Minimal Surfaces in Hyperbolic 3-Manifolds

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Mathematics Department

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Background and Motivation

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Every 2-sphere contains a closed geodesic.

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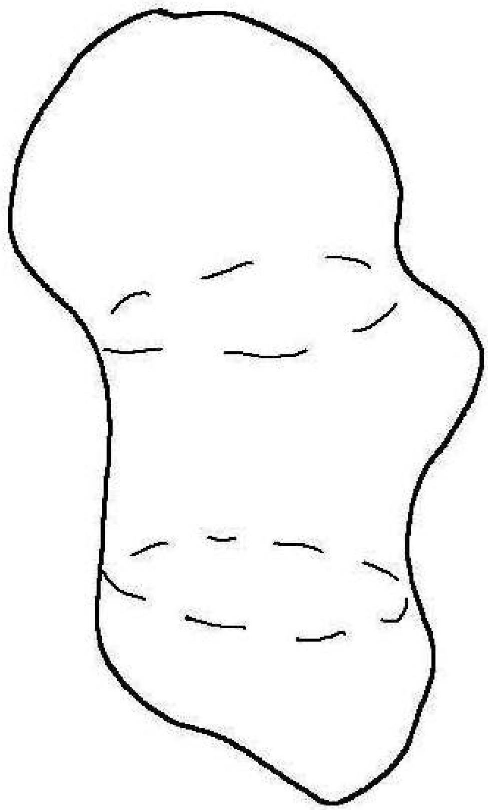

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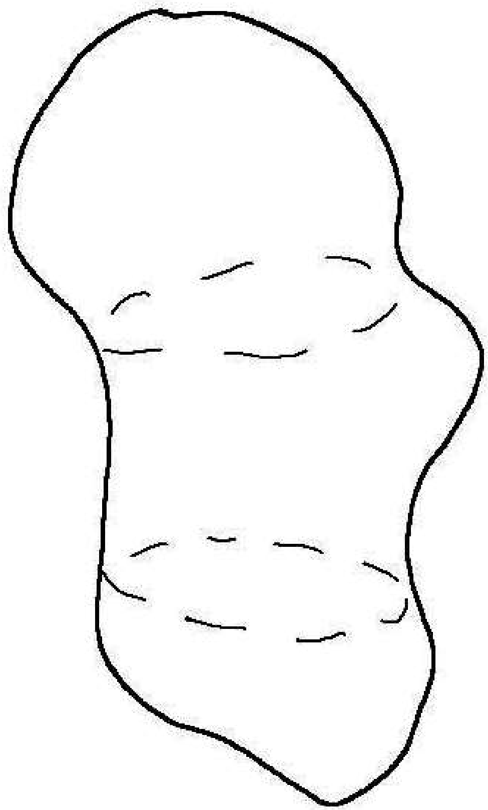

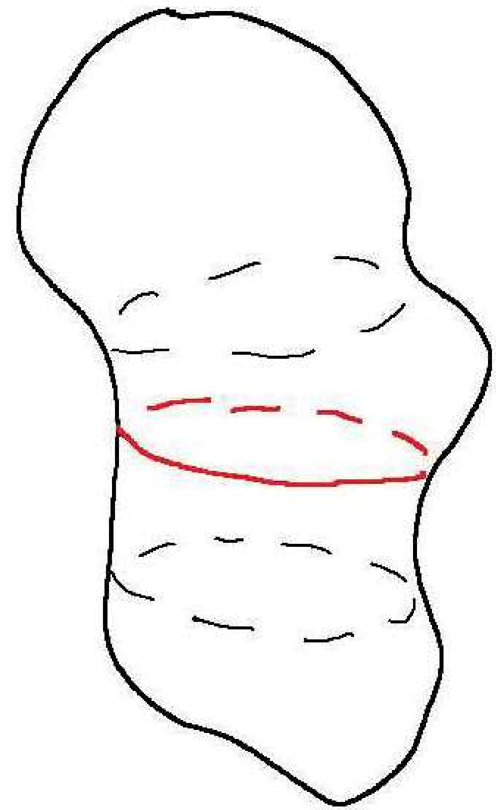

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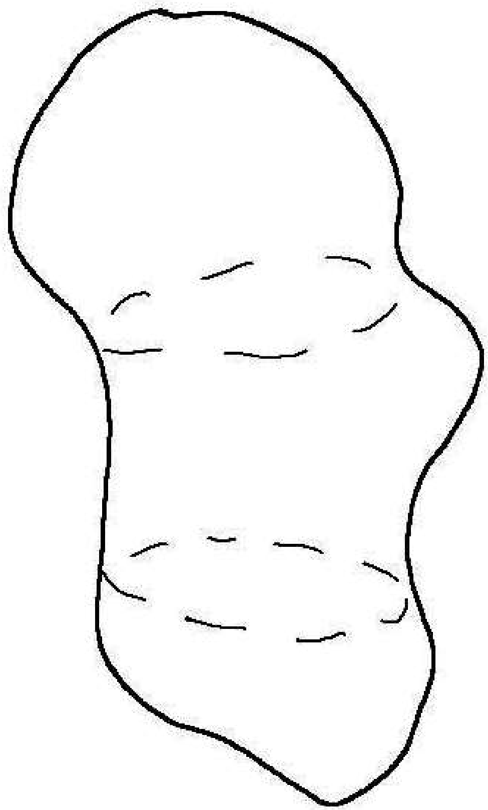

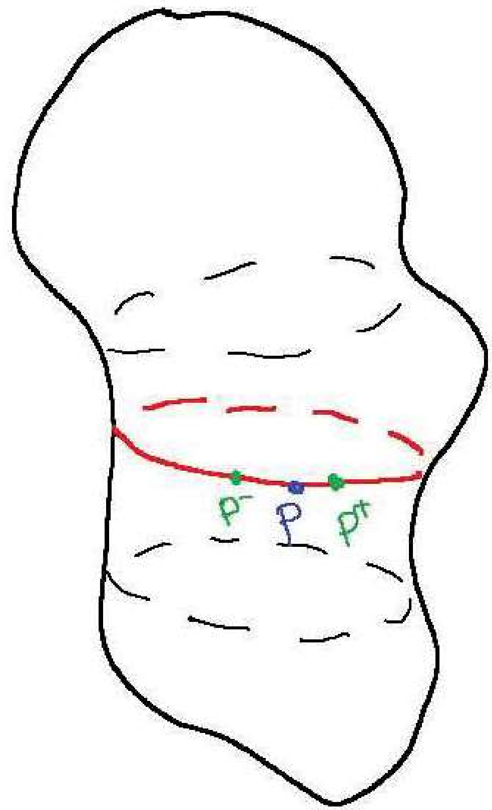

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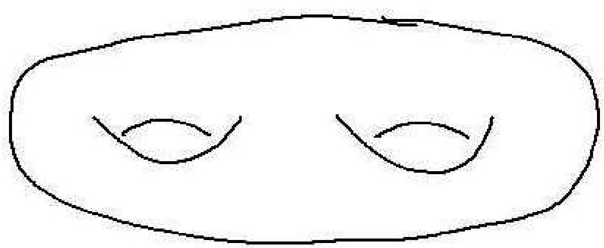

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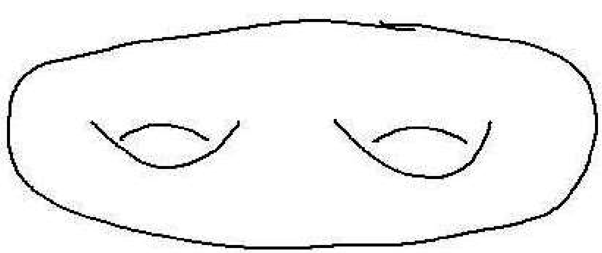

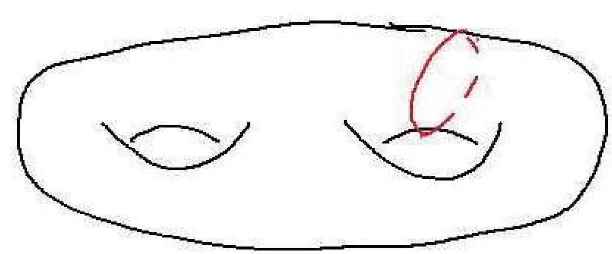

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Question:

Does every closed 3-manifold contain a closed, embedded minimal surface?

