A Strong Haken Theorem

Outgrowth of work with M. Freedman on Powell Conjecture

Warwick/ICMS/Zoom 12 May 2020

Suppose T is a Heegaard surface for a compact orientable 3-manifold M, so $M = A \cup_T B$. Recall:

Definition

- *(M, T) is reducible if there is a sphere in M intersecting T in a single essential circle.*
- *(M, T) is ∂-reducible if there is a properly embedded disk in M intersecting T in a single essential circle.*

Foundational: allows controlled reduction/∂-reduction of (M, T).

Theorem

- *(Haken, 1968) If M is reducible, so is (M, T).*
- *(Casson-Gordon 1983) If M is ∂-reducible, so is (M, T); moreover the ∂-reducing disks have the same boundary.*

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Only indirect relation between reducing sphere for M and reducing sphere for (M, T), & ditto for ∂-reducing disks.

Theorem (Strong Haken)

[Suppose M has no $S^1 \times S^2$ summands] and M contains a properly embedded surface S consisting of ∂-reducing disks and reducing spheres for M. Then T can be isotoped so that each component of S is a ∂-reducing disk or a reducing sphere for (M, T).

The condition that each 2-sphere is separating is used frequently in the proof, but may not be necessary.

For the talk we take S a disk with $\partial S \subset \partial_{-} B = \partial M$.
(Hence A is a handlebody.)

Let Σ denote a spine of B, that is (a thin regular neighborhood of) the union of ∂B and a graph in B such that B deformation retracts to Σ. $\Delta \subset A$ is a complete collection of meridians of A, so $A - \Delta$ consists of 3-balls.

Will think: $\Delta \subset A = M - \Sigma$.

Consider an edge e of Σ that is disjoint from Δ; that is, $\partial\Delta$ nowhere runs along e. A point on e corresponds to a meridian of B whose boundary lies on $A - \Delta = 3 - balls$.
So the boundary of the meridian also bounds a disk in A. Thus the point on e corresponds to a a reducing sphere for T. So call such an edge a reducing edge of Σ.

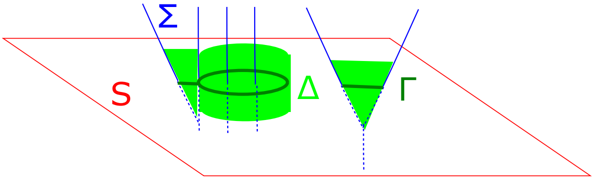
Lemma

Suppose a spine Σ and a complete collection of meridians Δ for A have been chosen to minimize $(|\Sigma \cap S|, |\partial\Delta \cap S|)$ Then Σ intersects S only in reducing edges.

Notes:

- We do not care about the number of circles in $\Delta \cap S$.
- If $S \cap \Sigma = \partial S$, S is a ∂-reducing disk for (M, T)

$(\Sigma \cup \Delta) \cap S$ can be viewed as a graph Γ in S in which $\Sigma \cap S$ are vertices and $\Delta \cap S$ are the edges. (Regard ∂S as 'vertex at ∞'.)



End of Phase I: Only reducing edges of Σ intersect S

Intermission: Lollypops in compression-bodies.

Let W be a 3-manifold and $\delta : (S^1, p) \to (\partial W, ^*)$ a generic immersion that is null-homotopic in W. Then $\exists$ crossing resolutions of δ so that δ, pushed into W rel *, bounds disk in W.

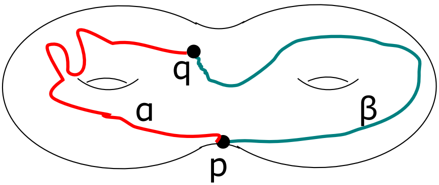
More technically:

Theorem (Freedman-S, 2017 - Lollypop Theorem)

Let $\partial W \times [0,1))$ be a boundary collar. There is a height function $h : S^1 \to [0,1)$ so that $h(p) = 0$, $h(S^1 - p) \subset (0,1)$ and the image of $\delta' : S^1 \to \partial W \times [0,1)$ defined by $\delta'(\theta) = (\delta(\theta), h(\theta))$ is an embedded curve bounding a disk in M.

Corollary

Suppose C is a compression-body with $p \in \partial_+ C$ and $q \in$ interior(C). Suppose α, β are two arcs from p to q in C. Then, perhaps first sliding the end of β at p around a closed path in $\partial_+ C$ and allowing points of the arc β to pass through the arc α, β can be isotoped rel endpoints to α in C.



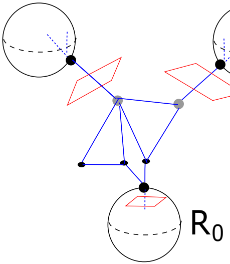
Proof uses two compressionbody - facts:

- $\pi_1(\partial_+ C) \to \pi_1(C)$ surjective and
- complement of *spine*(C) is boundary collar.

End of Intermission

Phase 2: Choosing reducing spheres disjoint from S

Let $\mathfrak{R}$ be the reducing spheres in M associated to all edges of Σ that intersect S. Let $M_{\mathfrak{R}}$ be a component of $M - \mathfrak{R}$.
$\mathfrak{R}$ appears in $M_{\mathfrak{R}}$ like flowers with blossoms (the reducing spheres) on $\partial M_{\mathfrak{R}}$, and stems (the reducing edges) mostly inside $M_{\mathfrak{R}}$.

