Isometry groups of infinite-genus hyperbolic surfaces: joint w/ T. Aougab + N. Vlamis

Theorem (Allcock '06) For any countable group G, there is a complete hyperbolic surface X s.t. $\mathrm{Isom}(X) \cong G$. When G is finite, X may be taken to be closed.

History regarding ↗ • Greenburg '72 proved G is finite $\Rightarrow \exists$ closed X.
• Kojima '88 – proved this in dimension 3
• Following Long–Reid '05 → Belolipetsky–Lubotzky '05: proved for mfds of any dim

Note: Allcock gives explicit construction and when G is not finite he gets an infinite type surface.

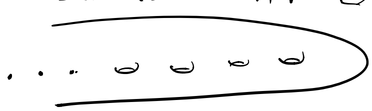
Def: A surface S is of finite type if $\pi_1(S)$ is fin. gen. Otherwise S is of infinite type.

Question: Fixing any infinite-type surface S, for which groups G does $\exists\ \rho$ a complete hyp. metric s.t. $G \cong \mathrm{Isom}(X)$ where $X = (S, \rho)$?

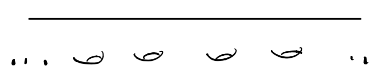
Below X will always denote a complete geod. hyp. structure on S.

Crash course in ∞-type surfaces

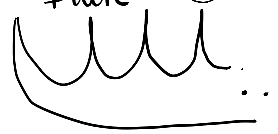
Exs: Loch ness monster ①



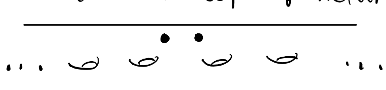
ladder ②



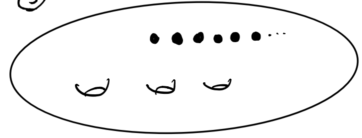
flute ③



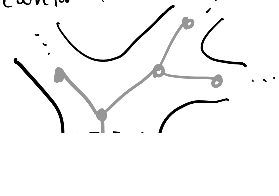
④ ladder w/ 2 punctures



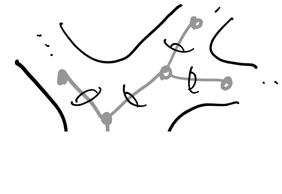
⑤

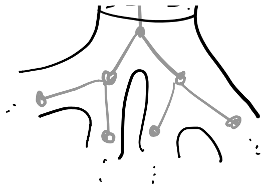


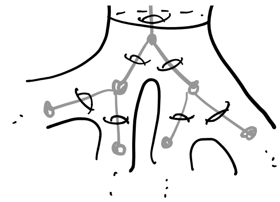
⑥ cantor tree



⑦







blooming cantor tree

Def: E = "space" of ends

totally disconn., metrizable, separable top. space

E^G closed subset of ends a.c.g.

Types of ends:

- accumulated by genus
- planar

Theorem (Kerekjarto, Richards '63) [Classification]

An infinite type surface S is determined up to homeomorphism by (E, E^G, g, b)

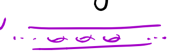
Today: Assume w/out boundary (for simplicity)
All results are for S w/ no planar ends so that $E = E^G$.

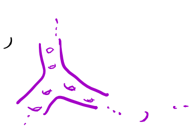
Theorem (Aougab - P. - Vlamis '20): Let S be an infinite-genus surface with no planar ends and let G be an arbitrary group. Then, there are 3 distinct cases:

1.) If the end space of S is self-similar, there exists X with $\mathrm{Isom}(X) \cong G$ <u>iff</u> G is countable.

Ex: loch ness , chimneys , ...

2.) If the end space of S is doubly pointed, then $\mathrm{Isom}(X)$ is virtually cyclic.

Ex: ladder , ...

3.) If S has a compact nondisplaceable subsurface, $\mathrm{Isom}(X) \cong G$ <u>iff</u> G is finite. Ex: tripod , ...

Note: This theorem covers countable <u>and</u> uncountable end spaces.

When E is countable, theorem is easier to prove <u>and</u> we can stregthen 2.).

Relies on the following classification theorem

Theorem (Aougab-P.-Vlamis) If S has infinite genus and no planar ends then exactly one of the following are true:

1.) E is self similar
2.) E is doubly pointed
3.) S has a compact non-displaceable subsurface.

Corollary of main thm: If S has ∞ genus and no planar ends, then $Map(S)$ contains every countable group.

Applications of Corollary: (Inheritence) when S has no planar ends and E is self-similar:

- $Map(S)$ does not satisfy Tits Alternative (Grigorchuk + Thompson's groups)
- $Map(S)$ contains free groups
- $Map(S)$ is not RF (previously proved P.-Vlamis)

(lots of others! incoherence, etc.)

Tools for main theorem: LOTS pointed symmetry, Mann-Rafi partial order on E, pointed symmetry iff self-similar, edits to Allcock construction, etc.