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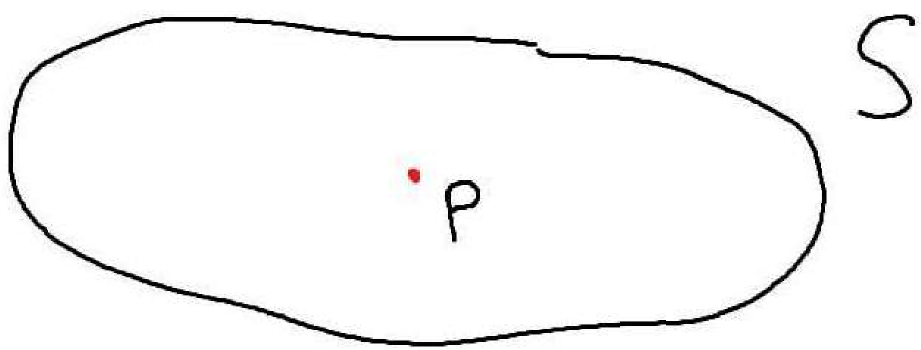

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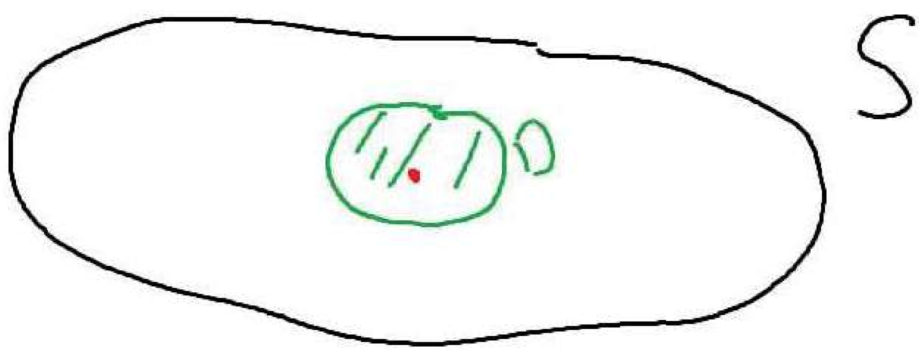

D is the least area disk among the disks with the same boundary

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  - Area Minimization in a restricted class.

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Infinite Volume Hyperbolic 3-Manifolds