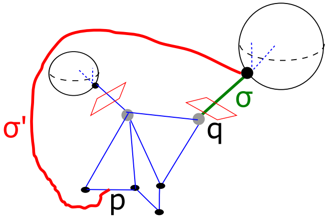


$M_{\mathfrak{R}}$

Important note: $M_{\mathfrak{R}} - \Sigma$ is handlebody A cut along reducing disks; i. e. still a handlebody.

Proposition (Stem Swapping)

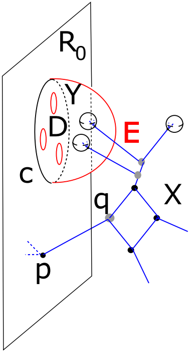
The complex Σ' obtained from Σ by replacing the stem σ with σ' is also a spine for T. That is, T is isotopic to a regular neighborhood of Σ'.



A stem swap

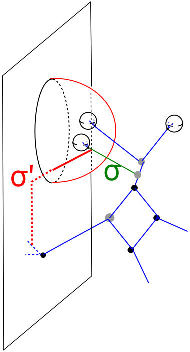
Proof: Apply Lollypop Corollary to σ and σ', arcs in handlebody $A \cap M_{\mathfrak{R}}$ ending at a very fat point: the blossom.

Let $c \subset \mathfrak{R} \cap S$ be innermost in S, bounding disk $E \subset S$, and let $D \subset R_0$ be the disk c bounds in $\mathfrak{R}$.



Problem: If replace D with E, R_0 is no longer reducing sphere.

Solution: first do stem swaps:



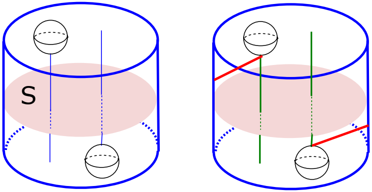
Then replace $D \subset R_0$ to get new reducing sphere R_1, reducing $|\mathfrak{R} \cap S|$. Eventually $\mathfrak{R}$ and S disjoint.

Note: $|\Sigma \cap \mathfrak{R}|$ may increase. We don't care.

Phase 3: Swap reducing edges off of S

Proposition

Suppose Σ intersects S only in reducing edges, and the associated set $\mathfrak{R}$ of reducing spheres is disjoint from Σ. Then T can be isotoped so that S is a ∂-reducing disk for T.



Swaps clearing S of final vertices

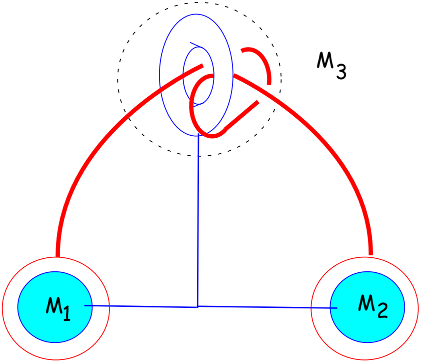
Proof: Swap stems as shown. Then S ∂-reduces T. QED

Note: It's tempting to pull blossoms through S, but this alters isotopy class of S.

Example (courtesy A. Zupan)

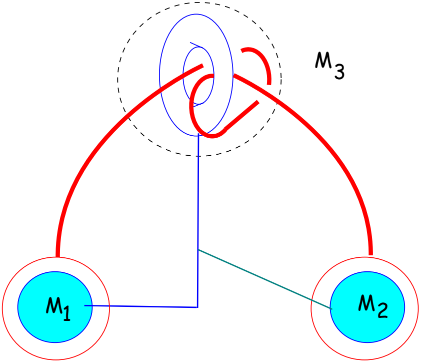
The initial setting is of a Heegaard split 3-manifold $M = M1\#M_2\#M_3$. Spine for M_3 shown as blue, including torus boundary conponent. Part of A shown is solid torus.

Target sphere S is sum of reducing spheres for M_1, M_2 along tube in M_3 shown in red.



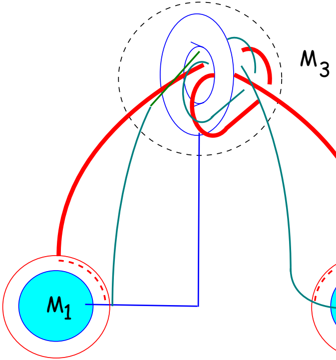
The initial setting

Rightmost edge turns color and begins to slide on the rest of the spine, towards a stem-swap:



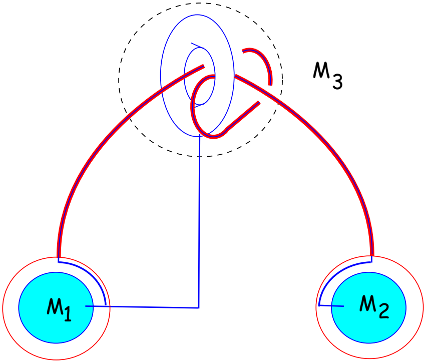
The slide begins

Because $\pi_1(\partial A) \to \pi_1(A)$ is surjective, and the slides take place in ∂A, one can slide end of arc on the rest of Σ until it is homotopic rel end points to the red path shown.



Edge now homotopic to (extended) red tube

Now apply Lollypop Theorem: edge now goes right through the tube, never intersecting S. S has become reducing sphere for (M, T).



Green edge isotoped into red tube