local maxima or minima. The open submanifold obtained by taking the union of all levels between a maximum immediately above and a minimum immediately below is called a *thick region.*

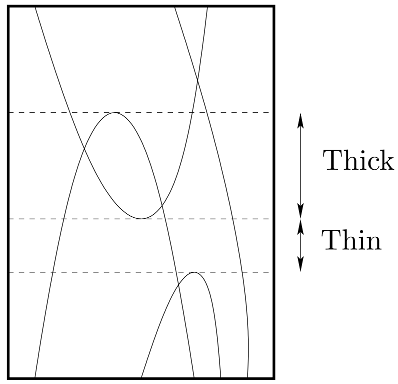


Figure 2.2: Thick and thin regions

Similarly a *thin region* is contained between a minimum above and a maximum below. See Figure 2.2.

The set $\{c \in [0,1] \mid F_c \text{ is not transverse to } K\}$ is the set of *critical points*. Chose now a collection of points in the interval, $R' \subset [0,1]$ such that for each adjacent pair of critical points there is exactly one point of R' between them. Let $R = R' \cup \{0,1\}$.

Definition. The *width* of a given product structure $\mathcal{F}$ with respect to a transverse K is

$$w_K(\mathcal{F}) = \sum_{r \in R} w_K(F_r).$$

Definition. $\mathcal{F}$ realizes *thin position for* K and K is *thin* if K is transverse to $\mathcal{F}$ and $w_K(\mathcal{F}) \leq w_{K'}(\mathcal{F})$ for all K' which are both transverse to $\mathcal{F}$ and ambiently isotopic to K relative to C.

Definition. $\mathcal{F}$ realizes *bridge position* and K is *in bridge position* if K is transverse to $\mathcal{F}$ and $\mathcal{F}$ contains no thin region.

Definition. Suppose that $\mathcal{F}$ is transverse to K and F is a level of $\mathcal{F}$. If D is a disk properly embedded in $M - (K \cup C)$ with $\partial D = \alpha \cup \beta$, α properly embedded in $F - K$, $\beta \subset \partial(\overline{\eta}(K))$, and D is attached to F's head (tail) then D is an *upper (lower)* disk for F. If $D \cap F = \alpha$ then D is a *strict* upper (lower) disk for F.

2.5 Triangulations and normal surfaces

This section develops a few rudiments of normal surface theory. For a more complete treatment see [18] or [33].

2.5.1 Common notions

Given a triangulated manifold (M, T) there is a standard notion of equivalence for submanifolds:

Definition. An isotopy $H : M \times I \to M$ is a *normal isotopy* if $H_r(\sigma) = \sigma$ for all $r \in I$ and for every simplex σ in T.

Two submanifolds in M are *normally isotopic* if there is a normal isotopy taking one to the other.

Let S be a surface properly embedded in a triangulated three-manifold M and assume that S is transverse to the skeleta of M. Denote the i-skeleton of (M, T) by T^i. At this point we overuse notation slightly and again define $w(S) = w_T(S) = |S \cap T^1|$ to be the *weight* of S.

Note that the triangulations considered need not be simplicial. In fact, most of the triangulations discussed in this thesis have a single vertex. Finally, when discussing a simplex in a triangulation, we routinely disregard its structure as a subset of M and treat it as the regular Euclidean simplex of side-length one.

2.5.2 Normal surfaces

Remark 2.5.1. If we wish to study a three-manifold M via the surfaces M contains then it is useful to have a well-behaved set of surfaces. Hopefully this set will include all surfaces of interest. We achieve this goal by picking a cell structure on M. The philosophy of normal surface theory now requires that every surface under consideration intersect cells in the simplest possible fashion.

Fix T, a triangulation of M. Let t be a tetrahedron of T and let f be a triangular face of t. Define a *normal arc* in f to be an arc properly embedded in f with its endpoints in distinct edges of f. A *normal curve* in ∂t is a simple closed curve, embedded in ∂t, which is transverse to the edges of t and whose intersection with each face of t is a collection of normal arcs.

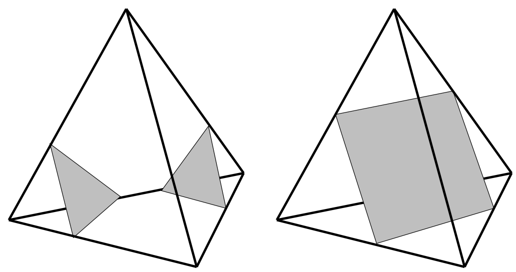


Figure 2.3: Normal disks

Define the *length* of a normal curve to be the number of normal arcs it contains. A normal curve is *short* if its length is four or less. Otherwise it is *long*. Here are a few standard facts about normal curves.

Lemma 2.5.2. *Every normal curve* α *bounds a disk in* ∂t *containing* 1 *or* 2 *vertices. In the former case the length of* α *must be three.*

Lemma 2.5.3. *If a nonempty normal curve* α *does not meet an edge of* ∂t *then* α *is short.*

Lemma 2.5.4. *All long normal curves have length a multiple of* 4*.*

Each normal curve bounds a disk in t. Disks with boundary of length three are *normal triangles* and those with boundary of length four are *normal quadrilaterals* or "normal quads." Collectively the normal triangles and quads are referred to as the *normal disks*. They are illustrated in Figure 2.3.

Definition. A surface S properly embedded in (M, T) is called *normal* if it intersects each tetrahedron of T in a collection of normal triangles and quads.

Definition. A surface S properly immersed in (M, T) is called *immersed normal* if S has no triple points and intersects each tetrahedron of T in an immersed collection of normal triangles and quads.

A reasonable example to bear in mind is the union of a pair of normal surfaces which meet transversely.

One of the main reasons why normal surfaces are interesting is:

Theorem 2.5.5 (Haken [5]). *If* F *in an incompressible and boundary-incompressible surface in* (M, T) *then* F *is isotopic to a normal surface.*

2.6 Haken sum

To state stronger results along these lines we must discuss the elementary linear algebra implicit in the definition of normal surfaces. See [5], [30], or the more recent paper [11].

Fix, for the remainder of this section, a triangulated three-manifold (M, T). To every one of the $7 \cdot |T|$ types of normal disk we now assign a variable, x_i, and thus fix once and for all an order on the normal disk types. Given a normal surface, F, we obtain a vector $v(F)$ of natural numbers, where the i^{th} entry counts the number of normal disks of that type. This is the *coordinate vector of* F.

If F_1 and F_2 are a pair of surfaces such that $v(F_1) + v(F_2) = v(F)$ then F is realized as the *Haken sum* of F_1 and F_2. In this case we write $F = F_1 + F_2$. Any pair of surfaces which have the property that their coordinate vectors may be added to obtain the coordinate vector of another normal surface are called *compatible*.

Definition. A normal surface is *fundamental* if it cannot be realized as a nontrivial Haken sum.

The following lemma is used throughout this thesis:

Lemma 2.6.1. *There is a constant $a_1 \in \mathbb{N}$ such that if (M, T) is a triangulated three-manifold and $S \subset M$ is fundamental then $w(S) < 2^{a_1 \cdot |T|}$.*

It suffices to take $a_1 \geq 16$. See Lemma 6.1 of [11] for a proof of this fact. There is a yet stronger notion of indecomposibility:

Definition. A normal surface F is a *vertex surface* if every positive integer multiple of F only decomposes as a sum of copies of F.

In many theorems it is possible to conclude that a particular surface is not only fundamental, but is in fact a vertex surface. This is a desirable improvement as, given a particular triangulation, there are often strictly fewer vertex surfaces and such are often strictly less complex in their intersection with the one-skeleton. However, in this thesis vertex surfaces will not play a major role.

Lemma 2.6.2. *A triangulated three-manifold contains only finitely many fundamental normal surfaces.*

This follows immediately from Lemma 2.6.1. Nonetheless, Lemma 2.6.2 eventually leads to many finiteness and algorithmic results in three-manifold topology. For example:

Theorem 2.6.3 (Jaco, Oertel [16]). *If M is a Haken manifold, with triangulation T, then there is a 2-sided incompressible surface among either the fundamental surfaces or their doubles.*

This quickly leads to:

Corollary 2.6.4 (Jaco, Oertel [16]). *There is an algorithm to decide whether or not an irreducible three-manifold is Haken.*

To see this one must understand that Haken sum is a geometric operation. Fix (M, T) a closed, triangulated three-manifold.

Definition. Suppose that F and G are normal surfaces in (M, T). F and G intersect *neatly* if they are compatible and the following conditions hold:

1. F and G intersect transversely.
2. $F \cap G$ intersects the two-skeleton of T transversely.
3. G minimizes the lexicographic complexity

$$(|F \cap G'|, |(F \cap G') - T^2|)$$

among all G' normally isotopic to G such that F and G' satisfy the first two conditions.

If F and G intersect neatly then the curves $\Gamma = \{F \cap G\}$ are *regular curves of intersection*. The following lemma is an immediate consequence of our definitions:

Lemma 2.6.5. *If $\alpha \in \Gamma$ is a regular curve of intersection then, for all $t \in T^3$, the connected components of $\alpha \cap t$ are arcs properly embedded in t. Each of these has its two endpoints on distinct faces of t.*

See Figure 2.4 for some of the ways $F \cup G$ may intersect a triangle of the two-skeleton.

Note that there is a natural cut-and-paste operation on each curve in Γ which is completely determined by the intersection of $F \cup G$ with the two-skeleton. These are called *regular exchanges*. See Figure 2.5 and [16]. Note further that if a regular exchange is performed on an intersecting pair of normal disks then another pair of normal disks, of the same type, is obtained.

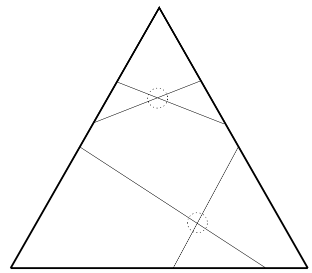


Figure 2.4: Intersecting surfaces

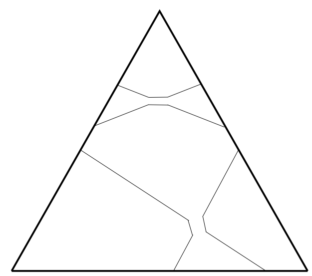


Figure 2.5: Regular exchanges

Lemma 2.6.6. *Suppose the F and G meet neatly. The surface obtained by doing regular exchanges along Γ is normally isotopic to $F + G$.*

Proof. Let H be the surface obtained by doing regular exchanges. It is enough to show that H is normal. This is because $F + G$ is normal and two embedded normal surfaces are normally isotopic if and only if they have the same weight on each edge of the one-skeleton.

Fix a tetrahedron $t \in T^3$. Again, it is enough to show that $t \cap H$ is an embedded collection of normal disks. This is easily done by inducting on the number of normal arcs in $F \cap \partial t$. □

The same proof gives:

Lemma 2.6.7. *Suppose the F and G meet neatly. The surface obtained by doing regular exchanges along $\Gamma' \subset \Gamma$ is an immersed normal surface.*

Carrying out the other cut-and-paste operation is called *performing an irregular exchange* and is depicted in Figure 2.6.

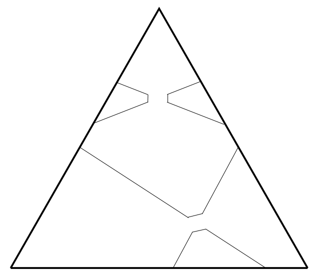


Figure 2.6: Irregular exchanges

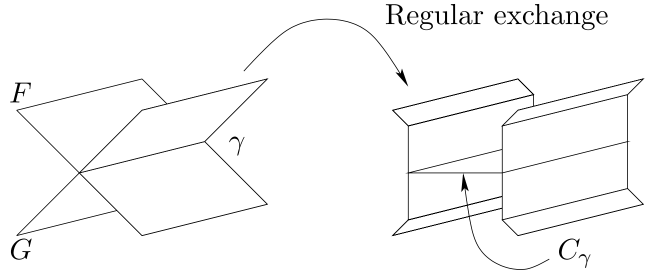


Figure 2.7: The exchange band for γ.

Let γ be a curve of Γ and set $N = \eta_M(\gamma)$, $A = N \cap F$, and $B = N \cap G$. Since N is an I^2-bundle over γ we may assume that $F \cup G$ locally cuts N into four pieces, N_0, N_1, N_2, N_3. Then the $N_i \cap \mathrm{fr}(N)$ are locally the four sides of the square fibre: $I \times 0, 1 \times I, I \times 1$, and $0 \times I$.

Suppose that the regular exchange along γ is obtained by replacing $A \cup B$ by $(0 \times I) \cup (1 \times I)$. After performing the regular exchange there is an *exchange band* C_γ such that C_γ's intersection, locally, with the square fibre is $I \times 1/2$. This band is an I-bundle over γ and thus is either a disk, annulus, or Mobius strip. See Figure 2.7.

Remark 2.6.8. In the case where C_γ is a Mobius band, both A and B are as well. If C_γ is an annulus then both of A and B are annuli. That is, γ always preserves orientation in M regardless of the orientability of M. This follows because F and G are distinct and γ admits a regular exchange.

Definition. If F and G are normal and meet neatly then the curves of

$$\Gamma' = \left\{ \overline{\partial C_\gamma \setminus \partial M} \;\mid\; \gamma \in \Gamma \right\} \subset F + G$$

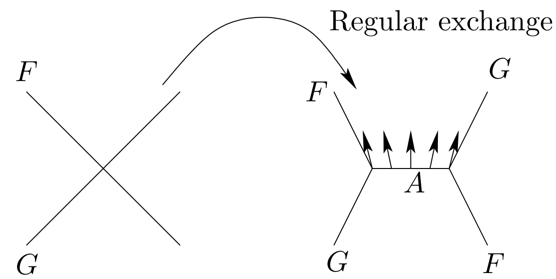


Figure 2.8: Alternation of patches

are called the *seams* (or *trace curves*) of $F + G$.

Definition. The connected components of $(F + G) - \Gamma'$ are called the *patches* of $F + G$.

Remark 2.6.9. Note that the patches of $F + G$ are labeled by the surface, F or G, which the patch comes from and that this labeling alternates as a seam is crossed.

Remark 2.6.10. Notice that if we choose a transverse orientation for a exchange annulus or disk A of $H = F + G$ then we will obtain a induced transverse orientation on $\overline{\partial A \setminus \partial M} \subset H$. This induced orientation points to a patch coming from F along one boundary component while pointing to a patch of G along the other boundary component. See Figure 2.8. This trick will even work for a boundary compressible Mobius strip: Double the Mobius strip to obtain an annulus, A. Transversely orient A so that the orientation points away from the Mobius strip and towards D, the boundary compressing disk. Again, the transverse orientation on the two ends of A point into differently labeled patches. This merely reflects the fact that the boundary of the Mobius strip is two-sided in H and the arc $D \cap H$ connects the two sides.

Remark 2.6.11. It follows from the above discussion that if two normal surfaces F and G can be made disjoint by a normal isotopy then they are compatible. The sum $F + G$ is the disjoint union of F and G if and only if such an isotopy exists.

We end this section with a notion due to Jaco and Rubinstein [17] and, independently, Casson.

Definition. If (M, T) is a closed triangulated three-manifold then we say T is *0-efficient* or simply *efficient* if every normal two-sphere in M is the boundary of a regular neighborhood of a vertex of T^0. Such two-spheres are called *vertex links.*

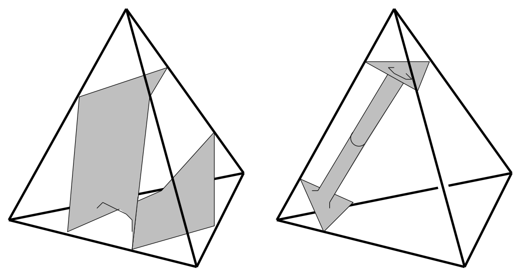


Figure 2.9: Almost normal pieces

Remark 2.6.12. Note that the only orientable prime three-manifolds which do not admit such triangulations are $\mathbb{RP}^3$ and $S^2 \times S^1$. This is due to Jaco and Rubinstein [17] and may also be found in unpublished work of Casson's. Jaco and Sedgwick [19] have further shown that all other lens spaces (including S^3) admit infinitely many efficient triangulations.