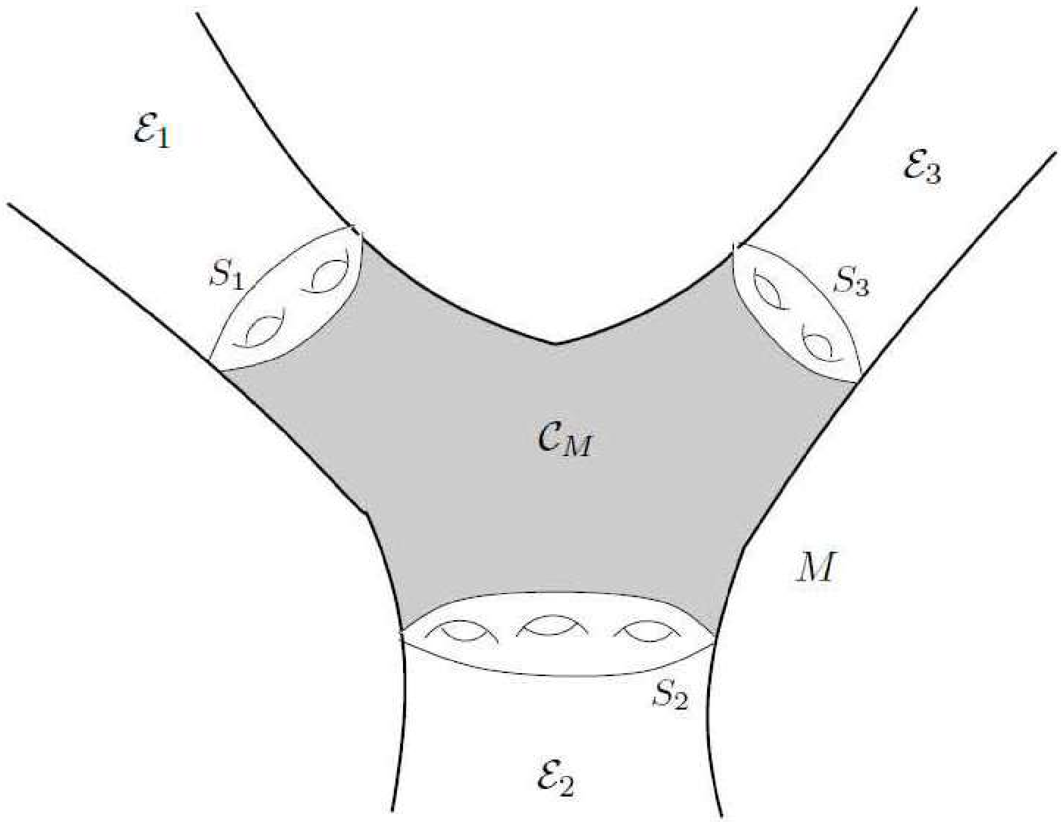


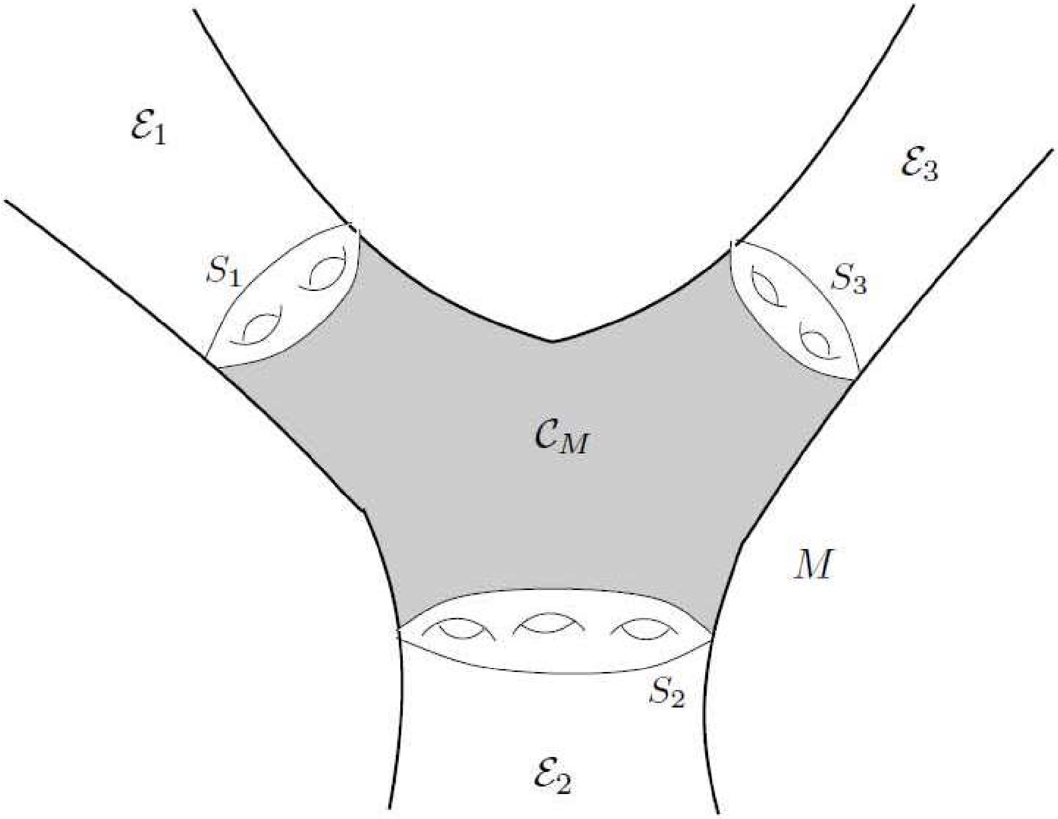
FIGURE 1. M is an infite volume hyperbolic 3-manifold with 3 ends. The shaded region is the compact core $\mathcal{C}_M$.

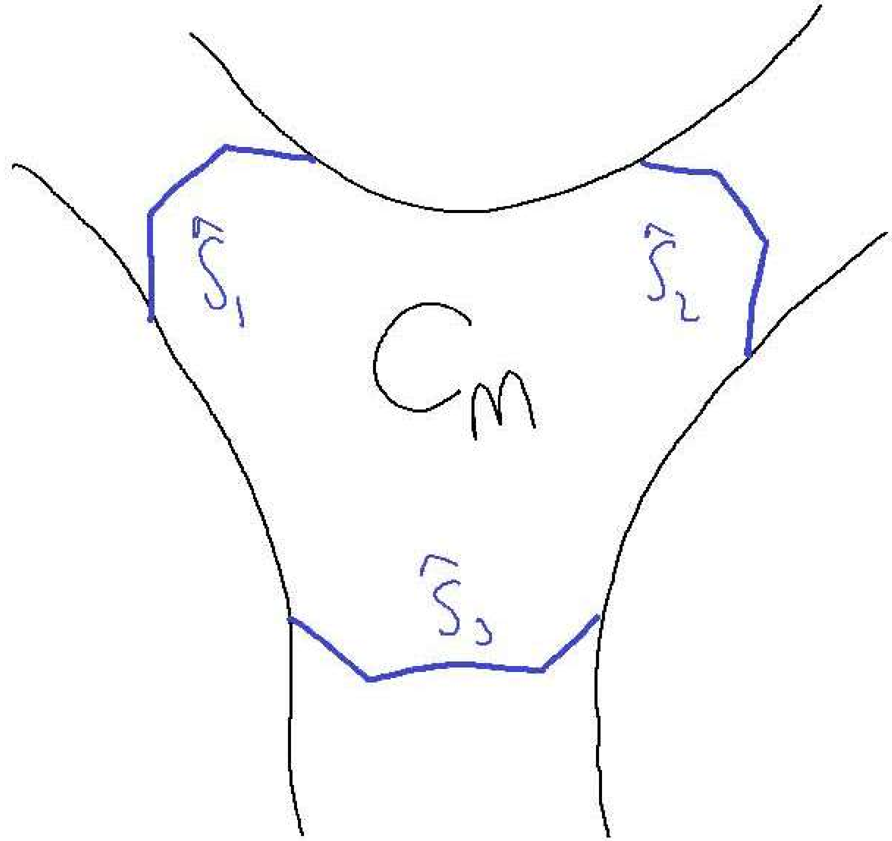
$$\mathcal{E}_i \simeq S_i \times [0,\infty)$$

Marden Conjecture

$$M \simeq \mathring{C}_M \qquad M - \mathring{C}_M = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$$

M geometrically finite: Compact core can be chosen CONVEX





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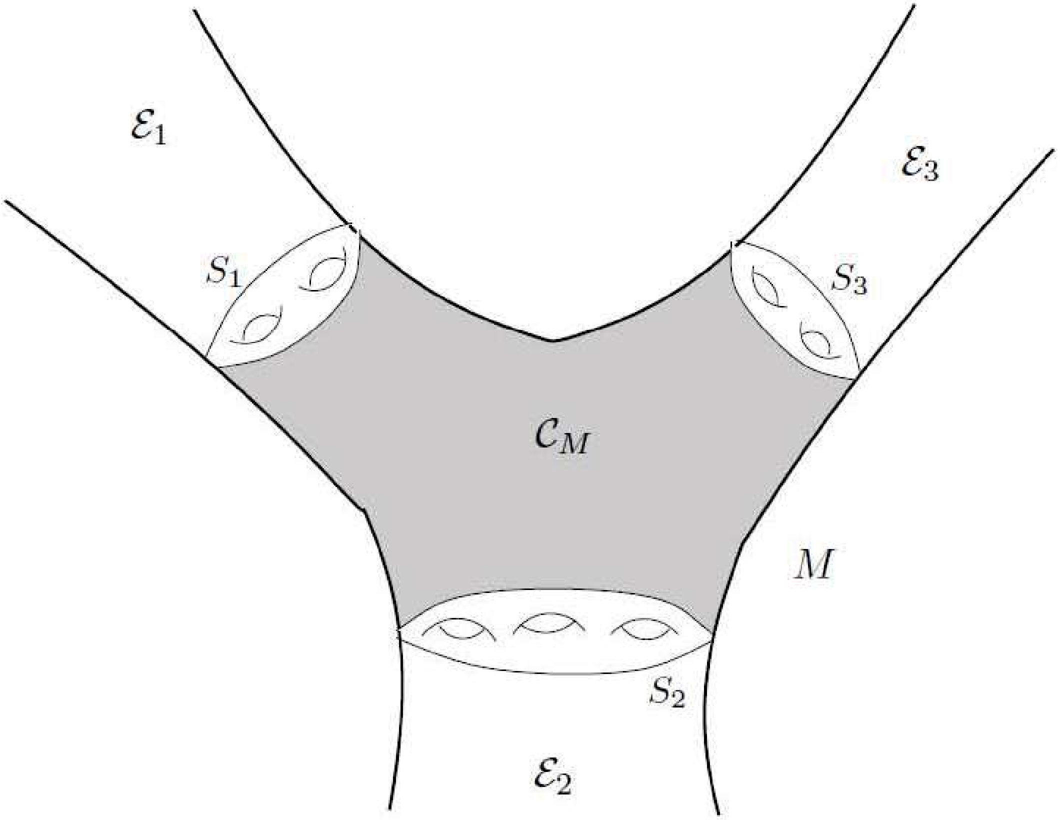

