High crossing knot complements with few tetrahedra

Neil Hoffman

Oklahoma State University

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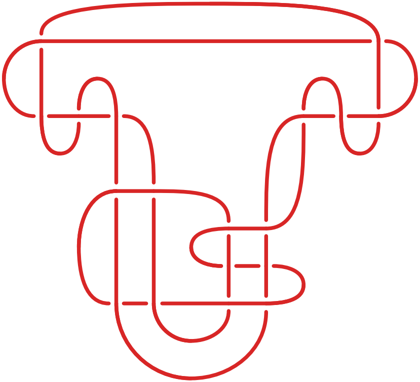
jt. work with Robert Haraway
Warwick Geometry and Topology Online / International Centre for Mathematical Sciences (ICMS)

Big Question(s)

Diagram to triangulation: Given a diagram D of a knot K how many tetrahedra are needed to make up a complement?

Triangulation to Diagram: Given a triangulation $\mathcal{T}$ of a knot complement $S^3 \setminus K$, how many crossings could K have?



Restatement:
$c(K)$ minimum crossing number over all diagrams of K.
$t(K)$ minimum number of tetrahedra needed to triangulate a complement of K.

Diagram to triangulation: Coarsely bound $t(K)$ by a function in $c(K)$.

Triangulation to Diagram: Coarsely bound $c(K)$ by a function in $t(K)$.

Octahedralization

Octahedral Decomposition (attributed to D. Thurston)
$t(K) \leq 4c(K)$ using octahedra.

Triangulation to Diagram: Is $c(K)$ bounded by a **polynomial** function in $t(K)$?

No!

Theorem (Haraway-H)

There is a constant C such that the complement of the torus knot $T_{F_{n+3},F_{n+2}}$ in S^3 can be triangulated with at most $(2n-1)+C$ tetrahedra and $c(T_{F_{n+3},F_{n+2}}) \geq \varphi^{2n}$, where $\varphi = \frac{1+\sqrt{5}}{2}$.

Triangulation to Diagram: If $S^3 \setminus K$ hyperbolic, is $c(K)$ bounded by a **polynomial** function in $t(K)$?

Still no!

Theorem (Haraway-H)

The complement of twisted torus knot $T(F_{n+5}, F_{n+4}, 2, 4)$ in S^3 can be triangulated with at most $2n - 1 + D_1 + D_2$ tetrahedra and $c(T(F_{n+5}, F_{n+4}, 2, 4)) \geq \varphi^{2n}$, where $\varphi = \frac{1+\sqrt{5}}{2}$.

Our construction here can be adapted to Satellite knot complements as well.

Bag of Tricks

Theorem (Murasugi)

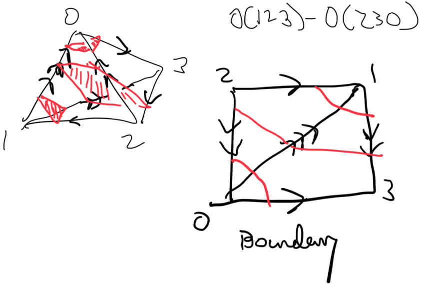
A p/q torus knot $K_{p,q}$ with $p \geq q \geq 2$ has at least $p(q-1)$ crossings. More generally, if K is any knot presented as a homogeneous n-braid with braid index n, $c(K)$ can be read from that diagram.

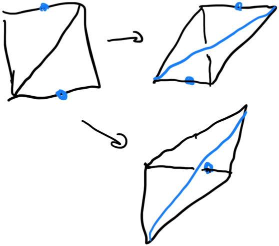
Two Gadgets

1.

2.

Jaco and Rubinstein's Layered Solid Tori





$t(K)$ bounding $c(K)$

Proposition (H-Haraway)

If K is a torus knot, there exists globally defined exponential function in $t(K)$ that bounds $c(K)$.

Theorem (Greene, Howie)

It is decidable if $\mathcal{T}$ is the triangulation of an alternating knot complement.

Corollary (Juhász–Lackenby)

If K is alternating, $c(K)$ is bounded by an function of $7t(K)^3 \cdot 2^{14t(K)+4}$.

Thank you for your attention!