Remark 2.6.13. As shown in Lemma 4.1.4 if M is a closed, efficiently triangulated three-manifold with more than one vertex then M is homeomorphic to the three-sphere. Jaco and Segdwick [19] have given an argument to prove that an efficient triangulation of S^3 has at most two vertices; Ben Berton has given an infinite collection of such triangulations of S^3 [15].

2.7 Almost normal surfaces

In order to capture certain behaviors we often allow surfaces to intersect the tetrahedra in a more varied fashion. This leads us to consider the *almost normal surfaces* introduced by Pitts and Rubinstein.

The *almost normal pieces* shown in Figure 2.9 are one of the three *almost normal octagons* and one of twenty-five *almost normal annuli.* The tubes of the almost normal annuli are required to be unknotted and parallel to an edge of the containing tetrahedron.

Definition. A surface properly embedded in M is *almost normal* if it intersects all tetrahedra but one in normal disks and it meets the exceptional tetrahedron in exactly one almost normal piece (and possibly some normal disks.)

Note that the linear algebra of normal surfaces carries through essentially unchanged for almost normal surfaces. We simply add variables for the almost normal pieces.

All of the notions regarding Haken sum and regular exchange are defined identically as above and the same lemmata are obtained.

The usefulness of almost normal surfaces is underlined by three theorems of Rubinstein:

Theorem 2.7.1 (Rubinstein [28], Stocking [32]). *Fix (M, T), a closed orientable triangulated three-manifold. Every strongly irreducible Heegaard splitting of M is isotopic to an almost normal surface.*

Theorem 2.7.2 (Rubinstein [28]). *Fix (M, T), a closed orientable atoroidal triangulated three-manifold and choose a number $g \in \mathbb{N}$. Then there are only finitely many strongly irreducible Heegaard splitting surfaces of genus g.*

Theorem 2.7.3 (Rubinstein [28], Thompson [33]). *There is an algorithm to decide whether or not a given a closed, triangulated three-manifold (M, T) is homeomorphic to the three-sphere.*

Remark 2.7.4. It should be noted that Thompson's proof of Theorem 2.7.3 introduced the idea of placing the one-skeleton of a triangulated three-manifold in thin position with respect to a product structure. Stocking's proof of Theorem 2.7.1 relies heavily on this technique as does our Chapter 6.

Remark 2.7.5. A stronger version of Theorem 2.7.2, which does not rely on normal surface theory, was previously obtained in the Haken case by Johannson [21]. The genus two hyperbolic case has been obtained by Hass [9].

Chapter 3

The Distance Conjecture

This chapter is aimed at obtaining the following theorem:

Theorem 3.0.1. *For every closed orientable three-manifold* M *there is a constant* $b_1(M) \in \mathbb{N}$ *such that if* $H \subset M$ *is a Heegaard splitting with genus greater than* $b_1(M)$ *then* $d(H) \leq 4$.

This somewhat technical result leads directly to:

Theorem 3.0.2. *Let* M *be a closed orientable three-manifold. Then, up to isotopy, there are only finitely many Heegaard splittings of* M *with distance greater than* 4.

Proof. Fix H in M, a Heegaard splitting of distance 5 or higher. Lemma 2.2.5 implies that M is atoroidal. Theorem 3.0.1 implies that the genus of H is less than or equal to $b_1(M)$. But by Theorem 2.7.2 the manifold M contains only finitely many Heegaard splittings, up to isotopy, of any given genus. The conclusion follows. □

As we will see in Chapter 7 more can be said about the distance of a splitting given some information about its almost normal structure. Nonetheless, I cannot currently answer the following:

Question. Can the constant in Theorem 3.0.2 be reduced to 2? That is, does M have only finitely many filling splittings, up to isotopy?

If true this would be sharp; there are Haken manifolds with infinitely many non-isotopic distance two splittings.

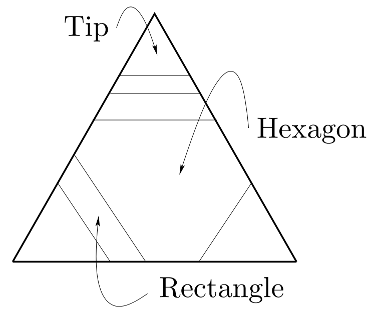


Figure 3.1: The skeletal faces

Remark 3.0.3. Suppose that T is a triangulation of M. The ideas of this chapter yield a rough counting argument which shows that if the genus of H is greater than $172 \cdot |T| + 80$ then H has distance less than 5. This bound on genus can probably be lowered.

Even better would be an answer to:

Question. Must all filling splittings of M be minimal genus splittings?

3.1 Blocks

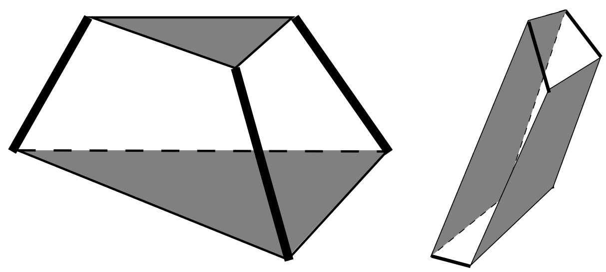
This section develops the definitions required to prove Theorem 3.0.1.

3.1.1 On the blocks

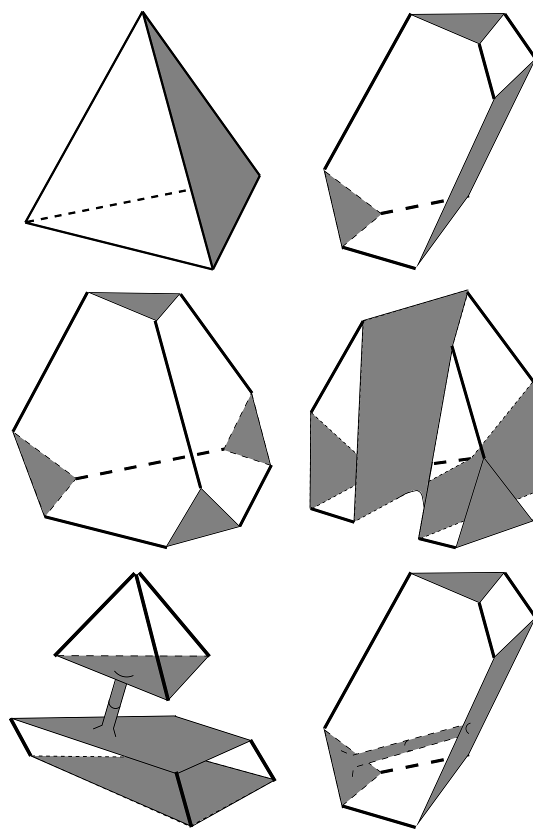
Fix $H \subset (M, T)$ a normal or almost normal surface. Let $f \in T^2$ be a face of the two-skeleton. Then, in general, $f - H$ is divided into three kinds of connected component: Any component meeting a vertex and exactly two edges of f is a *tip*. Any component meeting no vertices and two edges is a *rectangle*. All other components are called *hexagons*. See Figure 3.1. As we often need to discuss collections of these components we give them the general name of *skeletal faces*.

If $t \in T^3$ is a tetrahedron then the possibilities for $t - H$ are more varied:

Definition. A connected component of $t - H$ is called a *block*. If a block, B, is adjacent to exactly two normal disks of H of identical type then B is a *product block*. All other blocks are called *core blocks*.



The product blocks



A few core blocks

Figure 3.2: Some of the possible blocks

See Figure 3.2 for pictures of the two product blocks and six of the 52 possible core blocks. In the figures I have shaded some of the faces — these are the faces which lie in the surface H.

Definition. If B is a block then the faces $B \cap H$ are called the *superficial faces* of B.

A block is *exceptional* if one of its superficial faces is an almost normal octagon or annulus.

Note that the product blocks receive a cell structure in t which is identical to the cell structure on $N \times I$; here N is the corresponding normal disk. Take $\iota : t \to M$ to be the natural map of t into M and take $B \subset t$ to be a product block. It follows that $\iota(B)$ is an I-bundle over $\iota(N \times 1/2)$. Note that $\iota(B)$ is not necessarily a product.

We adopt the following naming convention: any block which is not exceptional, i.e. not adjacent to an almost normal piece of H, and which does not contain a vertex of t is called a *5, 6, 7, or 8-sided block*, depending on the total number of faces of the block. Again, see Figure 3.2. These blocks may be familiar to the reader of Haken's paper [6] where product blocks are referred to as "good regions" and the other non-exceptional blocks are referred to as "bad regions."

3.1.2 Blocked submanifolds

We will need to consider two kinds of submanifold of (M, T).

Definition. A three-dimensional submanifold $V \subset M$ is *blocked* if V is realized as a union of blocks and $\mathrm{fr}_M(V)$ is a normal or almost normal surface.

If X is contained in V, a blocked submanifold, and there exists a union of blocks $\widehat{X} \subset V$ such that $X = \widehat{X} - \mathrm{fr}_V(\widehat{X})$ then X is a *shrunken* submanifold of V. Note that if X is shrunken inside of V then X uniquely determines and is uniquely determined by $\widehat{X}$.

We will use the following lemma heavily:

Lemma 3.1.1. *There is a constant $b_2 \in \mathbb{N}$ such that if V is any blocked submanifold of any (M, T), X is any shrunken submanifold of V, and B' is any block of $\widehat{X}$ then $Y = X - B'$ is also shrunken and $\chi(Y) \leq \chi(X) + b_2$.*

Proof. The submanifold Y is clearly shrunken, as $\widehat{Y} = \overline{\widehat{X} \smallsetminus B'}$ is the required union of blocks.

Let $B = X \cap \overline{\eta}(B')$ and let $A = Y \cap B$. Note that B equals B' after shrinking slightly along the skeletal faces of B' which are in $\mathrm{fr}_V(\widehat{X})$ and expanding slightly across all of B''s other skeletal faces. As $X = Y \cup B$, $A = Y \cap B$, and X, Y, A, and B are all compact and cellular we have $\chi(X) = \chi(Y) + \chi(B) - \chi(A)$. Rearranging gives $\chi(Y) = \chi(X) + \chi(A) - \chi(B)$.

We will show that the quantity $\chi(A) - \chi(B)$ is bounded above by bounding the number of combinatorial types of A and B:

- There are only finitely many possibilities for the block B'.
- There are only finitely many possibilities for the map $\iota : B' \to (M, T)$.
- For every open cell $\sigma \subset B'$ either $\iota(\sigma)$ is in $\mathrm{fr}(\widehat{X})$ or it is not. Also, for each σ, there are only finitely many possibilities for the preimage $\iota^{-1}(\iota(\sigma))$ and the associated gluing data.

These choices determine A and B completely. Since there are only a finite number of possible results, regardless of M, T, V and X, we are done. □

Another constant merits attention: Let $b_3 = \max\{n \in \mathbb{N} \mid n$ is the number of skeletal rectangles of some block $B\}$. As we will see in the proof below, if T is a minimal triangulation of M, it suffices to take $b_1(M)$ greater than $8(5b_2 + b_3 + 16) \cdot |T| + 1$.

3.2 The proof

As a bit of notation, if X is an I-bundle, then let $\pi_X : X \to X/\sim$ be the natural projection map.

If Y is a shrunken submanifold of a blocked manifold V then take $\partial_V Y = Y \cap \mathrm{fr}(V)$; this is the *superficial boundary* of Y. When Y is also an I-bundle then the superficial boundary is referred to as the *horizontal boundary* of Y while $\mathrm{fr}_V(Y)$ is the *vertical boundary*. If α is arc or curve embedded in $Y/\sim$ then $\pi_Y^{-1}(\alpha)$ is a *vertical surface* in Y. All of the connected components of $\partial_V Y \cap \mathrm{fr}_V(Y)$ are called *corners* of Y.

We now are equipped to prove Theorem 3.0.1: that is, we can now show that if the genus of a Heegaard splitting is sufficiently large then the splitting has small distance.

Proof. Fix M, a closed, orientable three-manifold and let T be a minimal triangulation of M. Let H be a Heegaard splitting of M. Note that if H is weakly reducible then H has distance less than 2 and we are done. Assume, therefore, that H is strongly irreducible.

Using Theorem 2.7.1 isotope H to be normal or almost normal with respect to T. Cut along H to obtain $M - H = V \amalg W$, a disjoint union of blocked handlebodies. Recall that $\chi(V) = \chi(W) = 1 - g(H)$.

Remove all of the core blocks from V to obtain the shrunken submanifold V_I which is the "I-bundle part" of V. Define W_I similarly. As $|T| \geq 1$ and V contained at most $6 \cdot |T| + 2$ core blocks $\chi(V_I) \leq \chi(V) + 8 \cdot |T| b_2$ by Lemma 3.1.1. Also, $\mathrm{fr}_V(V_I)$ inherits a cell structure from $\mathrm{fr}_V(\widehat{V_I})$ which is a union of skeletal rectangles. Thus $|\mathrm{fr}_V(V_I)| \leq 8 \cdot |T| b_3$.

V_I is not yet small enough. Remove from V_I all product blocks of $\widehat{V_I}$ which are adjacent to a core block of W; call the shrunken submanifold obtained X. Again, X is produced by removing all product blocks of $\widehat{V_I}$ which are adjacent across a superficial face to a core block.

As each core block has at most four normal superficial faces, by Lemma 3.1.1, $\chi(X) \leq \chi(V_I) + 4(8 \cdot |T|) b_2$. Similarly, as each product block contains at most four skeletal rectangles, $|\mathrm{fr}(X)| \leq |\mathrm{fr}(V_I)| + 16(8 \cdot |T|)$.

Let $Q = X/\!\sim$ be the base space for the I-bundle X. Define Q' to be the surface obtained by capping off all of Q's boundary components with disks. We compute:

$$\begin{aligned}
\chi(Q') &= \chi(Q) + |\partial Q| \\
&= \chi(X) + |\mathrm{fr}(X)| \\
&\leq \chi(V_I) + |\mathrm{fr}(V_I)| + (4b_2 + 16)(8 \cdot |T|) \\
&\leq \chi(V) + (5b_2 + b_3 + 16)(8 \cdot |T|) \\
&\leq 1 - g(H) + (5b_2 + b_3 + 16)(8 \cdot |T|).
\end{aligned}$$

We deduce if $g(H) > 8(5b_2 + b_3 + 16)|T| + 1$ then $\chi(Q')$ is strictly negative. Assume that the genus of H is at least this large. It follows that Q' is non-empty and contains as a subsurface a once-punctured torus $\mathbb{T}^2 - \mathbb{D}^2$. We may assume that this torus lies inside of Q.

To be concrete, let R equal the connected component of Q which contains the punctured torus and let $Y = \pi_X^{-1}(R)$. Note that Y is again a shrunken submanifold of V. Let α and α' be any pair of curves in this torus summand of R such that $|\alpha \cap \alpha'| = 1$. Let $A = \pi_Y^{-1}(\alpha)$ and $A' = \pi_Y^{-1}(\alpha')$ be the vertical annuli lying above α and α'.

Using A' it is easy to deduce that A is not boundary parallel into H and that

∂A is essential in H. That is, A is *disk-like*. See also Lemma 5.1.2. Thus A compresses or boundary compresses in V yielding an essential disk, D_A. Note that $D_A \cap A = \emptyset$. A similar construction gives a disk-like vertical annulus $B \subset W_I$ and a disjoint essential disk $D_B \subset W$.

Let Z be the connected component of W_I which contains B. Let Γ be the set of corners of Z. B is disjoint from the corners of Z because B is vertical inside of Z. The annulus A is disjoint from Γ by the definition of the shrunken submanifold X.

Claim. There is a corner of Z, $\gamma \in \Gamma$, which is essential in H.

Suppose not. Then every $\gamma \in \Gamma$ bounds a disk $D \subset H$. These disks do not contain the horizontal boundary of Z because $\partial_W Z$ is nonplanar.

Note that every component of $\mathrm{fr}_W(Z)$ is a vertical annulus because each such component is two-sided in W. Let C be such a vertical annulus in $\mathrm{fr}_W(Z)$ with $\partial C = \gamma \amalg \gamma'$. Let D, D' be the subdisks of H which γ, γ' bound. Thus $S = D \cup C \cup D'$ is a two-sphere embedded in W. As W is irreducible S bounds a three-ball, $U \subset W$. Note that $U \cap Z = C$ as B cannot be contained in U.