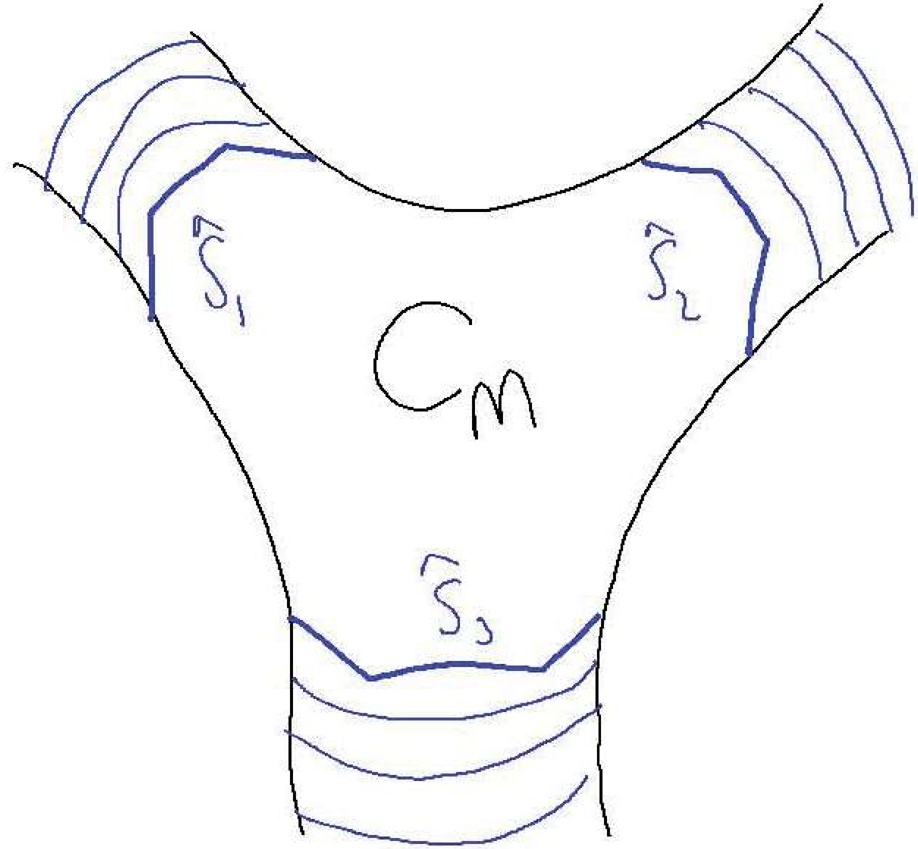

Ends are very simple.

Equidistant surfaces are convex

and foliates the ends.

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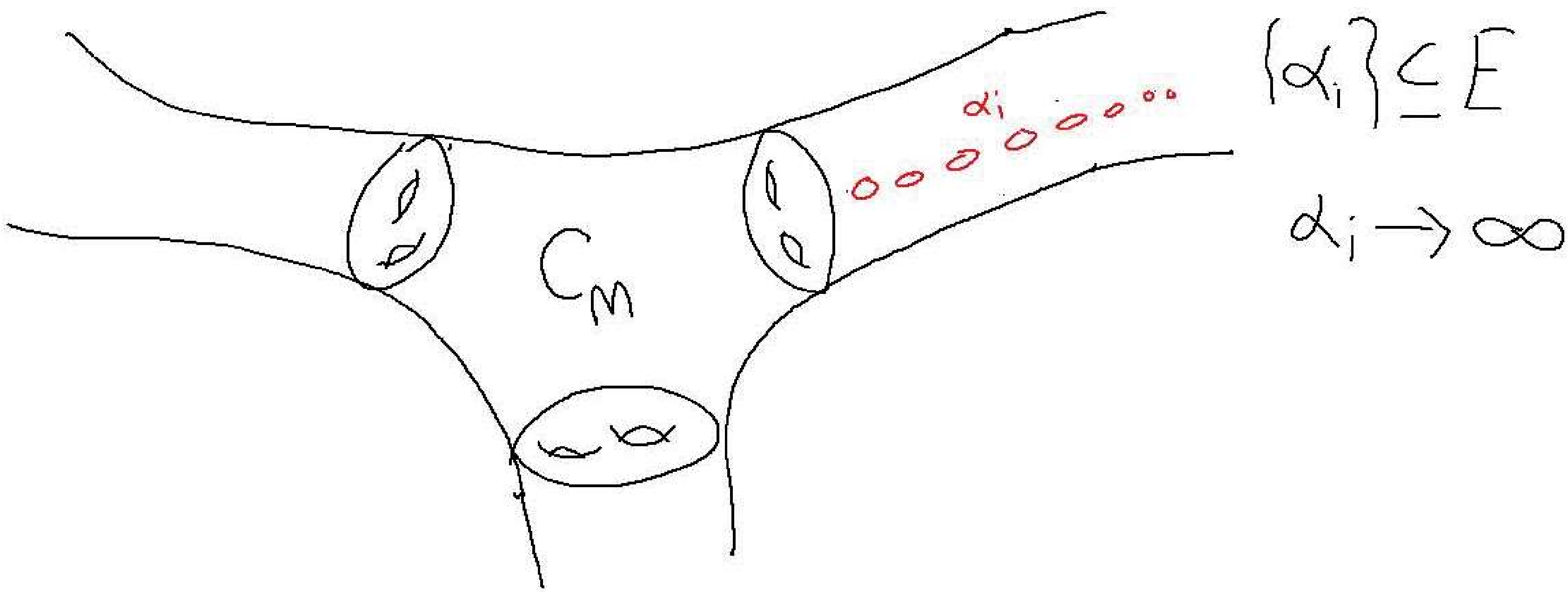

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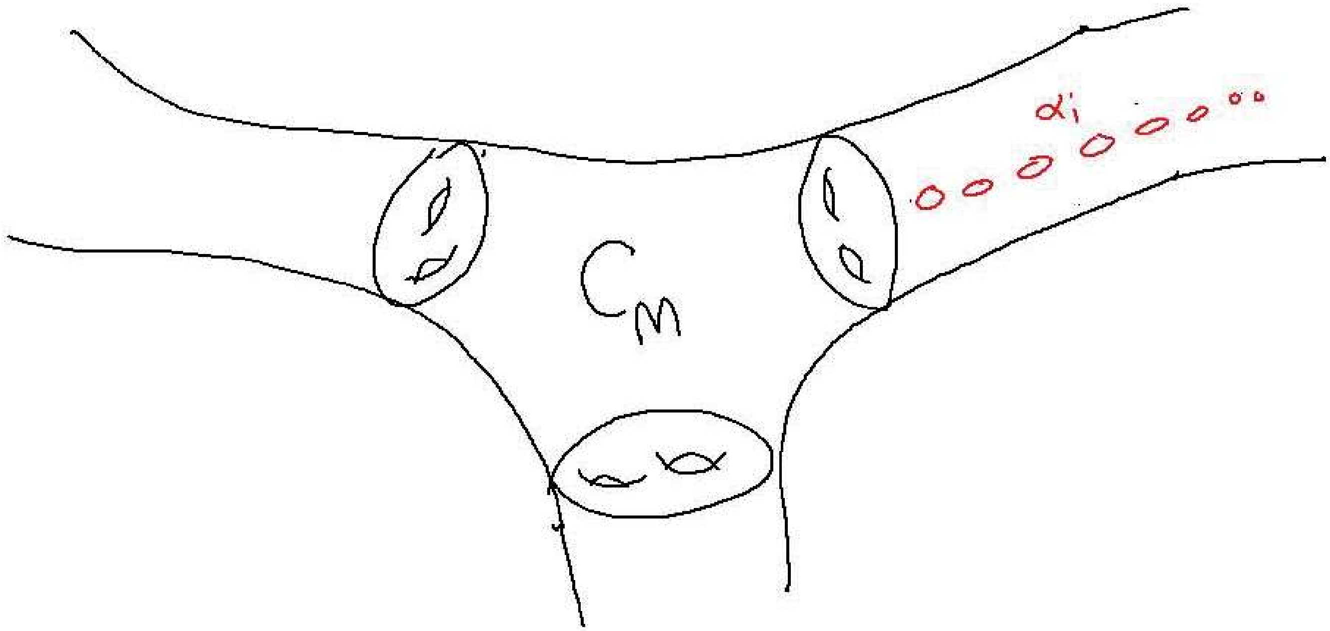

