Incoherence of free-by-free and surface-by-free groups

joint with Rob Kropholler and Stefano Vidussi

A group is coherent if every finitely generated subgroup is finitely presented

Some families of groups which are coherent:

- Free groups
- surface groups
- 3-manifold groups ! (Scott)
- Free-by-cyclic groups

Coherent → subgroups are algebraically well-behaved.

What about geometrically well-behaved??

Let G act geometrically on a geodesic metric space X. Then $H < G$ is K-quasi-convex (for the action on X) if every geodesic between points of Hx_0 is contained in a neighborhood of Hx_o It is quasi-convex if such a K exists.

For hyperbolic groups, this doesn't depend on X. Quasi-convex subgroups of hyperbolic groups are hyperbolic.

Locally Quasi-convex: every finitely generated subgroup is quasi-convex. (for hyperbolic groups $\rightarrow$ coherent)

Example: fundamental groups of 3-manifolds with totally geodesic boundary. non-example: closed three manifold groups!

Fibrations!

A group algebraically fibers if it admits a surjection to $\mathbb{Z}$:

$$1 \to K \to G \to \mathbb{Z} \to 1$$

with finitely generated kernel.

Stallings: if a 3-manifold group fibers, the 3-manifold fibers.

Often, a good source of incoherence.

Bieri: G has cohomological dimension $\leqslant 2$. If $N \trianglelefteq G$, N finitely presented $\implies$ N is finite index or free.

$$1 \to K \to G \to \mathbb{Z} \to 1$$

Hiillman - Kochloukova: If G is the fundamental group of an aspherical 4-manifold, K is f.p $\implies$ K is $PD(3)$ group

algebraic fibering $+\ \chi \neq 0 \implies$ incoherence.

Some interesting incoherent groups:
$F_2 \times F_2$
Bowditch-Mess: an incoherent hyperbolic 4-manifold; M. Kapovich

(Kropholler-Vidussi-W) $F_m \rtimes F_n$ $m, n \geqslant 2$. $S_g \rtimes F_n$, $g, n \geqslant 2$ are incoherent.

Some ingredients in the proofs:

$G = H \rtimes F_n$ has *excessive* homology if $H^1(G, \mathbb{Z}) \geqslant n$.

Theorem If $G = F_m \rtimes F_n$ or $G = S_g \rtimes F_n$ has excessive homology, then G is incoherent.

A group is RFRS (residually finite rationally solvable) if $\exists$ a sequence of f.i.n subgroups $G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq G_3 ...$ such that

1) $\cap_i G_i = e$

2) $Ker(G_i \to H_1(G_i; \mathbb{Q})) \leqslant G_{i+1}$

Theorem Let $G = F_m \rtimes F_2, m \geqslant 2$. If there exists $s \in F_2$ such that $F_m \rtimes < s >$ is RFRS, then G is incoherent.

Theorem Let $G = S_g \rtimes F_2$. If there exists $s \in F_2$ such that $S_g \rtimes < s >$ is RFRS, then G is incoherent.

Handle-Moser: $H < Out(F_m)$. Then one of the following holds:
1) H contains a fully irreducible element
2) There is a finite index subgroup of H which leaves a free factor invariant.
This allows us to use induction!!

$F_m \rtimes F_2, m \geqslant 2$ is incoherent.
induct on m (Base case: all $F_2 \rtimes F_m$ have virtually excessive homology)

$S_g \rtimes F_2, g \geqslant 2$ is incoherent.

Corollary (with a little work) The fundamental group G of an aspherical surface bundle over a surface is coherent iff $\chi(G) = 0$

Some questions:
Do free-by-free and surface-by-free groups virtually fiber?
Is coherence a quasi-isometry invariant? local quasi-convexity?