The ball U admits an I-bundle structure over $\mathbb{D}^2$ where $D \cup D'$ is the horizontal boundary, C is the vertical boundary, and the induced fibration of C agrees with the fibration of C coming from the I-bundle structure on Z. Thus $Z \cup U$ is again an I-bundle.

In this way cap off all of the vertical boundary components of Z with I-bundles over $\mathbb{D}^2$. It follows that W is homeomorphic to an I-bundle over a closed surface. This contradiction establishes the claim.

Let $\gamma \in \Gamma$ be an essential corner of Z. Then,

$$D_A \cap A = A \cap \gamma = \gamma \cap B = B \cap D_B = \emptyset.$$

It follows that the distance of H is four or less. □

Remark 3.2.1. It is possible to improve this 4 to a 3 with a finer analysis. I plan to present this stronger theorem in a later paper.

Remark 3.2.2. There is a folk-lore theorem that a closed three-manifold contains only finitely many acylindrical surfaces. A proof of this proceeds along the same lines as the above — however we no longer need worry about producing the annulus B or the curve γ. Theorem 2.7.2 is replaced by a similar result for incompressible surfaces (see Corollary 2.3 of [16].)

Chapter 4

Tightening Almost Normal Surfaces

This chapter investigates a technique for obtaining, from a given almost normal surface, a compression body with many pleasing properties. Our technique, the *tightening sequence*, is a specialization of Haken's normalization procedure to the case of almost normal surfaces.

As a justification of this seeming repetition, note that if the almost normal surface in question is compressible (as in the case of a Heegaard splitting) then we are not so interested in the normal surface obtained via Haken's procedure. Instead, we most likely are interested in how the surface compresses. Also, we wish to control the complexity of the compressing disk. Our main theorem, exactly capturing this notion, is stated in the first section after a technical preliminary.

It should be remarked that our theory has many formal similarities to the theory of *barrier surfaces* and *shrinking* as developed by Jaco and Rubinstein in [17]. They deal with a far wider range of surfaces. However, by restricting ourselves to almost normal surfaces we obtain more sensitive information in the form of the *canonical compression body*. See Section 4.1.1. The interested reader should consult both the chapter that follows as well as their paper.

Finally, I should point out that our Corollary 4.2.2 only became clear to me after early conversations with Professor Jaco on this subject.

4.1 Constructions

4.1.1 Canonical compression bodies

Let $S \subset M$ be a closed transversely oriented surface. Let $\mathcal{V}_S$ be the set of all compression bodies V based over S such that $\partial_- V = \partial V \smallsetminus (S \times 0)$ is normal. If we only consider elements of $\mathcal{V}_S$ up to normal isotopy then there is a natural partial order on the set $\mathcal{V}_S$. Namely, $V \leq V'$ if $V \subset V'$, perhaps after a normal isotopy. Note that the product neighborhood $S \times I$ is only an element of $\mathcal{V}_S$ when S itself is normal.

We may now state the main theorem of this chapter:

Theorem 4.1.1. *Let $S \subset M$ be a transversely oriented almost normal surface. Then the partially ordered set $\mathcal{V}_S$ has a unique minimal element.*

This is proved in Section 4.2.2. However, we may immediately deduce Lemma 1 of [32]:

Lemma 4.1.2. *Suppose that T is a one-vertex triangulation of M, a three-manifold. Let $S \subset M$ be a closed, two-sided, almost normal surface which is incompressible on one side. ($*$) Assume that M is efficiently triangulated and S is not a sphere. Then there is an embedding $\mathcal{F} : S \times I \to M$ with the following properties:*

1. *$\mathcal{F}(S \times 0) = S$.*
2. *$\mathcal{F}(S \times 1)$ is normal.*
3. *$\mathcal{F}(S \times I)$ is on the incompressible side of S.*

Proof. Choose the transverse orientation on S so that it points towards the incompressible side. Let V be the unique minimal element supplied by Theorem 4.1.1. We have $\partial_- V = \{S'$ together with a collection of normal spheres$\}$, where S' is homeomorphic to S and, by efficiency, all of the normal spheres are vertex links. It follows that there is at most one copy of each vertex link.

We may cap off these spheres (if they occur) with regular neighborhoods of the relevant vertices. Note that these neighborhoods do not intersect S' because S' is a not a sphere and thus contains normal quads. Call the capped off compression body V'. V' admits the desired product structure by the incompressibility of S and we are done. □

Remark 4.1.3. The technical assumption $(*)$ in the lemma above may be replaced by the following: M is irreducible and S is not contained in a three-ball which is embedded in M.

At this point we can also show that most efficient triangulations have exactly one vertex.

Lemma 4.1.4. *Suppose that (M, T) is an efficiently triangulated closed three-manifold. If $|T^0| > 1$ then M is homeomorphic to S^3.*

Proof. As M is connected, so is T^1. Suppose that x and y are distinct vertices of T^0 which are connected by an edge, $e \in T^1$. Let S_x and S_y be the vertex links about x and y , respectively. Let t be a tetrahedron adjacent to e. Connect S_x to S_y by an unknotted tube which is parallel to e inside of t. Call the almost normal two-sphere obtained S.

By Theorem 4.1.1, there is a pair of canonical compression bodies, V and W, based on S's head and tail. Each of these is homeomorphic to a three-ball, perhaps with a collection of smaller three-balls removed; i.e. is a "punctured three-ball." Also, the boundary of $V \cup W$ is a collection of vertex links. It follows that $M \cong S^3$. □

Remark 4.1.5. This is a version of Thompson's Lemma 2 in [33].

4.1.2 Non-normal surfaces and surgery

Let S be a surface properly embedded in a triangulated three-manifold M. Assuming that S is transverse to the skeleta of M we will characterize some of the ways S can fail to be normal.

A *bent arc* of S in a triangle $f \in T^2$ is a properly embedded arc of $S \cap f$ with both of its endpoints contained in a single edge of f. Also, a *simple curve* a properly embedded closed curve of $S \cap f$ in the interior of f. Both of these are drawn in Figure 4.1. *Outermost* bent arcs and *innermost* simple curves of S in f are defined in the natural way.

Definition. An embedded disk D is a *tightening disk* for S if $\partial D = \alpha \cup \beta$ where $D \cap S = \alpha$, $D \cap T^1 = \beta$ and β does not meet T^0.

There is a *tightening isotopy* of S across D: Push, via ambient isotopy of S, α along the disk D until we have moved α past β. This procedure reduces $w(S)$ by exactly two. Note that any outermost bent arc of S determines a tightening disk.

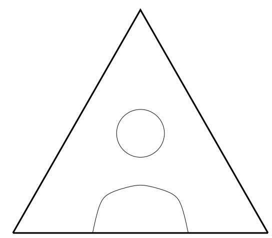


Figure 4.1: Non-normal curves

Definition. An embedded disk D is a *surgery disk* for S if $\partial D = \alpha$ is embedded in S, $D \cap S = \alpha$ and $D \cap T^1 = \emptyset$.

We may *surger* S along D: Remove a small neighborhood of α in S and cap off the boundaries thus created with disjoint, parallel copies of D. Any innermost simple curve of $S \cap T^2$ determines a surgery disk. Note that we do not require α to be essential in S.

Suppose S contains an almost normal octagon. There are two tightening disks on opposite sides of the octagon both giving tightening isotopies of S to a possibly non-normal surface of lesser weight. (A more complete description is given in [33].) These are the *exceptional tightening disks* . If S contains an almost normal annulus then the tube is parallel to at least one edge of the containing tetrahedron. For every such edge there is an exceptional tightening disk. Also, the disk which surgers the almost normal annulus is the *exceptional surgery disk*.

4.1.3 Tightening sequences

In this section we define this chapter's main tool, the *tightening sequence*.

Suppose that S is a transversely orientable almost normal surface with respect to some triangulation of M. We wish to isotope S off of itself while reducing the weight of S as efficiently as possible. We will later analyze the tracks of these isotopies and show that they yield bridge position for part of the one-skeleton and give rise to "nice" blockedsubmanifolds.

As S has at most one exceptional surgery disk choose a transverse orientation for S which points towards an exceptional tightening disk, D.

We construct a *tightening map*:

1. Thicken S to obtain $\mathcal{F}_0 : S \times I \to M$. Note that $\mathcal{F}_0(S \times 0) = S$. Set $F_0 = \mathcal{F}_0(S \times 1)$. F_0 is almost normal, transversely oriented, and has an exceptional tightening disk,

$D' = D - \mathcal{F}_0$, which does not intersect the image of $\mathcal{F}_0$.

2. Do a small normal isotopy of F_0 in the transverse direction. Tighten F_0 along D' to obtain a possibly non-normal surface F_1. Extend $\mathcal{F}_0$ to $\mathcal{F}_1$, with $F_1 = \mathcal{F}_1(S \times 1)$. F_1 inherits a transverse orientation from F_0.

3. Let $i \in \{1, 2, 3, \ldots\}$. If F_i has no outermost bent arc with transverse orientation pointing towards a tightening disk D then the construction is complete. Otherwise extend $\mathcal{F}_i$ to $\mathcal{F}_{i+1}$ by doing a small normal isotopy of F_i in the transverse direction and then tightening F_i across D. This produces F_{i+1} together with its induced transverse orientation.

Remark 4.1.6. As $w(F_{i+1}) = w(F_i) - 2$ this process terminates.

Let F_n be the last surface produced.

Definition. The map $\mathcal{F} = \mathcal{F}_n : S \times I \to M$ is called a *tightening map*. The *tightening sequence* corresponding to S and $\mathcal{F}$ is comprised of the ordered collection of tightening disks which were used to produce $\mathcal{F}$.

By construction $\mathcal{F}(S \times 0) = S$. We may assume that each F_i in the construction is represented by some $r_i \in I$, i.e. $\mathcal{F}(S \times r_i) = F_i$ and $r_n = 1$. Thus $\mathcal{F}_i = \mathcal{F}|[0, r_i]$.

4.2 Building a compression body

4.2.1 Tracking an isotopy

In this section we analyze how $\mathcal{F}$ intersects the skeleta of the triangulation. Let $S \subset M$ be a transversely oriented almost normal surface. Let $\mathcal{F}$, $\mathcal{F}_i$, and F_i be as defined in Section 4.1.3.

In Figures 4.2 and 4.3 we display a few of the possible types of intersection, image$(\mathcal{F}_i) \cap T^2$, were $\mathcal{F}_i$ an embedding. Abusing notation we have:

Definition. The shaded regions in Figures 4.2 and 4.3 are called *skeletal faces*.

Lemma 4.2.1 below shows that this collection is complete up to symmetry. Note that the arcs bounding the skeletal faces receive a transverse orientation from the surface they lie in. Arcs of S are always pointed towards while arcs of F_i are pointed away from by

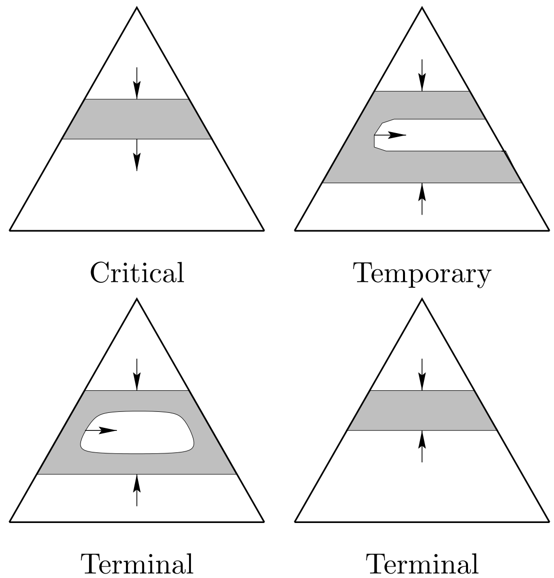


Figure 4.2: Rectangles

the transverse orientation. The two faces with a normal arc of F_i are called *critical*. Those with a bent arc are called *temporary* while the rest are called *terminal*.

As we will see, the critical skeletal faces can be combined in various ways while a temporary face always results in a terminal face which is stable. Note also that there is a second critical rectangle which "points upward." The non-critical faces may be foliated by $\mathcal{F}_i$ in multiple ways, depending on the order of the tightening disks in the sequence.

Lemma 4.2.1. *The tightening map $\mathcal{F} : S \times I \to M$ is an embedding. Furthermore, $F_n = \mathcal{F}(S \times 1)$ only intersects the two-skeleton in normal arcs and innermost simple curves. The latter must have transverse orientation pointing toward the bounded surgery disk.*

Proof. We will show by induction that for all i;

1. $\mathcal{F}_i$ is an embedding.

2. For every $f \in T^2$ the connected components of image$(\mathcal{F}_i) \cap f$ are given, up to symmetry, by Figures 4.2 and 4.3.

These claims hold trivially for $i = 0$, as all components of image$(\mathcal{F}_0) \cap f$ are critical rectangles.

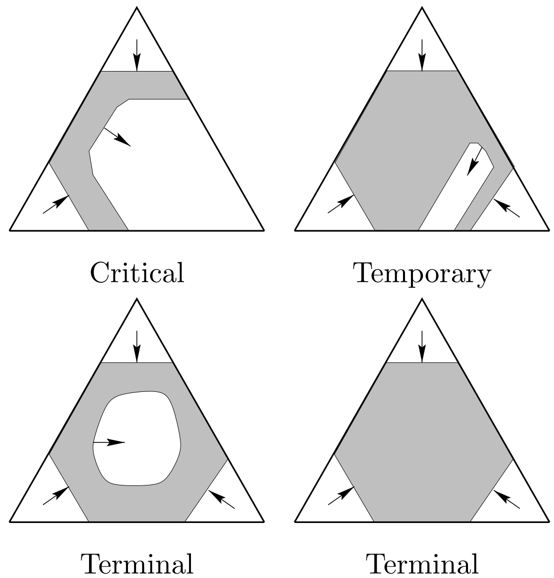


Figure 4.3: Hexagons

Suppose that they hold at $i = k$. We now verify the first claim for $i = k + 1$: Suppose that α is the bent arc on the boundary of D, the next tightening disk in the sequence. If interior(D) meets image($\mathcal{F}_k$) then by the second induction hypothesis some skeletal face, s, of image($\mathcal{F}_k$) must meet the interior of D. It follows that the interior of s must meet α. Thus $\mathcal{F}_k$ was not an embedding, a contradiction.

It follows that $D \cap \text{image}(\mathcal{F}_k) = \alpha$. Since the $k + 1^{\text{th}}$ stage of the isotopy is supported in a small neighborhood of D it follows that $\mathcal{F}_{k+1}$ is an embedding.

The transverse orientation on F_k gives rise to a transverse orientation on F_{k+1}. To verify the second claim we list the possible situations arising in a single triangular face $f \in T^2$. Note that a collection of two or more skeletal faces may be combined only if each of the faces contains a normal arc of F_k:

1. Two critical rectangles may be combined to produce a temporary rectangle, a terminal rectangle with a hole, or a critical hexagon.

2. Three critical rectangles may be combined to produce a temporary hexagon or a terminal hexagon with a hole.

3. A critical rectangle and critical hexagon may be combined to produce a temporary hexagon or a terminal hexagon with a hole.

4. A temporary face can lead to either terminal face.

This completes the induction step.

Now, by maximality of the tightening sequence, $F_n = \mathcal{F}(S \times 1)$ has no outermost bent arcs with outward orientation. $F_n \cap T^2$ cannot contain a bent arc with inward orientation, a simple curve with outward orientation, or a non-innermost simple curve as that would violate the classification of skeletal faces. □

Given that $\mathcal{F}$ is an embedding, in the sequel image($\mathcal{F}_i$) is denoted by $\mathcal{F}_i$.

Corollary 4.2.2. *If S' is any normal or almost surface in M which does not intersect the exceptional tightening disk of S then $\mathcal{F} \cap S' = \emptyset$, perhaps after a normal isotopy.*

Proof. By hypothesis, we may tighten along the exceptional tightening disk of S and obtain an embedding, $\mathcal{F}_1$. We now proceed as in Lemma 4.2.1. □

Note that $\mathcal{F}$ naturally imposes a product structure on the pair $(M, M - \mathcal{F})$, as defined in Section 2.4. This allows us to examine the intersection of $\mathcal{F}$ with the one-skeleton.