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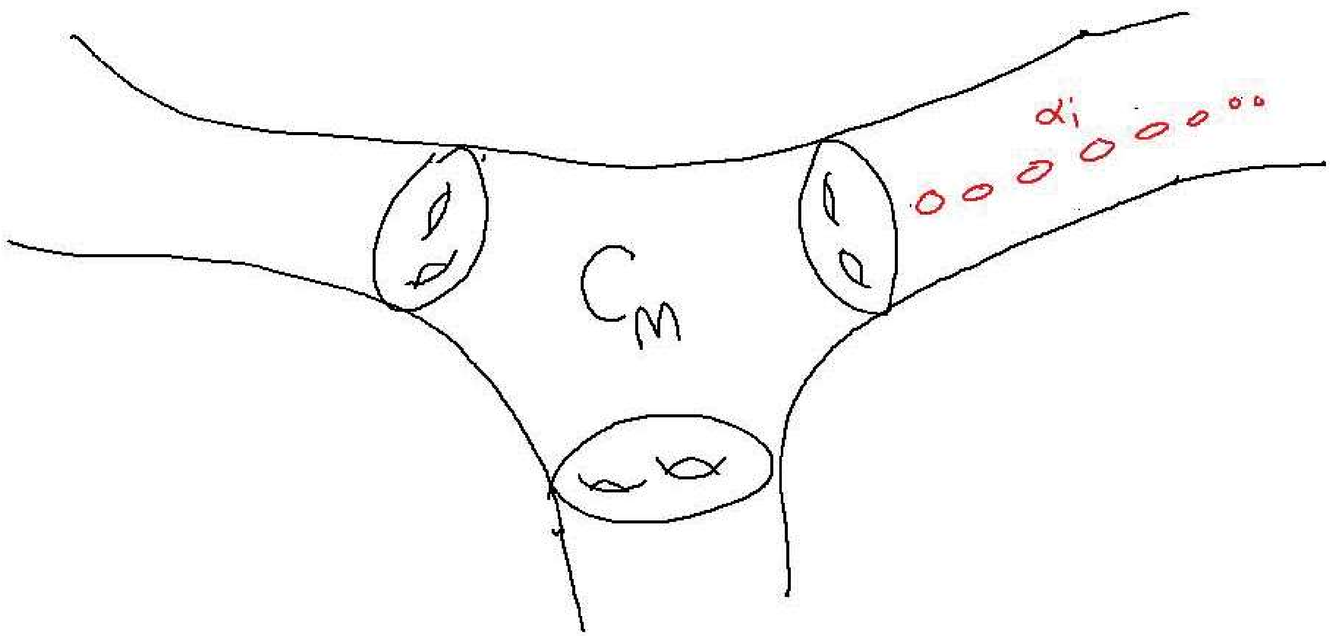

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E has unbounded geometry if injectivity radius = 0

⇨ $\exists \{\alpha_i\} \subseteq E$ s.t. $\alpha_i \to \infty$ & $|\alpha_i| \searrow 0$

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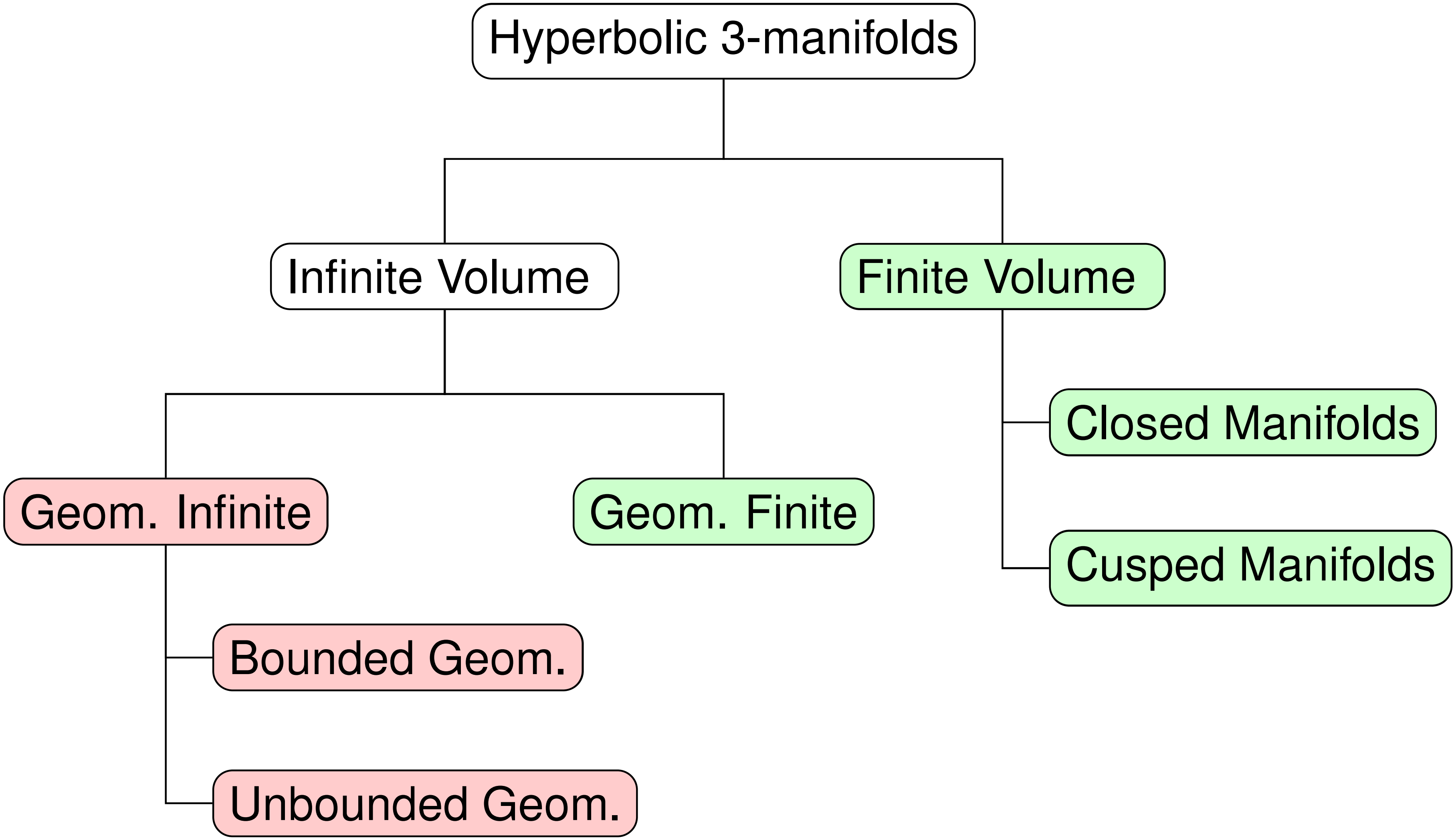

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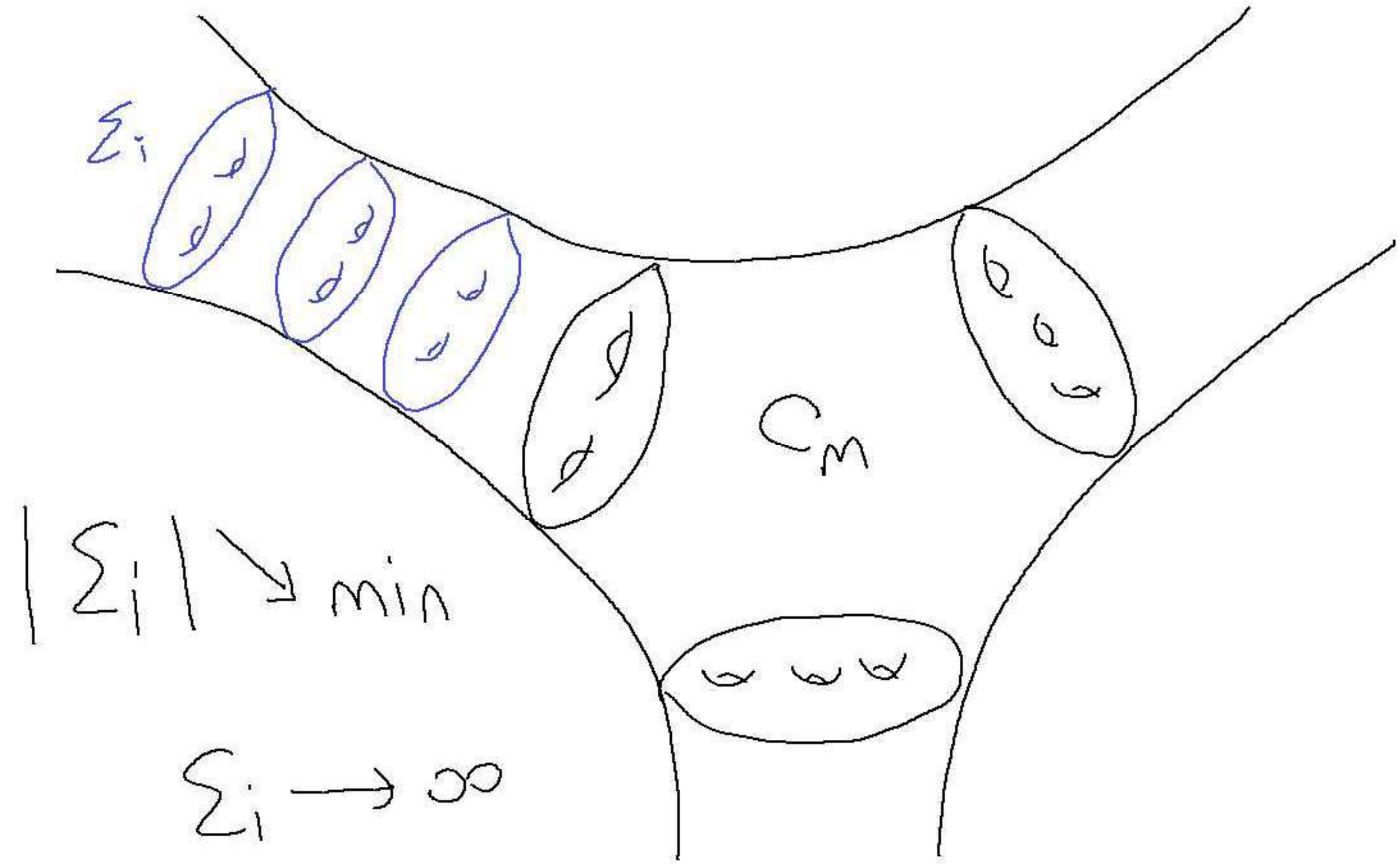

minimizing sequence escaping to infinity

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Tools

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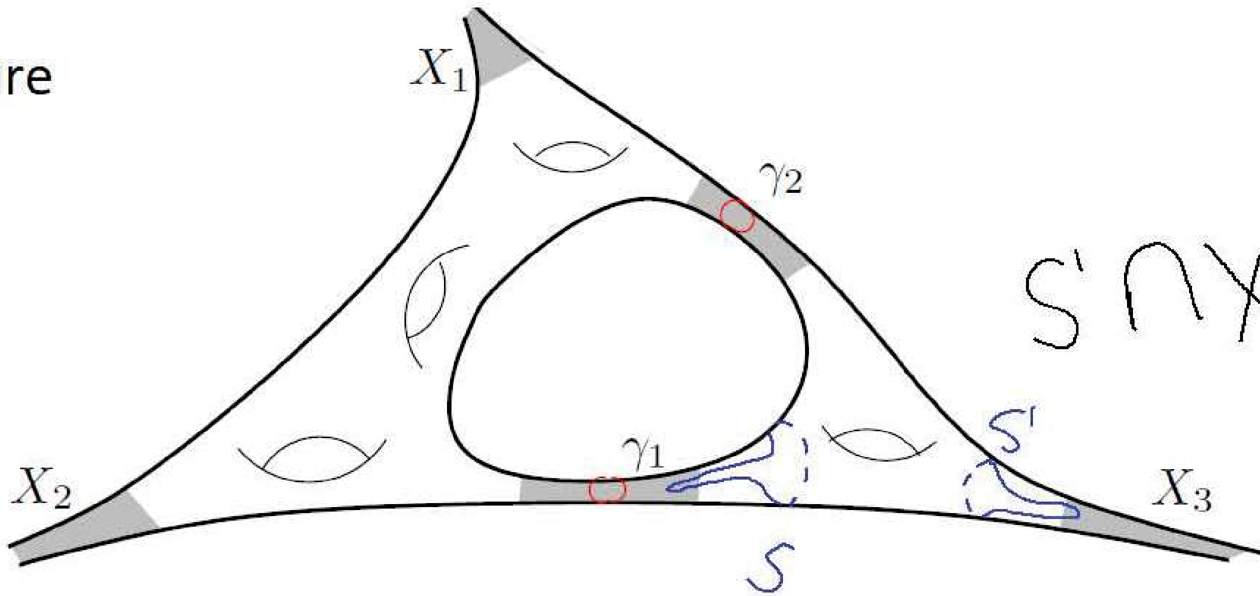
2D picture