Lemma 4.2.3. *$T^1 \cap \mathcal{F}$ meets the nontransverse levels of $\mathcal{F}$ only in maxima.*

Proof. Lemma 4.2.1 shows that $\mathcal{F}$ gives a foliation of some submanifold of M. The second induction hypothesis shows that any bent arc of any F_i is outermost and has a transverse orientation pointing toward its tightening disk. It follows that all nontransverse levels occur at maxima of the one-skeleton with respect to $\mathcal{F}$. □

Let t be any tetrahedron in the given triangulation of M.

Lemma 4.2.4. *For all i, $t - \mathcal{F}_i$ is a disjoint collection of balls.*

Proof. Again we use induction. Our induction hypothesis is as follows: $t - \mathcal{F}_i$ is a disjoint collection of balls, unless $i = 0$ and t contains the almost normal annulus of S.

Let B be a component of $t - \mathcal{F}_k$. There are two cases to consider. Either B is cut by an exceptional tightening disk or it is not. Assume the latter. After the $k+1^{\text{th}}$ stage of the isotopy $B \cap \mathcal{F}_{k+1}$ is a regular neighborhood (in B) of a collection of disjoint arcs and disks in ∂B. Hence $B - \mathcal{F}_{k+1}$ is still a ball.

If B is adjacent to the almost normal piece of F_0 then let D be the exceptional tightening disk. Set $B_\epsilon = B - D$. Each component of B_ϵ is a ball, and the argument of the above paragraph shows that they persist in the complement of $\mathcal{F}_1$. □

A similar induction argument proves:

Lemma 4.2.5. *For all i, $t \cap \mathcal{F}_i$ is a disjoint collection of handlebodies.*

This lemma is not used in what follows and its proof is accordingly left to the interested reader. Recall that $\partial \mathcal{F}_i = S \cup F_i$. A trivial corollary of Lemma 4.2.4 is:

Corollary 4.2.6. *For all i, the connected components of $t \cap F_i$ are planar.*

The connected components of $t \cap F_n$ warrant closer attention:

Lemma 4.2.7. *Each component of $t \cap F_n$ has at most one normal curve boundary component. This normal curve must be short.*

Proof. Let $t \in T^3$ be a tetrahedron. Let P be a connected component of $t \cap F_n$. By Lemma 4.2.1 ∂P is a collection of simple curves and normal curves. Let α be any normal curve in ∂P. Let $\{\alpha_j\}$ be the normal arcs of α.

Claim. α has length 3 or 4.

By the definition of a critical face, each α_j lies on a critical rectangle or hexagon. If no α_j is on a hexagon, then α is parallel to a normal curve $\beta \subset S$. The first step of the tightening procedure prevents β from being a boundary of the almost normal piece of S. It follows that α must be short.

Otherwise α_1 is on the boundary of a critical hexagon $h \subset f$. Let β be a normal curve of S incident on h and let $\beta_1 \subset \beta$ be one of the normal arcs in ∂h. Let e be the edge of f which α_1 does not meet. This edge is partitioned into three pieces; $e_h \subset h$, e', and e''. We may assume that β_1 separates e_h from e'. Note that a normal curve of length ≤ 8 has no parallel normal arcs in a single face. Thus β meets e' exactly once, at an endpoint of e'. Since α and β do not cross it follows that β separates α from e' in ∂t.

Similarly, α is separated from e''. Thus α does not meet e at all. Lemma 2.5.3 implies that α is short.

Claim. P, the component of $F_n \cap t$ in question, has at most one boundary component which is a normal curve.

Suppose that P has two such: α and β. Let A be the annulus cobounded by α and β in ∂t. Suppose the transverse orientation F_n induces on α points away from A. There are several cases, depending on the length of α and the types of skeletal faces to which α is adjacent.

1. Suppose α has length three:
   (a) If α meets only critical rectangles then a normal triangle of S separates α from β.
   (b) If α meets one critical hexagon then the almost normal octagon and the exceptional tightening disk together separate α from β.
   (c) If α meets two critical hexagons then either a normal triangle or normal quad of S separates α from β.
   (d) If α meets only critical hexagons then a normal triangle of S separates α from β.
2. Suppose α has length four:
   (a) If α meets only critical rectangles then a normal quad of S separates α from β.
   (b) If α meets one critical hexagon then S could not have been an almost normal surface.
   (c) If α meets two critical hexagons then a normal triangle of S separates α from β.

In all cases except 1(b) and 2(b), observe that $S \cap P \neq \emptyset$ and thus $S \cap F_n \neq \emptyset$. This contradicts the fact that $\mathcal{F}$ is an embedding (Lemma 4.2.1.) In case 1(b), P must intersect either S or the exceptional tightening disk whereas in case 2(b) S could not have been almost normal. Both are impossible.

We deduce that the transverse orientation which F_n gives α must point toward A. Let γ be an arc which runs along P from α to β. Let α' be a push-off of α along A, towards β. This push-off bounds a disk in one of the components of $t - \mathcal{F}$, by Lemma 4.2.4. This disk does not intersect $P \subset F_n \subset \mathcal{F}$ and hence fails to intersect γ. This is a contradiction. □

Remark 4.2.8. By Lemma 4.2.1 all simple curves of F_i are innermost. It follows that the "tubes" analyzed in the lemma above do not run through each other. Furthermore, Lemma 4.2.5 implies that these tubes are unknotted, but this fact is not needed in the sequel.

We have a corollary which is easy to deduce from Lemma 4.2.4, Lemma 4.2.7, and Corollary 4.2.6:

Corollary 4.2.9. *The surface obtained by surgering all simple curves of F_n is a disjoint collection of two-spheres, disjoint from the two-skeleton of T, and normal surfaces.*

4.2.2 Main theorem and corollaries

In this section we give the proof of Theorem 4.1.1 and a few curious corollaries.

Proof. (of Theorem 4.1.1) There are two cases: If the surgery disk of the almost normal annulus is on S's head then thicken S and attach a thickened copy of the exceptional surgery disk, D, to obtain the submanifold V. We must show that V is the desired minimal element as defined in Section 4.1.1.

Let V' be any other compression body in $\mathcal{V}_S$. Note that $\partial_- V'$ is normal and disjoint from $S \cup D$. It follows that we may normally isotope V inside of V'.

Now suppose that there is a tightening disk above S. Let $\mathcal{F}$ be the tightening map. We attach thickened surgery disks along every simple curve of F_n. As in Corollary 4.2.9, this cuts F_n into surfaces which are either normal or contained in a single tetrahedron. The latter are all spheres by Lemma 4.2.4 so we may cap them off with balls. Call this compression body V.

Again, suppose that V' is another element of $\mathcal{V}_S$. Let $F' = \partial_- V'$. By Corollary 4.2.2 we may assume that $F' \cap \mathcal{F} = \emptyset$. Also, as F' is normal it cannot meet the given surgery disks of F_n. It follows that F' is disjoint from V and we are done. □

Let S be a separating almost normal surface containing an almost normal octagon. There are two exceptional tightening disks, one above and one below S. These allow us to construct a pair of product structures, $\mathcal{F}_+$ and $\mathcal{F}_-$. These intersect only at S by Lemma 4.2.1. Let $\mathcal{F}_S = \mathcal{F}_+ \cup \mathcal{F}_-$.

Lemma 4.2.10. *Let T be a one-vertex efficient triangulation of M. Suppose that M contains S, an almost normal two-sphere with exceptional piece an octagon. Then $M = S^3$ and $\mathcal{F}_S$ is isotopic (rel T^1) to a product structure realizing bridge position for $K = T^1 \cap \mathcal{F}_S$.*

Proof. By Theorem 4.1.1, we can form the minimal compression bodies above and below S. Call these V and W respectively. Since both of these must have normal boundary consisting of a disjoint union of spheres we may assume that V has normal boundary equal to the vertex link and W has empty normal boundary. Since S was not contained in B, a regular neighborhood of the vertex, we see that $M = V \cup W \cup B$ and conclude that $M = S^3$.

Remove the last ball which was added to W. Note that V and W are homeomorphic to $S^2 \times I$ and that both are alterations of $\mathcal{F}_\pm$ only off of a regular neighborhood of the one-skeleton. It follows that $\mathcal{F}_{\pm}$ are isotopic to V and W via an isotopy fixing T^1. By Lemma 4.2.3, K must be in bridge position with respect to $\mathcal{F}_S$ and we are done. □

Theorem 4.2.11. *Suppose that T is an efficient, one-vertex triangulation of S^3. Then any thin position of the one-skeleton realizes bridge position.*

Proof. Suppose that T is an efficient triangulation of S^3. Let $\mathcal{F}$ be a product structure on $(S^3, B \cup B')$ where B is a regular neighborhood of the unique vertex and B' is a small ball inside some tetrahedron. Assume that $\mathcal{F}$ realizes thin position of $K = T^1 \cap (M - (B \cup B'))$. Suppose there exists a minimum at level F_b immediately above a maximum at level F_a. That is, $\mathcal{F}|[a, b]$ is a thin region.

By Claims 4.1 – 4.5 of Thompson's paper [33] there is a level F' in the first thick region of $\mathcal{F}$ which contains (after surgering F' along simple curves and possibly other surgery disks each contained in the interior of a tetrahedron) a connected component which is an almost normal sphere S. Since none of the surgery disks meet the one-skeleton we have $w(S) \leq w(F)$.

Now we must estimate the width of $\mathcal{F}$. Suppose that the number of edges in the one-skeleton is k. The weight of $F_0 = \partial B$ is $2k$. Recalling that there is a minimum *above* F', we have:

$$w(\mathcal{F}) \geq (w(F') + (w(F') - 2) + \ldots + 2k) + ((w(F') - 2) + \ldots + 2) + 4$$

The $+4$ in the above sum is contributed when the weight goes up at the minimum at level F_b.

As above consider $\mathcal{F}_S$. By Lemma 4.2.10, $\mathcal{F}_S$ can be isotoped (rel T^1) until it realizes bridge position for the one-skeleton. The width of $\mathcal{F}_S$ is:

$$w(\mathcal{F}_S) = (w(S) + (w(S) - 2) + \ldots + 2k) + ((w(S) - 2) + \ldots + 2)$$

Recalling that $w(S) \leq w(F)$ this gives a contradiction to the assumed thinness of $\mathcal{F}$. □

Remark 4.2.12. It follows from this that there is an algorithm to compute the bridge number of the one-skeleton of a one-vertex, efficient triangulation T of S^3. By the above theorem, minimal bridge position is thin. Every such may be isotoped relative to T^1 so that the thick region contains an almost normal S^2. By Lemma 5 of Thompson's paper [33], all almost normal (with octagon) two-spheres of least weight are fundamental.

Thus to find the bridge number of the one-skeleton we need only list the fundamental almost normal two-spheres and pick one with smallest weight. One-half of this weight is the bridge number.

4.3 Generalizations

We end this chapter with a collection of remarks. The lemmata proved in Section 4.2.1 are quite forgiving — there are several possible generalizations. Here are two which will be used later in this thesis.

Remark 4.3.1. Suppose that S is a properly embedded, two-sided surface inside of (M, T) such that, for all but one tetrahedron $t' \in T^3$, S intersects the tetrahedra of T in disjoint collections of normal disks. $S \cap t'$ is a disjoint collection of normal triangles together with exactly two almost normal octagons of the same type.

In this case the the construction of the tightening sequence goes through without change and almost all of the lemmata of Section 4.2.1 are unaltered. We cannot obtain Lemma 4.2.7 as F_n may contain exactly one of the two original octagons. However we still find that all other normal curves of F_n are short and all planar components of $F_n \cap t$ have at most one normal boundary component, which must be short.

If this occurs then tighten along the surgered F_n in the direction indicated by the induced transverse orientation. We finally obtain a possibly empty normal surface cobounding a compression body with S. This technique is used in Lemma 5.1.7.

Remark 4.3.2. Suppose that S is a normal, two-sided surface inside of (M, T) such that S decomposes as a Haken sum. Suppose that A is an exchange annulus which is a *tunnel*; A is boundary parallel in $M - S$, relative to ∂A. Let X be the solid torus which S and A cobound. Form S' by doing an irregular exchange along A. S' is a disjoint union of a surface F_0 isotopic to S and the surface ∂X. See Figure 4.4.

As S is two-sided, we may chose a transverse orientation on S which points towards A. This induces on F_0 a transverse orientation pointing away from X. We wish to tighten in this direction. However, it is easy to see that the second induction hypothesis of Lemma 4.2.1 does not hold.

We proceed as follows. Let $\mathcal{F}_0 : S \times I \to M$ be an embedded isotopy of S to F_0 such that image$(\mathcal{F}_0)$ is equal to a regular neighborhood of $S \cup X$. The classification

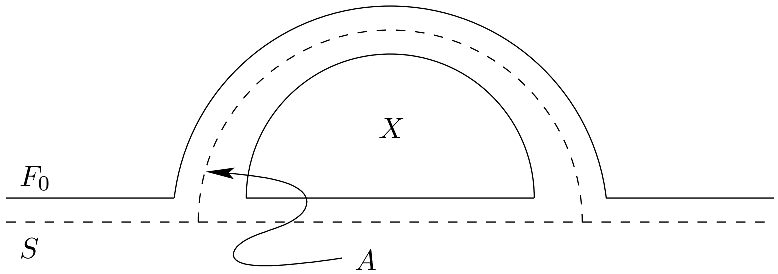


Figure 4.4: An irregular exchange along A

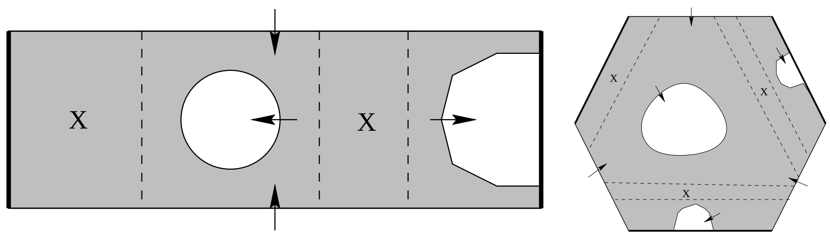


Figure 4.5: A few complicated faces

of skeletal faces is now more complicated, as indicated by Figure 4.5. (The dotted lines indicate an intersection between the annulus A and the face $f \in T^2$.) Nonetheless, using this enlarged classification we may again prove Lemmata 4.2.1, 4.2.4, and 4.2.7. Thus we obtain a tightening sequence with all of the desired properties. This technique is used in Theorem 7.2.3.

Remark 4.3.3. I believe that the surfaces discussed in this section, as well as almost normal surfaces, are all *barrier surfaces* in the sense of of Jaco and Rubinstein [17] (See Lemma 3.1 and Corollary 3.5 of that paper.) I cannot resist posing the following question, which surely has an affirmative answer:

Question. Suppose that (M, T) is a triangulated irreducible three-manifold and that B is a two-sided barrier surface for some connected component $N \subset (M - B)$. *Shrink* B inside of N. The track of this shrinking process gives an (embedded?) submanifold $C \subset N$. Is C a canonical compression body?

A “yes” answer would indicate that the theory developed in this chapter is, in part, subordinate of that of [17].

Chapter 5

Decomposition Lemmata

This chapter studies an issue of crucial importance to normal surface theory. Namely, how do the exchange annuli of a decomposition lie in the given three-manifold? There is a straight forward classification possible for the *trivial* exchange annuli.

The classification in the last section, together with standard normal surface theory techniques, will prove that certain surfaces inside a given three-manifold must be fundamental. For example, given weak conditions on the triangulation of the three-manifold in question, it follows that acylindrical surfaces are always fundamental.

5.1 Annuli

5.1.1 Trivial annuli

Fix M, an irreducible, orientable three-manifold with non-empty boundary.

Definition. An annulus A properly embedded in M is *trivial* if it is compressible or boundary compressible in M and performing some compression or boundary compression on A yields a collection of trivial disks in M.

An annulus A is a *tunnel* if it is parallel relative to its boundary into ∂M. A is a *tube* if A is the boundary of a regular neighborhood of an arc properly embedded in M. Finally, A is a *tent* if ∂A bounds an annulus B inside of ∂M, $A \cup B$ bounds a "cube with a knotted hole" inside of M, and $\partial_+ A$ bounds a disk in ∂M.

Note that these possibilities are not mutually exclusive. For example, if a tent A cuts a solid torus from M instead of a cube with a knotted hole then A is also a tunnel.

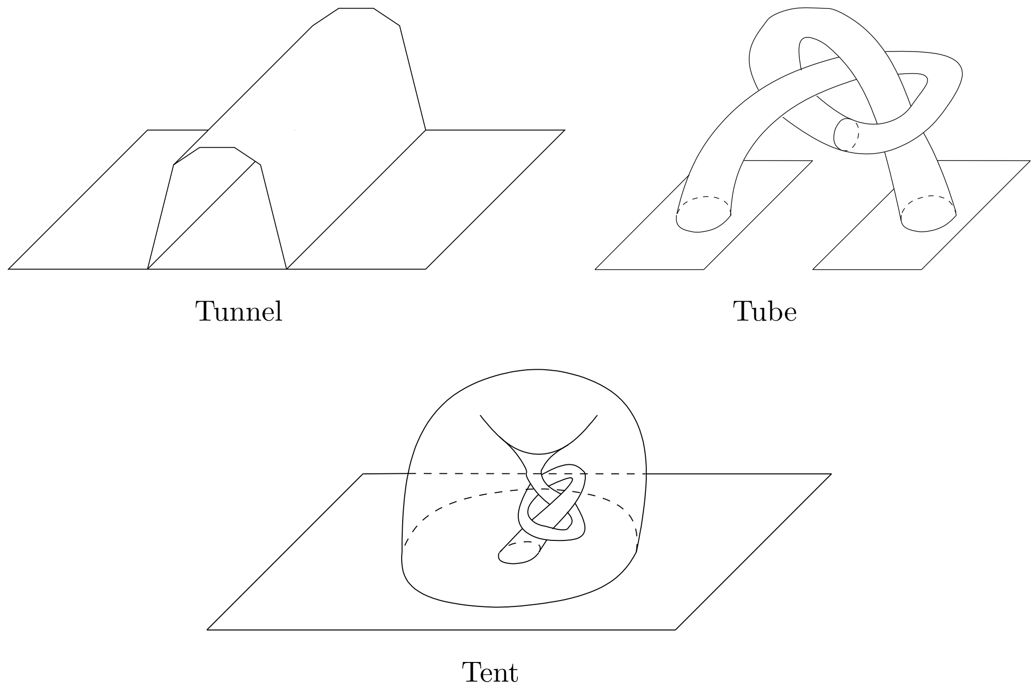


Figure 5.1: The trivial annuli

Also, if M is a three-ball then an unknotted tube is also a tunnel while a knotted tube is also a tent, etc. See Figure 5.1.

Lemma 5.1.1. *Every trivial annulus is either a tunnel, tube, or tent.*

Proof. Suppose that A is a trivial annulus in M. Suppose that A may be boundary compressed along a disk D to obtain a trivial disk D'. D' cuts a ball, X, from M. There are two cases:

1. If $D \subset X$ then A is an unknotted tube, contained inside of X.

2. If D is not contained in X then A is a tunnel.

On the other hand, suppose that A compresses along a disk D yielding a pair of trivial disks, D', D'' cutting out the balls X', X''. Again there are two cases:

1. If $X' \subset X''$ then A is a tent.

2. If the two balls are disjoint then A is a tube.

□

There are several simple obstructions which prevent an annulus from being trivial. Here is a useful one:

Lemma 5.1.2. *If A is an annulus, properly embedded in M, and there is a closed curve $\gamma \subset \partial M$ such that γ meets ∂A once transversely then A is not trivial.*

Proof. Suppose that A is a trivial annulus. The two boundary components of A either bound disks or cobound an annulus in ∂M. Thus if γ is any closed curve in ∂M which is transverse to ∂A then γ meets ∂A an even number of times. □

The situation for Mobius strips is very simple:

Definition. A Mobius strip, properly embedded in M, is *trivial* if its double is a trivial annulus.

Remark 5.1.3. Note that trivial Mobius strips only appear in components of M which are solid tori or once punctured $\mathbb{RP}^3$'s.

5.1.2 Exchange annuli

We will need the following structure lemma for the exchange bands of a neat Haken sum:

Lemma 5.1.4. *Suppose that H is a normal or almost normal surface (with exceptional piece an octagon) and H is a neat Haken sum $H = F + G$. Let C' be an exchange band of $F + G$. Fix a tetrahedron $t \in T^3$ and a block B of $t - H$. Let C be a component of $B \cap C'$. Then:*

1. *C is properly embedded in B.*

2. *$|C \cap H| = |C \cap T^2| = 2$.*

3. *$C \cap T^1 = \emptyset$.*

4. *If H is normal then C meets each face of t in at most one connected component.*

5. *C meets each disk of H in at most one connected component.*

Proof. This follows directly from the definition of an exchange band, the minimality assumption in the definition of a neat intersection, and Lemma 2.6.7. □

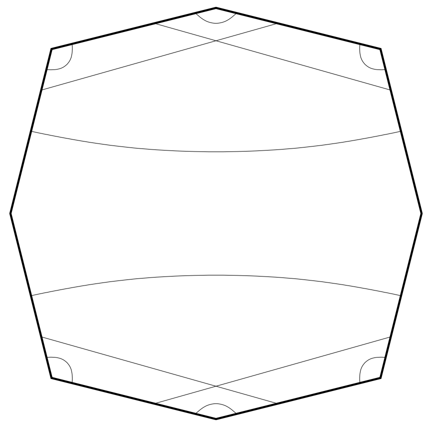


Figure 5.2: Possible seams on an octagon

Remark 5.1.5. If the hypotheses of Lemma 5.1.4 hold and f is a face of t then we immediately deduce that $C' \cap f$ is a disjoint collection of arcs properly embedded in $f - H$ which avoids T^1. Furthermore, each of these arcs has endpoints on distinct normal arcs of H.

Lemma 5.1.6. *Suppose H has almost normal piece an octagon and that t, B, C', and C are defined as above. Suppose further that H is two-sided, C is a tent or a tunnel, B is an exceptional block, and $B \cap \text{interior}(C)$ is nonempty. Then Figure 5.2 shows all possible intersections of C with the octagon in H.*

Proof. It follows from our hypotheses that C is attached to only one side of each normal or almost normal disk of H. The conclusion may now be obtained by enumerating all curves in ∂B which meet the one-skeleton of B four times and obey the constraints of Lemma 5.1.4. □

5.1.3 Trivial exchange annuli

Fix (M, T), an orientable triangulated three-manifold. An exchange annulus A of the Haken sum $H = F + G$ is *trivial* if A is a trivial annulus inside of $M - H$. Define trivial exchange Mobius strips similarly.

We now restrict our attention to efficient triangulations of M and to surfaces which are either normal or have almost normal piece an octagon:

Lemma 5.1.7. *Suppose that*

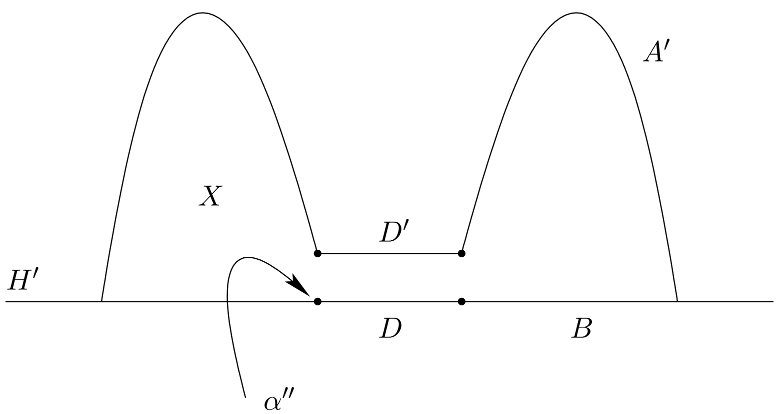


Figure 5.3: A cross-sectional view of S

1. *(M, T) is a orientable, efficiently triangulated three-manifold not homeomorphic to S^3 and*

2. *H is a normal surface or almost normal surface with exceptional piece an octagon and*

3. *H is two sided.*

Then no neat Haken decomposition of H admits an exchange annulus which is a tent in $M - H$.

Proof. The first condition forces T to have a single vertex, v, by Lemma 4.1.4, and also forces M to be irreducible.

Suppose that H admits a tent A as an exchange annulus of the neat Haken decomposition $H = F + G$. Set $\partial A = \alpha' \amalg \alpha''$ and suppose that A arises as the exchange annulus along the curve of intersection $\alpha \subset F \cap G$. Choose the notation so that $B \subset H$ is the annulus bounded by ∂A, α'' bounds a disk D in H, and $D \cap \alpha' = \emptyset$.

Let D' be a disk in M such that $\partial D' \subset \text{interior}(A)$ and D' is normally isotopic and parallel to D. Let A' be the component of $A \setminus \partial D'$ containing α'. Take H' equal to the closure of $H \setminus (B \cup D)$.

Let $S = D' \cup A' \cup B \cup D$. Note that S is an embedded two-sphere which is disjoint from H. Furthermore S bounds a three-ball, X, which does not meet H. A cross-sectional view is shown in Figure 5.3. Our aim is to show that the existence of S contradicts the efficiency of T.

Claim. There is a connected arc $\epsilon \subset T^1$, which does not meet the unique vertex v of T, such that ϵ is properly embedded in X.

To see this restrict attention to a single face $f \in T^2$ which meets A. Let E be a 4-gon of $f - (H \cup A)$ which has one edge in $e \in T^1$, two edges in H, and one edge in A. There are two possibilities: either D (and thus D') meets the edge e or B meets the edge e. In either case we obtain the desired edge ϵ.

It follows that X is not a regular neighborhood of v.

Claim. S is not normal.

Suppose that S were normal. Then S is a copy of the vertex link and bounds a regular neighborhood of the vertex v. Since $X \neq Y$ it follows that $M - S$ is a disjoint union of two balls and $M \cong S^3$, a contradiction.

Claim. For every face $f \in T^2$, $f \cap S$ is a collection of normal arcs.

This is straightforward: Every component of $f \cap S$ is contained inside of a single component of $f - H$ because H is two-sided. Also, each component of $f \cap S$ meets each component of $t \cap A$ at most once.

Claim. The disk types of S are a subset of the disk types of H.

This is similar to the previous claim.

Claim. S contains at most two almost normal octagons. If S contains exactly two then they are parallel and the region between them is contained inside of X.

Suppose that H contains an almost normal octagon, N. By Lemma 5.1.6 there is exactly one component of of $N - A$ which meets opposite edges of N. Call this component N'. Depending on whether N' is contained inside of H', B, or D the two-sphere S must contain zero, one, or two octagons. In the latter situation the two octagons are parallel through X because D and D' are parallel.

We have shown above that S cannot be normal.

1. If an almost normal octagon is contained in B then S is an almost normal two-sphere. We may tighten S away from X, the bounded three-ball. By Theorem 4.1.1 and the efficiency of the triangulation we obtain either a normal two-sphere S_1 or the empty surface. In the latter case we deduce that $M \cong S^3$, a contradiction. In the former

case, S_1 must be the vertex link. Now, the vertex cannot be on the same side as X as that side contained an almost normal octagon. It follows that the vertex is on the other side and, again, $M \cong S^3$.

2. If S contains a pair of almost normal octagons, one in D and one in D', then we may tighten away from X by Remark 4.3.1. We now proceed as in the second case.

This completes our argument. □

Tunnels and tubes cannot be ruled out by an assumption of efficiency. However, if we assume that all exchange bands are annuli and that they are all trivial then we obtain the following:

Lemma 5.1.8. *Suppose that*

1. *(M, T) is a orientable, efficiently triangulated three-manifold not homeomorphic to S^3 and*

2. *H is closed, connected, two-sided surface embedded in M which is not a sphere and*

3. *H is normal or almost normal (with octagon) and*

4. *$H = F + G$ is a neat Haken decomposition with all exchange bands being trivial annuli.*

It follows that there is a Haken decomposition $H = H' + G'$ of H where G' is nonempty and H' is isotopic to H.

Proof. By Lemma 5.1.7 none of the exchange annuli of $F + G$ are tents. Suppose that A is an exchange annulus which is a tunnel. Note that the boundary curves of A are essential in H as A is not a tent. Let B be a annulus component of $H - \partial A$ such that $A \cup B$ bounds a solid torus, X, and H is not contained in X. D be a boundary compressing disk for A which is a meridian disk for X.

Pick a transverse orientation on A which points towards D. By Remarks 2.6.9 and 2.6.10 we deduce that there must be an odd number of seams in B which are all isotopic to the boundary components of A. It follows that there must be a seam in B which is parallel to ∂A and which is a boundary component of an exchange annulus, C, on the other side of H. Since C is trivial C must be another tunnel, as one component of ∂C is essential in H. See Figure 5.4

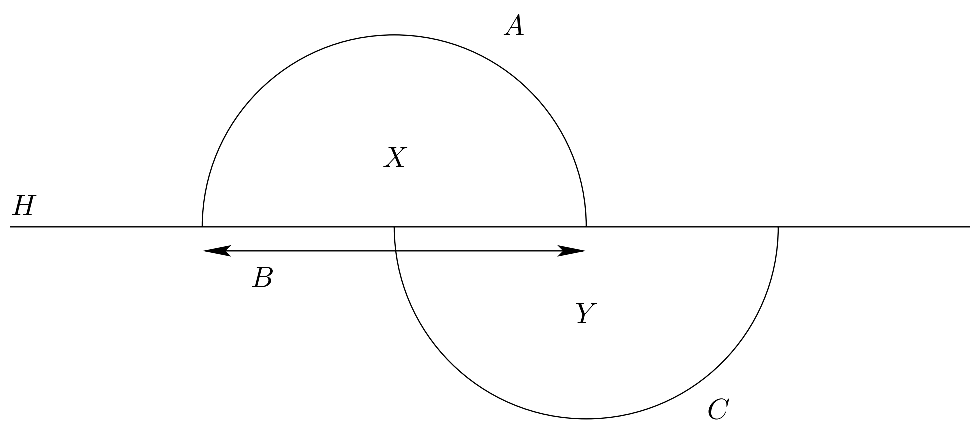


Figure 5.4: A pair of linking tunnels

Let Y be the solid torus cut out of $M - H$ by C. We may suppose that $\alpha, \gamma \in \Gamma = F \cap G$ are the regular curves of intersection which give rise to the exchange annuli A and C. Form the surface H'' by taking $F \cup G$ and performing regular exchanges along all curves of $\Gamma \setminus \{\alpha, \gamma\}$. By Lemma 2.6.7 H'' an immersed surface, with double curves along α and γ.

H'' is in fact the union of two normal (almost normal) surfaces: H' which is isotopic to H and G' which is a torus bounding the solid torus $X \cup Y$. See Figure 5.5. This is the desired conclusion.

Suppose now that all exchange annuli of $F + G$ are tubes. It follows that all of the seams of $F + G$ bound disks in H. Fix a maximal one of these, $\partial_+ A_1$, bounding a disk D_1; this is well-defined because H is not a two-sphere. Fixing a transverse orientation on H, we may assume that A_1 is attached to H along $\partial_+ A_1$ on H's head side. Note that $\partial_- A_1$ cannot attach A_1 to H *inside of* D_1 on H's head side as A_1 is not a tent.

Pick a transverse orientation on A_1 pointing away from the compressing disk (of A_1) which D_1 defines. Note that $\partial_- A_1$ also bounds a disk $D' \in H$. The transverse orientation on A_1 also points away from D'. (If not, then A_1 compresses on both sides. Take the union of the two compressing disks to obtain a sphere. This sphere is non-separating, as it is disjoint from H, a connected surface, and it meets A along the core circle of A. But a efficient triangulation cannot contain such a sphere.)