$S' \cap X_3 \subseteq T^2 \times [0,c]$

$S \cap N_{\delta_i}(\gamma_i) = \emptyset$

3D: Margulis tube = Solid Torus Neighborhood of a short geodesic

γ short $\Rightarrow N_r(\gamma)$ $\quad |\gamma| \searrow 0 \Rightarrow r \to \infty$

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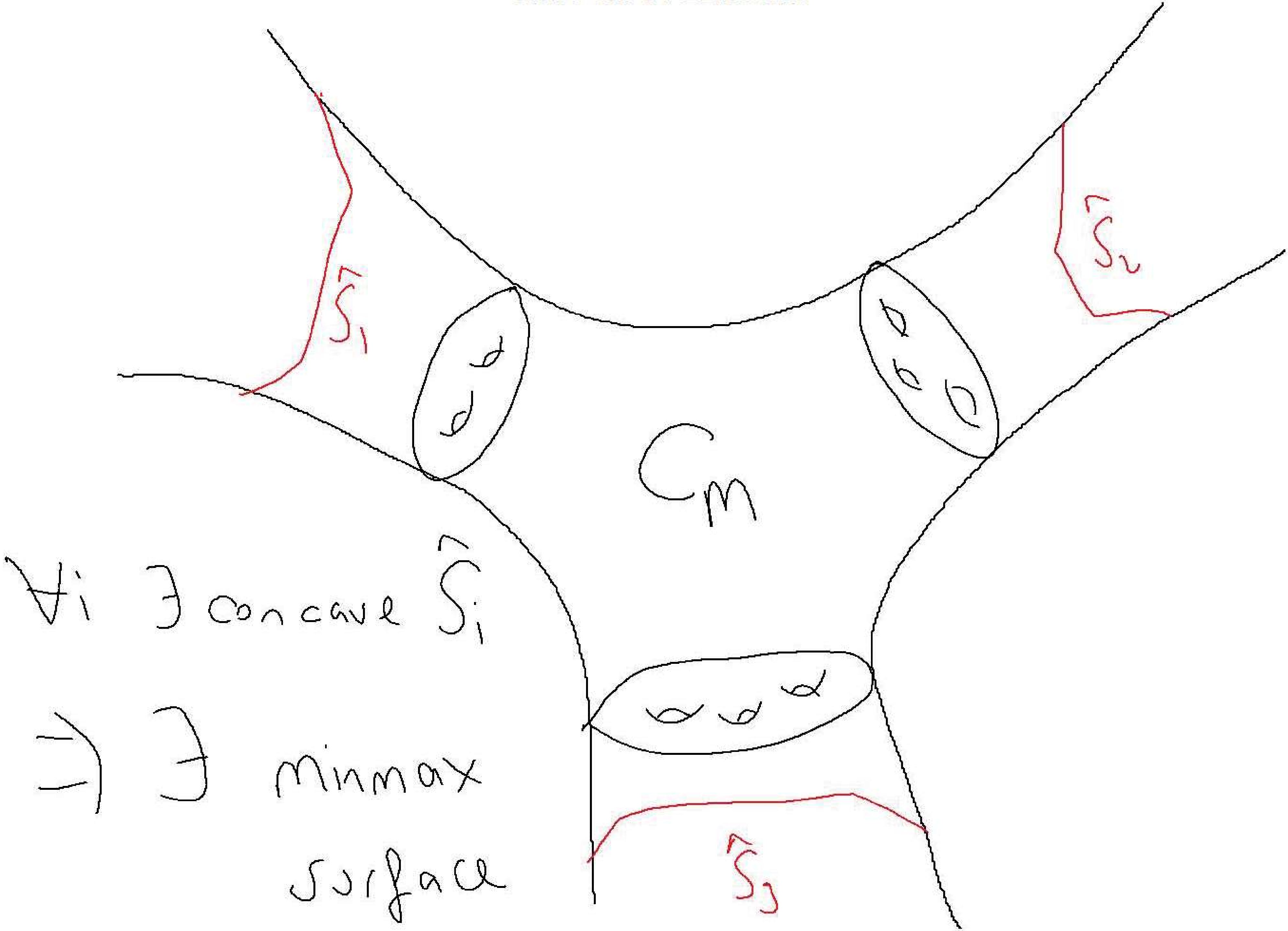
Min-Max for Noncompact Manifolds: [Montezuma, Song 2018]

Let M be a complete, noncompact 3-manifold.If M contains a bounded open set Ω such that $\overline{\Omega}$ has stricly mean concave boundary, then there exists a closed, embedded minimal surface in M.

MIN-MAX LEMMA



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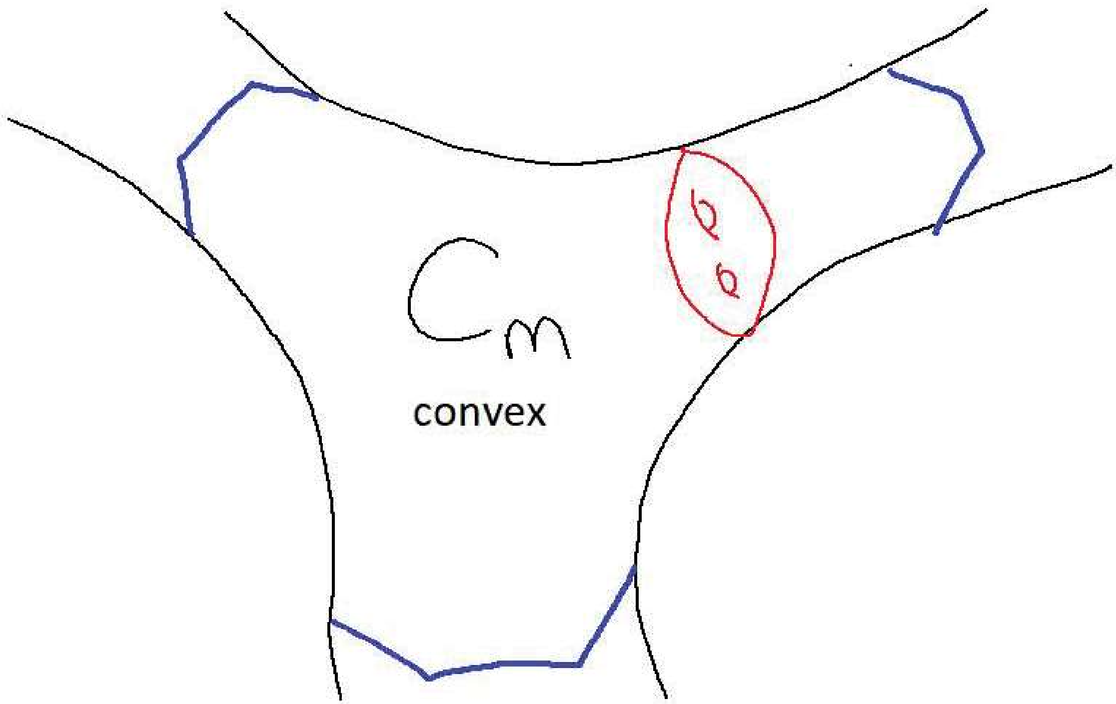
Compact core is convex.

Area minimizer of a homology class stays in the convex core.