From Remark 2.6.10 and the connectedness of H it follows that $\partial_- A_1$ cannot be outermost. Let $\partial_+ A_2$ be the outermost seam which bounds a disk, D_2, which contains

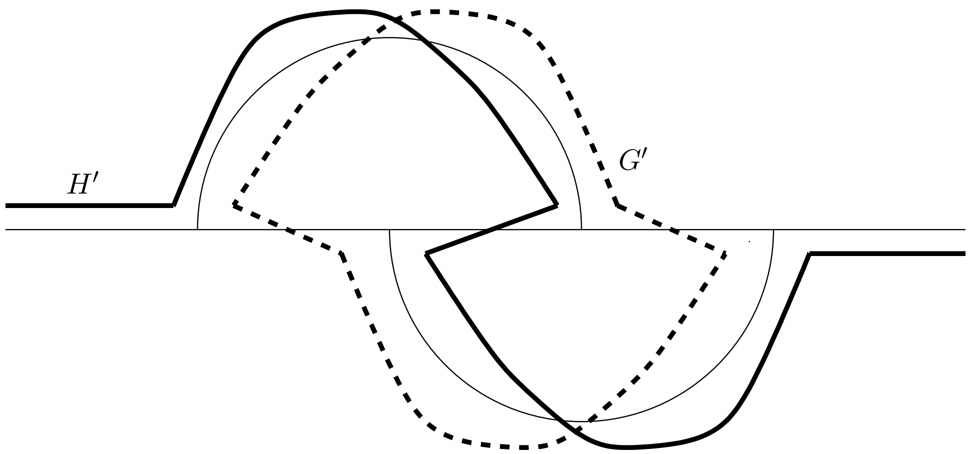


Figure 5.5: A new Haken decomposition

$\partial_- A_1$. In this fashion build a sequence of tubes $\{A_i\}$ with $\partial_+ A_i$ outermost and bounding a disk D_i which contains $\partial_- A_{i-1}$. Since there are finitely many exchange annuli, this sequence must eventually become periodic. Relabel the annuli so that the sequence starts with A_1 and ends with A_n such that $\partial_- A_n \subset D_1$. See Figure 5.6.

Let α_i be the curve of intersection of $F \cap G$ corresponding to A_i and let X_i be the three-ball bounded by $A_i \cup H$. Form a surface H'' by performing regular exchanges along all double curves of $F \cap G$ except the α_i.

By Lemma 2.6.7 H'' is in fact the union of two normal (almost normal) surfaces: H' which is isotopic to H and G' which is a torus bounding the solid torus $\cup X_i$. This is the desired conclusion. □

Remark 5.1.9. We will use the above lemma in Chapters 6 and 7 and in the next section. Note that the lemma is well suited to comment on normal or almost normal (with octagon) Heegaard splittings.

5.1.4 Non-trivial annuli

Again, let M be an irreducible, orientable three-manifold with non-empty boundary. We now consider the non-trivial annuli:

Definition. An annulus properly embedded in M is *essential* if it is incompressible and boundary incompressible in M.

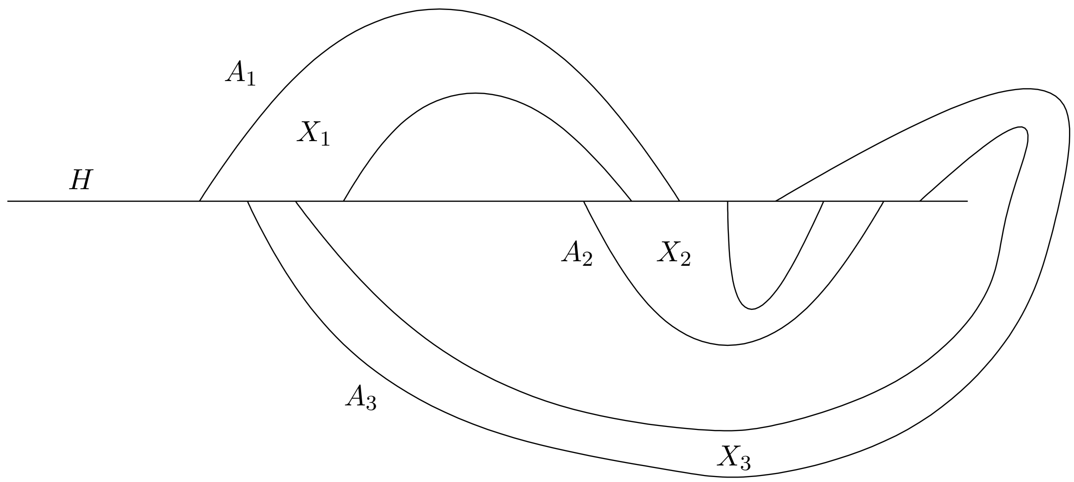


Figure 5.6: A sequence of three tubes

Annuli which are properly embedded in M and which are neither trivial nor essential are called *disk-like* . Note that if M contains a disk-like annulus then it follows from the definitions that ∂M is compressible. We will call a Mobius strip *essential* or *disk-like* if and only if its double is.

Remark 5.1.10. When M is closed and triangulated we will abuse notation and call exchange bands essential or disk-like depending on their status in the cut open manifold $M - H$.

5.2 Applications

Assume that M is a closed, irreducible, orientable three-manifold not homeomorphic to a lens space. Further suppose that M is given with an efficient triangulation, T.

5.2.1 Acylindrical surfaces

Definition. A surface, H, embedded in M is *acylindrical* if it is two-sided, incompressible and $M - H$ admits no essential annuli.

Lemma 5.1.8 thus has the following:

Corollary 5.2.1. *All acylindrical surfaces in M are isotopic to fundamental normal surfaces.*

Proof. Fix an acylindrical surface H. By Theorem 2.5.5 isotope H to be normal. Pick a least weight such normal surface, which we will again call H. Suppose that this surface is not fundamental.

Let $H = F + G$ be a neat Haken decomposition for H. Note that, as H is acylindrical, all exchange bands must be trivial. If there is a exchange band which is a Mobius strip, then by Remark 5.1.3 H is either the boundary of a solid torus or M is homeomorphic to $\mathbb{RP}^3$, both contradictions.

It follows that all exchange bands are annuli. By Lemma 5.1.7 none of these are tents. If there are any tubes or tunnels then by Lemma 5.1.8 there is a surface H' isotopic to N of lesser weight. But this is another contradiction. We conclude that there are no exchange annuli at all, i.e. H is fundamental. □

Remark 5.2.2. The fact that a three-manifold contains only finitely many acylindrical surfaces is well-know. See, for example, Hass's paper [10]. In fact, as we shall see, the folklore normal surface theory proof of this fact is a simpler version of the proof of Theorem 3.0.1.

5.2.2 Surface bundles

Take M to be a closed, orientable surface bundle which is also irreducible. Again, let T be an efficient triangulation of M. Recall that a surface bundle structure on M is a choice of monodromy $h : H \to H$ together with a homeomorphism between M and the mapping torus of h. A homeomorphism $h : H \to H$ is strongly irreducible if $\gamma \cap h(\gamma) \neq \emptyset$ for every essential simple closed curve $\gamma \subset H$.

Remark 5.2.3. This behavior is fairly generic. This follows from Lemma 4.6 of Masur and Minsky's paper [24]. In fact, they give a much stronger result: Let h be a pseudo-Anosov map. As $n \in \mathbb{N}$ goes to infinity the translation distance of h^n also goes to infinity.

Corollary 5.2.4. *M has only finitely many surface bundle structures with strongly irreducible monodromy and for any of these a fibre can be found amongst the fundamental surfaces of (M, T).*

Proof. The argument is identical to that of Corollary 5.2.1 with a slight exception: Note that $M - H$ is homeomorphic to $H \times I$ and all essential annuli are thus vertical. It follows that no exchange annulus, A, can be essential. This is because $\partial_+ A$ and $\partial_- A$ are disjoint curves on H. □

This finiteness result raises several natural questions. For example:

Question. How are the strongly irreducible bundle structures of M distributed on the faces of the Thurston norm ball in $H_2(M, \mathbb{R})$?

Remark 5.2.5. Corollary 5.2.4 may be improved: Fix M a closed, orientable three-manifold. There is a positive constant $b_4 \in \mathbb{R}$ such that if $\mathcal{F}$ is a surface bundle structure on M with monodromy h and fibre of genus g then h^i is *not* strongly irreducible for $1 \leq i \leq b_4 \cdot g$. (If $b_4 \cdot g < 1$ then the conclusion holds vacuously.)

Chapter 6

Bundle Recognition

This chapter investigates manifolds which admit surface bundle structures. As the reader might expect, our discussion will focus producing bundle structures with "many normal fibres." In particular, we will examine how complicated these normal fibres need be with respect to a fixed triangulation. Our main result is:

Theorem 6.0.1. *There is an algorithm to decide whether or not a given closed, orientable, irreducible, atoroidal, and triangulated three-manifold* (M, T) *is a surface bundle over the circle.*

This theorem is not entirely new; the careful reader of the papers of Tollefson and Wang [36] and of Jaco and Tollefson [20] may have deduced it. However, our approach yields additional information. In particular, we provide explicit bounds on the complexity of the one-skeleton with respect to the foliation.

As this chapter is fairly dense we will now give an outline of the sections: In Section 6.1 we further develop our understanding of blocks. We briefly discuss the behavior of normal surfaces inside of blocked submanifolds. In Section 6.2 we bound the complexity of a surface bundle structure on M as a function of the weight of a given fibre. We do this by finding a *dicing* set of normal and almost normal fibres. We finish in Section 6.3 by providing the desired algorithm and indicating why it works.

6.1 More on blocks

This section discusses the normal surfaces which may arise inside of a blocked submanifold $V \subset (M, T)$. Recall that a *blocked* submanifold $V \subset (M, T)$ is a union of closed blocks such that $\mathrm{fr}(V)$ is a properly embedded normal or almost normal surface (which need not be connected.)

6.1.1 Allowed surfaces

Definition. A surface F properly embedded in V is *normal* or *almost normal inside of* V if F is normal or almost normal inside of M and F is disjoint from $\mathrm{fr}_M(V)$.

Remark 6.1.1. Note that this implies that in every product block the normal disks of F are parallel to the normal disks of $\mathrm{fr}_M(V)$. It follows that inside every connected components of $V_P = V - \{\text{the core blocks of } V\}$ the surface F is a collection of parallel copies of $\mathrm{fr}_M(V) \cap V_P$.

Definition. If F is normal or almost normal inside of V and F does not decompose as a sum of normal and almost normal surfaces inside of V then F is *fundamental inside of* V.

We can now deduce that the fundamental surfaces in V are not significantly more complicated than those in M.

Lemma 6.1.2. *There is a constant, $c_1 \in \mathbb{N}$, such that if F is a fundamental surface inside of V then $w(F) \leq w(\mathrm{fr}_M(V))2^{c_1 \cdot |T|}$*

Proof. This follows directly from Lemma 2.6.1 and our Remark 6.1.1. ☐

6.1.2 Decomposition inside of blocked manifolds

Lemmata 5.1.7 and 5.1.8 still apply to normal and almost normal (with octagon) surfaces contained in V. That is:

Lemma 6.1.3. *Suppose that F is a normal surface or almost normal surface with exceptional piece an octagon inside of V. Suppose that F is not normally isotopic into $\mathrm{fr}_M(V)$ and that F satisfies the hypotheses of Lemma 5.1.8. Then $F = F' + G'$ where G' is nonempty, F' is isotopic to F, and F' is not normally isotopic into $\mathrm{fr}_M(V)$.*

Proof. Suppose that F' is normally isotopic into $\mathrm{fr}_M(V)$. Then, after this isotopy, F' and G' are disjoint and it follows that $F = F' + G'$ is not connected, a contradiction. □

This is essentially the same trick as that used in Lemma 5 of Thompson's paper [33]. See also our Remark 2.6.11.

6.2 Collecting surfaces

In this section we outline an inductive procedure which proves:

Theorem 6.2.1. *There is a constant $c_2 \in \mathbb{N}$ such that if (M, T) is a closed, orientable, efficiently triangulated three-manifold and $F \subset M$ is a normal surface which is a fibre of a surface bundle structure $\mathcal{F}$ on M then there is a collection of surfaces $\{F_i\}$ such that:*

1. *$F = F_0$ and for all i each of the F_i is a fibre of $\mathcal{F}$.*
2. *For all i, F_{2i} is normal and F_{2i+1} is almost normal. Also each F_{2i+1} has exceptional piece an octagon except for F_1 which contains an almost normal annulus.*
3. *For all i, there is a pair of surfaces, F_{2j} and F_{2k}, such that F_{2i+1} tightens to F_{2j} and F_{2k}. Furthermore, the union of all of these tightening maps gives a foliation isotopic to $\mathcal{F}$.*
4. *There are at most $2 \cdot |T| + 2$ of the F_i's.*
5. *For all i, $w(F_i) < w(F_0)2^{c_2 \cdot |T|^2}$.*

The constant c_2 can be taken to be $3c_1 + 4$, with c_1 as in Lemma 6.1.2. As usual, this is an overestimate. Figure 6.1 gives a schematic picture of the desired collection of surfaces. Such a collection of normal and almost normal surfaces is a *dicing of M with respect to $\mathcal{F}$*.

The rest of the section is devoted to the proof of Theorem 6.2.1.

6.2.1 Removing the vertex

To start, we set $F_0 = F$. Let S_0 be the link of the unique vertex in T. Let t be any tetrahedron in T^3 which F_0 meets and let $e \in t \cap T^2$ be an oriented edge which intersects F_0.

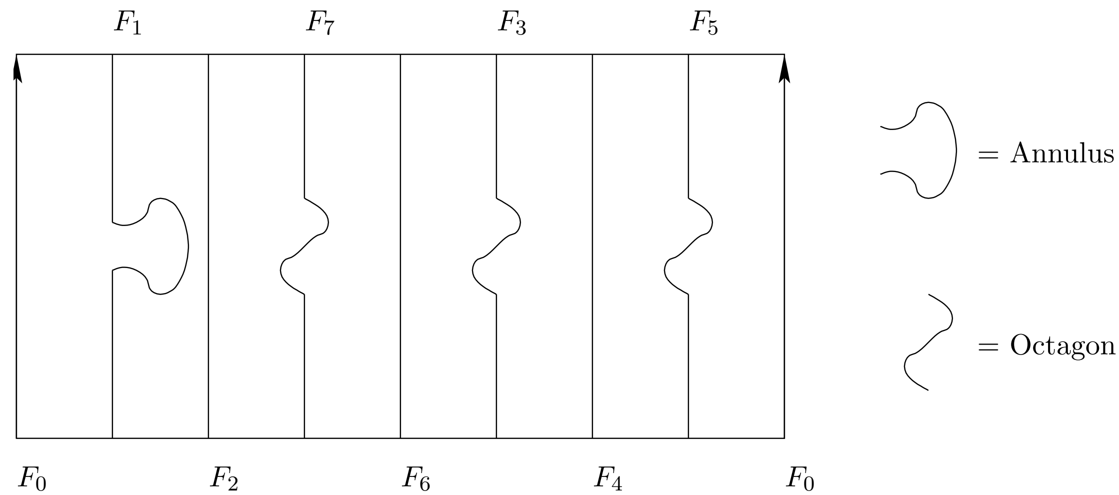


Figure 6.1: A dicing of M by eight fibres

Let x be the first point of $e \cap S_0$ and let y be the first point of $e \cap F_0$. Let $X \subset S_0$ and $Y \subset F_0$ be the pair of normal disks in t meeting x and y. Connect X and Y inside of t by a small unknotted tube parallel to the arc $[x, y] \subset e$. This constructs F_1, an almost normal surface with annular exceptional piece. F_1 is isotopic to a fibre of $\mathcal{F}$ because it is isotopic to F_0. Note that $w(F_1) = w(F_0) + 2 \cdot |T| + 2$.

By Lemma 4.1.2 we may tighten F_1 away from the exceptional surgery disk. As F_1 is incompressible, we obtain an isotopic normal surface. Now, if this surface is normally isotopic to F_0 then the collection of surfaces is complete and the construction is finished. If not then label the new surface F_2. Note that $w(F_2) \leq w(F_1) - 2$.

6.2.2 Beginning the induction step

We are now prepared for the general induction step. Let $\{F_i\}$ be the collection of surfaces constructed thus far. Suppose that $|\{F_i\}| = r$. Let F_{2j} and F_{2k} be a pair of normal fibres which have already been constructed. We may assume that one of the two components of $M - (F_{2j} \cup F_{2k})$ contains none of the F_i's in its interior. Call that component V. Note that V is not a union of product blocks as F_{2j} is not normally isotopic to F_{2k}.

V has several nice properties — V is efficient and does not contain the unique vertex of T. $\mathcal{F}|V$ is a product structure on V. Abusing notation use $\mathcal{F}$ to denote this product structure. Take $V \cong F \times I$ with $\mathcal{F}|0 = F_{2j}$ below and $\mathcal{F}|1 = F_{2k}$ above.

With $\mathcal{F}$ fixed set $K = T^1 \cap V$. After isotoping K rel(∂V) assume that $\mathcal{F}$ realizes thin position for $K \subset V$ as defined in Section 2.4. We now wish to find an almost normal fibre F_{2i} in V.

6.2.3 Following Gabai and Thompson

We set aside the proof of Theorem 6.2.1 temporarily. The following theorem is a reworking of Claims 4.1 – 4.5 of Thompson's paper [33]:

Theorem 6.2.2. *Suppose that (M, T) is a closed triangulated three-manifold. Let $V \subset M$ be a blocked submanifold such that:*

1. *V is not a union of product blocks.*

2. *V contains no normal spheres.*

3. *V is homeomorphic to $F \times I$, where F is a closed surface.*

4. *∂V is a disjoint union of two normal surfaces.*

Let $C = M - V$ and let $\mathcal{F}$ denote the product structure on (M, C). Then there is a closed almost normal surface S in V such that:

1. *S is isotopic to a level of $\mathcal{F}$.*

2. *The exceptional piece of S is an octagon.*

Proof. We only sketch a proof as this sort of thin position argument has appeared several times in the literature (see [1], [32], or [33].)

Let $K = T^1 \cap V$. Isotope K (rel(∂V)) to make K thin with respect to $\mathcal{F}$.

We now proceed in a series of steps.

Claim. $\mathcal{F}$ has a thick region.

To see this, suppose the opposite. As V is not a union of product blocks there must be either a maximum of K directly above $\partial_+ V$ or a minimum directly below $\partial_- V$ That is, one of the boundary components of V admits a high (low) disk. Recall however that if F is a normal surface then $F - T^1$ is incompressible in $M - T^1$ (see Claim 1.1 of [33].) Thus the high (low) disk cannot be adjacent to a normal boundary component of V. To avoid contradiction, $\mathcal{F}$ must have a thick region.

Let S be a level inside of a thick region, R and let t, t' be distinct tetrahedra in T. By Claims 4.1–4.4 of [33] we have:

Claim. $S \cap \partial t$ contains no parallel normal curves of length greater than or equal to eight.

Claim. $S \cap \partial t$ contains no normal curve of length greater than eight.

Claim. $S \cap \partial t$ and $S \cap \partial t'$ do not both contain normal curves of length eight.

Claim. We may assume that S in R has no bent arcs.

Note that this last is identical to Thompson's application of Lemma 4.4 of Gabai's paper [4]. Here Gabai's spanning surface P is the surface $V \cap T^2$.

Finally we have a claim which requires proof:

Claim. There is some tetrahedron $t \in T^3$ such that $S \cap \partial t$ contains a normal curve of length eight.

For every tetrahedron $t \in T^3$ the collection of surfaces $S \cap t = \mathcal{C}$ has all boundaries on ∂t. Since these curves bound disks in ∂t we may push these disks slightly inside of t and use them to compress all non-disk components of $\mathcal{C}$. As all of these compressions take place in the product V (which contains no normal spheres) and as S is incompressible in V these compressions must be pinching off spheres which do not meet the one-skeleton.

Thus there is an ambient isotopy of S inside of V which leaves the one-skeleton and the boundary of V pointwise fixed while making S normal or almost normal with an octagon. However, as S is in a thick region, S must have an upper disk D. As S contains no bent arcs D is not contained in the two-skeleton. Now, S cannot be normal as normal surfaces are incompressible in the complement of the one-skeleton. (Again, see Claim 1.1 of [33].) We conclude that S is the desired almost normal surface and Theorem 6.2.2 is proved. □

6.2.4 Finishing the induction

We pick up the thread of Theorem 6.2.1. Recall that we have a blocked submanifold V with boundary $F_{2j} \cup F_{2k}$. By Theorem 6.2.2 we have an almost normal surface S inside of V which is isotopic to the fibre. By Lemma 6.1.3 there is a normal or almost normal surface in V which is fundamental and not normally isotopic into ∂V. Call this surface S' and note that $w(S') \leq w(\partial V) 2^{c_1 \cdot |T|}$ by Lemma 6.1.2.

If S' is normal we set $S' = F_{2i}$ (where $2i$ is the smallest even number not yet used) and begin the next step of the construction.

If S' is almost normal we set $S = F_{2i+1}$ (where $2i+1$ is the smallest odd number not yet used) and tighten S in both directions to obtain a pair of normal surfaces, S_- and S_+. Note that $w(S_\pm) < w(S')$.

We must check to see if either of S_- or S_+ is parallel to a surface already in our collection of surfaces. For $\epsilon = \pm$, if S_ϵ is not normally parallel into our collection then we add it in, setting $S_\epsilon = F_{2i}$ (where $2i$ is the smallest even number not yet used) and begin the next step of the construction.

To finish the proof of Theorem 6.2.1 we must show that the construction terminates and that the weights of the F_i are appropriately bounded. However, each odd surface except for the first contains an almost normal octagon. Since every pair of odd surfaces is separated by a pair of even surfaces, the two octagons must be in distinct tetrahedra. It follows that there are at most $|T|$ odd surfaces containing an almost normal octagon.

Finally, at the r^{th} step of the construction the surface produced, F_i, has weight bounded by

$$w(F_i) \leq w(\mathrm{fr}(V)) 2^{c_1 \cdot |T|} \leq (w(F_j) + w(F_k)) 2^{c_1 \cdot |T|} \leq \max\{w(F_j), w(F_k)\} 2^{c_1 \cdot |T| + 1}$$

where F_j and F_k were constructed at an earlier stage. Thus $w(F_i) \leq w(F_0)(2^{c_1 \cdot |T|+1})^r$. Since the construction halts after at most $2 \cdot |T| + 1$ stages we obtain the desired bound and the desired set of dicing surfaces. This concludes the proof of Theorem 6.2.1.

6.3 The algorithm and its correctness

6.3.1 Description of the algorithm

Fix a closed, triangulated three manifold (M, T).

Definition. Call a collection $\{S_{2i+1}\}$ of almost normal surfaces in (M, T) *good of weight less than* K if it satisfies the following conditions:

1. After a normal isotopy the S_{2i+1} are pairwise disjoint, none of the S_{2i+1} are normally parallel, and all have weight less than K.

2. Only S_1 contains an almost normal annulus.

3. The octagons of the other surfaces are in distinct tetrahedra.

4. Each of the S_{2i+1} are connected and two-sided.

5. All of the S_{2i+1} have the same genus, $g > 1$.

Remark 6.3.1. Note that for fixed $K \in \mathbb{N}$ there are only finitely many good collections of weight less than K. The list of such collections may be algorithmically generated.

We now give our algorithm for recognizing surface bundles. Suppose that (M, T) is a given closed, efficiently triangulated, orientable, atoroidal three-manifold.

Algorithm. Enumerate all good collections of weight less than $2^{c_2 \cdot |T|^2 + c_3 \cdot |T|}$. Check each of these to decide if it is a dicing collection. If so, then M is a surface bundle over the circle. If no such collection exists then M is not a surface bundle over S^1.

The constant c_3 is defined below.

Remark 6.3.2. To decide if a good collection can be extended to a dicing of M construct both canonical compression bodies for each of the S_{2i+1}. By Corollary 4.2.2 the compression bodies produced, $\{V_i\}$, have pairwise disjoint interiors. $\{S_{2i+1}\} \cup \{\partial V_i\}$ is dicing if and only if the normal surface boundaries of the V_i may be matched in normally parallel pairs.

6.3.2 Proof of correctness

To demonstrate correctness we must show that if M admits some surface bundle structure then it admits a dicing of weight less than $2^{c_2 \cdot |T|^2 + c_3 \cdot |T|}$.

Theorem 6.3.3. *There is a constant, $c_3 \in \mathbb{N}$ such that if (M, T) is a closed atoroidal three-manifold which admits a surface bundle structure $\mathcal{G}$ then M has a surface bundle structure $\mathcal{F}$ with normal fibre F and $w(F) < 2^{c_3 \cdot |T|}$.*

It suffices to take $c_3 \geq a_1 + 2$, where a_1 is the constant mentioned in Lemma 2.6.1. The proof of Theorem 6.3.3 follows the ideas of Tollefson and Wang [36]. Please refer to that paper for an explanation of the concepts used below. The reader might also consult [35] for an introduction to the Thurston norm ball in $H_2(M, \mathbb{R})$.

Proof. Let G be a fibre of the given fibre bundle structure. Let $[G]$ be G's homology class. Since G is a fibre it can be represented by a least-weight taut normal surface, which we will also refer to as G. Let C be the minimal face of $\mathcal{P}$ carrying G. Note that every surface carried by C is thus a multiple of a lw-taut surface, by Theorem 3.3 of [36].

In fact, by Theorem 3.7 of that paper, C is a complete lw-taut face of the projective solution space $\mathcal{P}$. That is, C carries every lw-taut normal surface representing the class $[G]$.

Theorem 5.1 of the same paper provides a linear map h_G from C to a face C' on the boundary of the Thurston norm ball, inside of $H_2(M, \mathbb{R})$. Since G is carried by C every normal surface carried on the interior of C is carried on the interior of C'. C' is a fibred face because C' carries $[G]$. Thus every normal surface carried by the interior of C is a fibre for some bundle structure on M.

Let g be the point which G projects to in $C \subset \mathcal{P}$. We may express g as a convex sum $\sum a_i v_i$ where the a_i are non-negative real numbers and the v_i are vertices of C. Note that the vertex surfaces lying above the v_i are compatible as they are all vertices of the face C.

Choose a subset $\{h_G(v_j)\} \subset \{h_G(v_i)\}$ which is maximal with respect to the property of linear independence. Note that $|\{h_G(v_j)\}| \leq \text{rank}(H_2(M, R))$. This subset cannot be contained in a subface of C' as that would imply that $h_G(g) = \sum a_i h_G(v_i)$ was not in the interior of C', a contradiction.

Finally, let F' be the normal surface obtained by Haken summing the vertex surfaces which project to the v_j's. Since C was a lw-taut face, F' is a disjoint union of, say, k-many lw-taut normal surfaces all representing the same homology class, $\frac{1}{k}[F']$. Let F be one of these. By our choice of $\{v_j\}$ the homology class $\frac{1}{k}[F']$ lies above the interior of C'. We deduce that F is a fibre of a surface bundle structure on M and that

$$w(F) \leq w(F') \leq \text{rank}(H_2(M, \mathbb{R}))2^{a_1 \cdot |T|} \leq (|T| + 1)2^{a_1 \cdot |T|}$$

as desired. □

The effectiveness of our algorithm now follows directly from Theorem 6.3.3 and Theorem 6.2.1.

Chapter 7

Effective Bounds on Distance

In this chapter we will explore how the combinatorics of an almost normal Heegaard splitting can be used to bound the distance of the splitting. This adds a great deal of information to the broad outline sketched in Chapter 3. In almost all cases, we find *a priori* upper bounds, depending only weakly on the combinatorics.

However, as is detailed in Section 7.3 these techniques fail in a peculiar case. Thus the theorems in this chapter cannot completely bound the behavior of the distance of low genus splittings.

7.1 Using the normal surface structure

7.1.1 A brute force bound

Suppose that (M, T) is a closed triangulated three-manifold. Using pure brute force we can bound the distance of a normal or almost normal Heegaard splitting, in terms of its weight. Recall that $d(H)$ is the distance of a Heegaard splitting H while $w(H)$ is the weight of the surface H.

Theorem 7.1.1. *There is a constant $d_1 \in \mathbb{N}$ such that if H is a normal or almost normal Heegaard splitting then $d(H) \leq d_1 \cdot w(H)$.*

Proof. Cut M along H to obtain a pair of blocked handlebodies V and W. Note that $|V|$, the number of blocks in V, is bounded above by a linear function of the weight of H, as is $|W|$. Recall that there are only finitely many kinds of blocks and a finite number of ways in which a pair of blocks may be glued along a face. Thus there is a constant $d_2 \in \mathbb{N}$ and a subdivision of V (W) yielding a triangulation T_V (T_W) such that $|T_V|$ and $|T_W|$ are both bounded above by $d_2 \cdot w(H)$. Furthermore, we may insist that these triangulations induce identical triangulations on H.

By Theorem 6.2 of [20] there are essential disks $D_V \subset V$ and $D_W \subset W$ which are fundamental with respect to these new triangulations. By Lemma 2.6.1 there is a constant, $a_1 \in \mathbb{N}$, such that $w(D_V) < 2^{a_1 \cdot |T_V|}$ and the same holds for D_W. This also is a bound on the length of ∂D_V and ∂D_W as normal curves in the triangulation of H.

Since a pair of normal arcs meets at most once in a face

$$|\partial D_V \cap \partial D_W| \leq 2^{a_1(|T_V|+|T_W|)} \leq 2^{2a_1 d_2 \cdot w(H)}.$$

Finally, applying Lemma 2.3.1

$$d(H) \leq d(\partial D_V, \partial D_W) \leq 2\log(2^{2a_1 d_2 \cdot w(H)}) + 2 = 4a_1 d_2 \cdot w(H) + 2$$

and we are done. □

Remark 7.1.2. This theorem could be thought of as a "haven of last resort." For example, suppose we are given a normal or almost normal Heegaard splitting (with exceptional piece an octagon) of small genus which is fundamental and we are asked to bound the distance. We cannot apply Theorem 3.0.1 or any of the techniques in the rest of this chapter; they rely on the given surface either admitting a non-trivial Haken decomposition or containing an almost normal annulus. As a last ditch measure we might combine the bound on the weight of a fundamental surface (see Lemma 2.6.1) with Theorem 7.1.1 in order to obtain a bound which is solely a function of $|T|$.

7.1.2 Light normal surfaces

Suppose that (M, T) is a closed, triangulated three-manifold. We will say that a normal or almost normal surface H is *light* if it has the smallest weight amongst all of the normal and almost normal surfaces isotopic to H.

Remark 7.1.3. As an immediate consequence, if H is a light almost normal surface then neither of the two canonical compression bodies provided by Theorem 4.1.1 may be homeomorphic to $H \times I$ (with some collection of three-balls removed.) That is, H must be compressible in M and in fact there must be a compressing disk contained in each of the canonical compression bodies. It follows that if the almost normal piece of H is an annulus then the exceptional surgery disk is actually a compression disk for H.

Remark 7.1.4. Suppose for the moment that (M, T) is orientable, not S^3, and efficiently triangulated. It follows that if H is a light Heegaard splitting, of genus two or more, does not contain an almost normal annulus, and is not fundamental then every Haken decomposition of H admits a nontrivial exchange band. This is an immediate consequence of Lemma 5.1.8 and Remark 5.1.3.

7.1.3 The case of the almost normal annulus

Definition. If V is a blocked submanifold of (M, T) then the *two-skeleton* $K \subset V$ is the union of an exceptional tightening disk for ∂V (if one is present) together with all skeletal faces of the blocks of V.

We will need a bit of notation for bookkeeping purposes. Suppose that H is an almost normal surface and V is the canonical compression body on H's head. (As given by Theorem 4.1.1.)

Recall that a tightening isotopy requires the presence of a tightening disk, D. Recall that D is *adjacent* to a skeletal face $s \subset K$ if D intersects the boundary of s. D *causes* s if D is adjacent to s but interior$(s) \cap D = \emptyset$.

Remark 7.1.5. Note that, by the classification given in Figures 4.2 and 4.3, a skeletal face s contains at most one tightening disk. Note also that any critical skeletal face, as well as any face which had a hole, contains no tightening disk at all.

Construct a graph Γ by taking the skeletal faces of V, together with the exceptional tightening disk, as the vertex set and connecting a pair s and s' by a directed edge from s' to s if s contains the tightening disk which causes s'. See Figure 7.1.

Observe that Γ contains no directed cycles (see Remark 7.1.5.) Also, all vertices of Γ have out-degree zero, one, two, or three. To finish this discussion about Γ notice that the exceptional tightening disk is the only vertex in Γ with positive in-degree and zero out-degree.

Note that the two-skeleton K naturally has the structure of a branched surface where the *branch locus*, the set of points which do not have a disk neighborhood in K, is contained inside of $T^1 \cap K$. Also, the branch locus is a one-manifold. Lastly, the *branch*

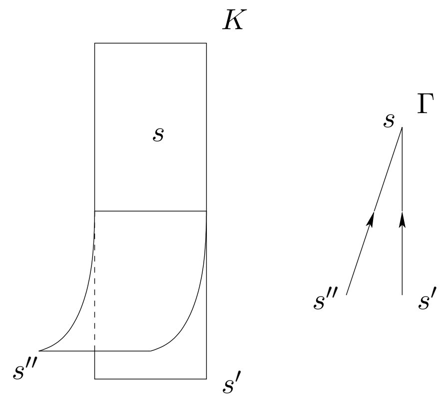


Figure 7.1: A bit of a K and Γ

direction is given by the graph Γ. That is, faces of K caused by the tightening disk $D \subset s$ "feed into" the face s.

Lemma 7.1.6. *All of the connected surfaces carried by this branched surface are disks.*

Proof. Let D be a connected surface carried by K. Note that D must have nonempty boundary, as every sector of K has nonempty boundary. Let γ be a simple closed loop in D. Note that D inherits a decomposition from the branch locus of K. If γ does not intersect the arcs of this decomposition then, as all sectors of K are disks, γ must be contractible.

If γ does intersect the preimage of the branch locus, L, then let x be a point of intersection such that no point of intersection is on an arc of L strictly closer to a preimage of the tightening disk. Since the sectors of K are disks it follows that γ must meet this component of L at least twice. That is, we may reduce $|\gamma \cap L|$ until we realize that γ is contractible. □

It also follows that each disk is carried at most once by any *source sector*; a sector with in-degree zero. We require a final definition:

Definition. Given a skeletal face $s \subset K$ the *carrier* of s, K_s, is the subcomplex of K which is the union of all skeletal faces $s' \subset K$ which are connected to s in Γ by a directed path starting at s and ending at s'.

The face s is contained in its own carrier, as s is connected to itself by a path of length zero. Borrowing a bit a terminology from the language of partially ordered sets, we will refer to the natural subgraph $\Gamma_s \subset \Gamma$ as the *downset* of s.

Theorem 7.1.7. *There is a constant, $d_3 \in \mathbb{N}$ such that if (M, T) is a closed, orientable, triangulated three-manifold and $H \subset M$ is an almost normal, light, Heegaard splitting with almost normal piece an annulus then $d(H) \leq d_3 \cdot |T|$.*

It suffices to take $d_3 \geq 8$.

Proof. Let W be the handlebody component of $M - H$ which contains the exceptional compressing disk D_W. By Remark 7.1.3 D_W is a essential disk in W.

Transversely orient H so that the orientation points towards the exceptional tightening disk of H. Let $V' \subset V$ be the canonical compression body on H's head. Let K be the two-skeleton of V'.

Recall the construction of V': We performed a sequence of tightening isotopies to form an isotopy $\mathcal{F}$ of H to a non-normal surface H'. This H' intersected the boundaries of the tetrahedra of T in short normal curves and in simple curves. We then surgered along all of the simple curves of H' and capped off all two-spheres contained entirely inside of a tetrahedron. Because H is light, one of the surgery curves must in fact be an essential curve in H', by Remark 7.1.3. Let s' be the skeletal face of V' which contains that simple curve.

Let K' be the carrier of s' and let D'_V be the surgery disk contained in s'. We may isotope H' back to H, undoing the tightening isotopy $\mathcal{F}$, and carrying $\partial D'_V$ to an essential curve on H. The reverse of $\mathcal{F}$, extended to an ambient isotopy, also pulls D'_V to a compressing disk for H in V. Denote this disk by D_V. Note that D_V is carried by the branched surface K'. Every face of K' thus receives a positive integer, its *multiplicity*, which counts the number of times D_V is carried by that face.

Note that $|D_V \cap D_W|$ equals the multiplicity of the exceptional tightening disk. Thus, by Lemma 2.3.1, a bound on this multiplicity gives a bound the distance of H.

Let $\widetilde{K'}$ be the "directed" universal cover of K'; the path lifting property holds only for directed paths. A moment's thought shows that $\widetilde{K'}$ is a tree with a copy of s' as its root.

The leaves of $\widetilde{K'}$ are all copies of the exceptional tightening disk. Finally, observe that there are at most $2 \cdot |T|$ hexagons in M and the tree contains a node of degree three only at a hexagon. It follows that $\widetilde{K'}$ has at most $3 \cdot 2^{2 \cdot |T| - 1}$ leaves. Thus,

$$d(H) \leq 2 \log(3 \cdot 2^{2 \cdot |T| - 1}) + 2 \leq d_3 \cdot |T|.$$

□

Remark 7.1.8. We should pause to note that this approach cannot work when the almost normal piece is an octagon. In this case we can still find a pair of compressing disks, each carried on a subcomplex of the skeletal faces. We may even show that the multiplicities of the two exceptional tightening disks is not too large, as above. However, we have no control over how many rectangles each of the carriers contains. Thus we have no real control over how many times the two disks intersect.

7.2 Relations among the annuli

7.2.1 A pair of non-trivial annuli

At this point we should pause and note the following:

Lemma 7.2.1. *Suppose M is a closed, orientable three-manifold and that $H \subset M$ is a Heegaard splitting of M. Suppose further that $A \subset V$, $B \subset W$ are non-trivial annuli or Mobius strips (or perhaps one of each.) Then $d(H) \leq 3$.*

Proof. As A and B are contained in handlebodies they cannot be essential. It follows that they are both disk-like. Compress or boundary compress both A and B to obtain a pair of essential disks, D_A and D_B.

Since $\partial D_A \cap \partial A = \partial A \cap \partial B = \partial B \cap \partial D_B = \emptyset$ we have the desired conclusion. □

Remark 7.2.2. This could be thought of as the underlying theme of this chapter. If such a pair of annuli exists then the distance of the given Heegaard splitting H is less than 4. Stated another way: For all splittings with sufficiently high genus Theorem 3.0.1 bounds the distance. For all splittings with sufficiently low weight Theorem 7.1.1 will bound the distance.

Thus, we may assume the remaining splittings admit a variety of Haken decompositions. Each of these admits a collection of exchange annuli and Mobius bands. In simple cases, one of these collections will submit to Lemma 7.2.1. However, if such favourable conditions do not obtain then more technical methods, as in Theorem 7.1.7, must be used.

Theorem 7.2.3. *There is a constant $d_4 \in \mathbb{N}$ such that if (M, T) is a closed, orientable, irreducible, triangulated three-manifold and H is a light normal or almost normal Heegaard splitting, admitting a non-trivial exchange band in W and a tunnel in V, then $d(H) \leq d_4 \cdot |T|$.*

It suffices to take $d_4 = d_3 + 2$ where a is the constant satisfying Theorem 7.1.7.

Proof. Let $D_W \subset W$ be the disk obtained by compressing or boundary compressing the non-trivial band in W. Let A denote the tunnel in V. Set X equal to the solid torus which A cuts out of V.

By Theorem 7.1.7, if H contains an almost normal annulus then we are done. There are two remaining cases: either H contains an almost normal octagon or H is normal. In either case we chose a transverse orientation of H pointing into V.

For the first case suppose that H contains an almost normal octagon. Apply Theorem 4.1.1 to form the canonical compression body, V', on H's head. As H is light, $\partial_- V'$ cannot be homeomorphic to H and there must be a compressing disk for H contained in V'. As in Theorem 7.1.7, let s' be the skeletal face in V in which the compression occurs and let K' be the carrier of s'.

We will now analyze how K' and X intersect. Note that K' is not contained inside of X, as K' carries an essential disk. Fix an identification of X with $D \times S^1$ where ∂D is decomposed into two connected arcs α and β. Arrange matters so that $\alpha \times S^1 = A$ and so that $K' \cap A$ is a disjoint union of copies of the fibre, $\mathcal{A} = \{\alpha \times c_i \mid c_i \in I\}$.

Note that the product structure on A respects the structure imposed by the triangulation. That is, $A \cap T^2$ appear as levels of the product structure. However, the disks $D \times \{\mathrm{pt}\}$ do not respect the triangulation of either interior(X) or of H.

Cut K' along $\mathcal{A}$ and discard the components which now lie inside of X. We glue a copy of $D \times c_i$ to all of the remaining components of $K' - \mathcal{A}$. Call this new branched surface K''. Note that K'' is not a subcomplex of the two-skeleton as it contains many copies of the disk D.

K'' is still decomposed by its branch locus into a collection of disks. Furthermore, from the proof of Lemma 7.1.6, observe that K'' only carries disks. These disks may be obtained from the disk carried by K' via a (large) number of boundary compressions. It follows that K'' carries at least one essential disk, D_V.

We may now argue as in Theorem 7.1.7: Let τ be one of the boundary components of the tunnel A. Note that

$$d_H(\partial D_V, \tau) \leq d_3 \cdot |T|$$

as above. Because $d_H(\partial D_W, \tau) \leq 2$ we have $d_H(\partial D_V, \partial D_W) \leq (d_3 + 2) \cdot |T|$ and are done with the octagon case.

For the second case suppose that H is normal. As in Remark 4.3.2, perform an irregular exchange along A, obtaining a surface H' isotopic to H and a torus bounding the solid torus X. Let τ' be the seam on H' which records where the cut and paste operation took place.

We tighten H' to obtain a canonical compression body, V'. Again, as H was light, $\partial_- V'$ cannot be homeomorphic to H'. Thus the two-skeleton of V' must contain a face s such that the carrier of s, K', carries an essential disk, D_V. Note that, as in Theorem 7.1.7, $d_{H'}(\partial D_V, \tau') \leq a \cdot |T|$. Since H' is isotopic to H via an isotopy which places τ' on top of τ, we are done as in the octagon case. □

7.2.2 Ordering the annuli

Suppose that M is a bounded irreducible three-manifold. Let $\mathcal{A}$ be a collection of pairwise disjoint compressible annuli, all properly embedded in M. There is a natural relation on $\mathcal{A}$, namely A relates to B if A compresses in the complement of B.

There is a subrelation of the inverse relation which is a partial order:

Definition. Suppose that A compresses in the complement of B. The annulus B *contains* A (written $B > A$) if interior(D) intersects A nontrivially for any compressing disk D of B.

Lemma 7.2.4. *If M is a three-manifold and $\mathcal{A}$ is a collection of compressible, properly embedded annuli inside of M then $(\mathcal{A}, >)$ is a partially ordered set.*

Proof. Clearly we cannot have $A > A$. Suppose that $C > B > A$. Pick any compression disk, D, for the annulus C. After making D transverse to B and A we have the collection of simple closed curves, $D \cap B$. Let γ be an innermost such bounding the disk $D' \subset D$. If γ is not isotopic (in B) to the core curve of B then γ bounds a disk $E \subset B$. As $E \cup D' = S$ is a two-sphere in M we deduce that S bounds a ball. This ball defines an ambient isotopy of B which leaves D fixed. This allows us to reduce $|B \cap D|$ by at least one. This operation may also decrease the quantity $|A \cap D|$ but certainly does not increase it.

After a finite number of these ambient isotopies we must arrive at a curve, again called γ, which is innermost in D and is isotopic (in B) to the core curve of B. In this case the disk $D' \subset D$ is a compressing disk for B. As such $D' \cap A$ must be nonempty, by our original assumption. We deduce that, before any of our isotopies took place, $D \cap A$ was nonempty and we have $C > A$. □

We now give each of the annuli in $\mathcal{A}$ a transverse orientation which points away from that annulus' compressing disk. We will call the connected components of $\partial M - \mathcal{A}$ *regions*.

Lemma 7.2.5. *Suppose that $\mathcal{A} \subset V$ is a collection of compressible annuli inside of a handlebody, V. If $A \in \mathcal{A}$ is maximal with respect to $>$ then the boundary components of A are adjacent to a common region R and the induced transverse orientation on ∂A points into R.*

Proof. Suppose that A is maximal with respect to $>$. Let R be the region pointed to by the transverse orientation of $\partial_+ A \subset \partial V$. Let γ by an arc, properly embedded in A, with $|\gamma \cap \partial_+ A| = |\gamma \cap \partial_- A| = 1$.

For each boundary component, β_i, of R choose a compressing disk D_i for the annulus, $B_i \supset \beta_i$. We choose these disks so that $A \cap \text{interior}(D_i) = \emptyset$. After compressing the B_i along these disks we obtain a family of compressing disks for V. After cutting V along this family we discard all connected components except for the one which contains R. Call this handlebody V_R.

Since the boundary of $V_R = R$ together with a collection of disks, and since $\gamma \subset V_R$ we deduce that $\partial_- A$ was also a boundary component of R. Finally, the transverse orientation on $\partial_- A$ must point towards R, again, because $\gamma \subset V_R$. □

We end this section with the trivial:

Lemma 7.2.6. *Suppose that α and β are disjoint essential simple closed curves on a connected surface S and τ is a simple arc with either one end point in α and the other end point in β or both end points on α. Suppose further that γ is a simple closed curve disjoint from $\alpha \cup \beta$ which meets τ an odd number of times. Then γ is essential.*

Proof. This follows directly from the Jordan Curve Theorem. □

7.3 A final bound

We conclude this chapter with the following theorem:

Theorem 7.3.1. *Take d_4 as defined in Theorem 7.2.3. Suppose that:*

1. *(M, T) is a closed, orientable, irreducible, triangulated three-manifold and*

2. *H is a genus two, light, normal or almost normal Heegaard splitting admitting a non-trivial exchange band $A \subset W$ and*

3. *A is either a Mobius strip or an annulus with both boundary components essential in H.*

Then the distance of H is less than or equal to $d_4 \cdot |T|$.

Proof. Let H be the given Heegaard splitting. Note that if H contains an almost normal annulus then the conclusion follows by Theorem 7.1.7.

Begin with the boundary compressible case: that is, suppose that H is given a Haken decomposition such that W contains a non-trivial boundary compressible band A with ∂A essential in H. Let D be a boundary compressing disk for A and set $\tau = D \cap H$. Now by Remark 2.6.10 there must be a seam, β, of H which comes from a exchange band $B \subset V$, is disjoint from ∂A, and meets τ an odd number of times.

By Lemma 7.2.6 it follows that β is an essential curve and hence that B is either a nontrivial band or is a tunnel. In the former situation, we are done by Lemma 7.2.1 and in the latter situation we are done by Theorem 7.2.3. (Note that this deals with Mobius strips, if they appear.)

Suppose now that there is a tunnel, C, in W. If there is an non-trivial band in V, then we would again be done by applying Theorem 7.2.3. If there is no such band as in Lemma 5.1.8 there is a tunnel in V whose boundary components alternate with those of C. We conclude that H was not light, a contradiction.

We find ourselves in the compressible case: all exchange bands in W are compressible annuli. Let $\mathcal{A}$ be the set of all of these annuli. As above, give this set the partial order $>$. Let B be a maximal element with respect to $>$ such that $B > A$. Note that both $\beta' = \partial_+ B$ and $\beta'' = \partial_- B$ are essential. If this were not the case then one of them would bound a disk, $D \subset H$, which does not contain the other. D, or rather a parallel copy of D is a compressing disk for B. As such $D \cap A \neq \emptyset$ because $B > A$. But this implies that one of A's boundary components is trivial, a contradiction.

Finally, Lemma 7.2.5 implies that there is an arc $\tau \subset H$ connecting β' to β'' such that τ meets no other annulus in $\mathcal{A}$. By Lemma 7.2.6 there is an exchange band in V with nontrivial boundary. As above, we may now obtain the desired bound. □

Remark 7.3.2. We end by outlining the remaining case, which has thus far resisted all of our efforts: Suppose H is a light normal or almost normal Heegaard splitting of genus greater than 1, which is not fundamental, of a closed, orientable, irreducible, efficiently triangulated three-manifold. We may further assume that the genus of H is not too large (by Theorem 3.0.1) and that the weight of H is not too small (by Theorem 7.1.1.)

Lemma 5.1.8 implies that, for every Haken decomposition of H, one of the two handlebodies, say W, must contain a nontrivial exchange band. Theorem 7.3.1 implies that if one of these nontrivial bands is either a Mobius strip or an annulus with both boundaries essential then we achieve the desired bound.

We are left with the specter that in *every* Haken decomposition of H *all* of the exchange bands have at least one boundary component trivial. I do not know how to deal with this remaining case.

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