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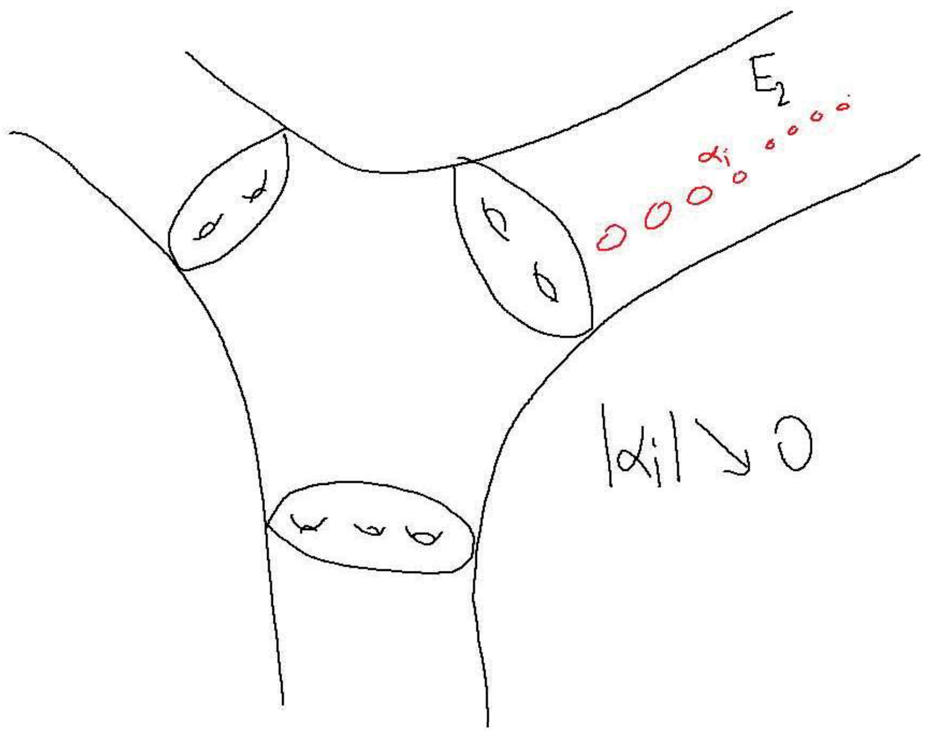

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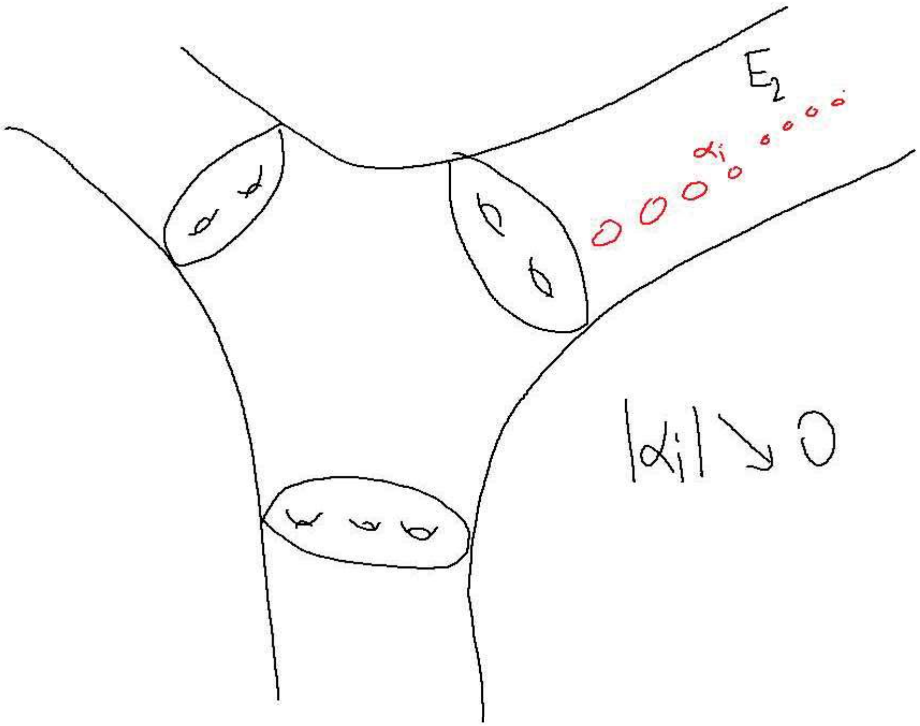

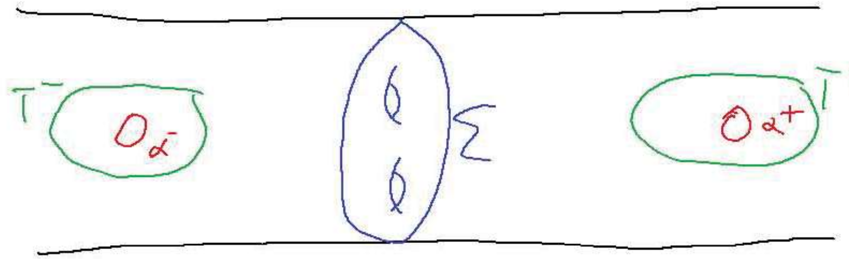

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$\Sigma =$ area minimizer in X

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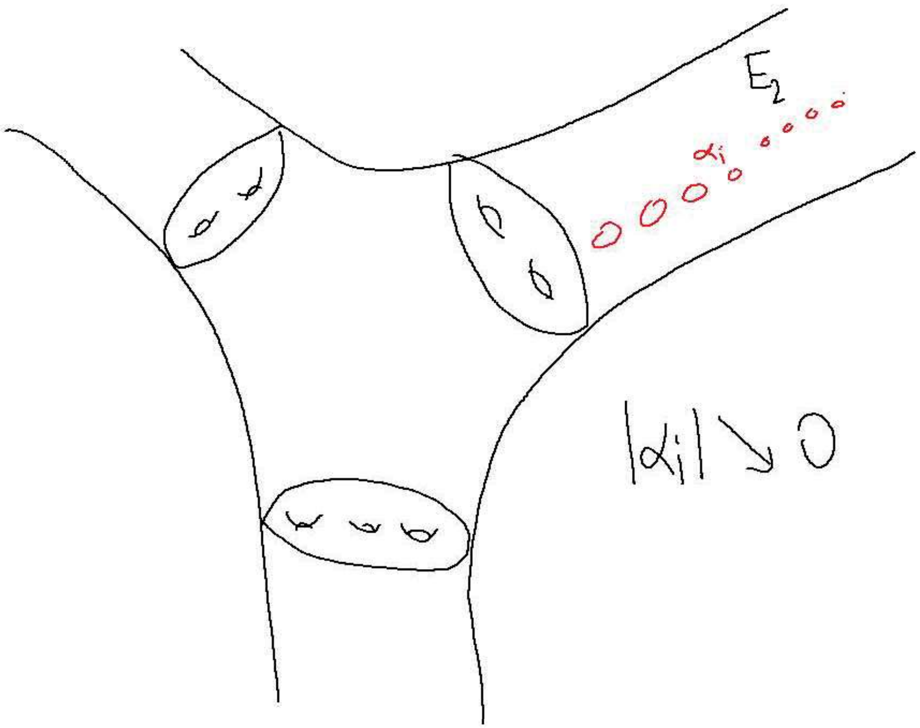

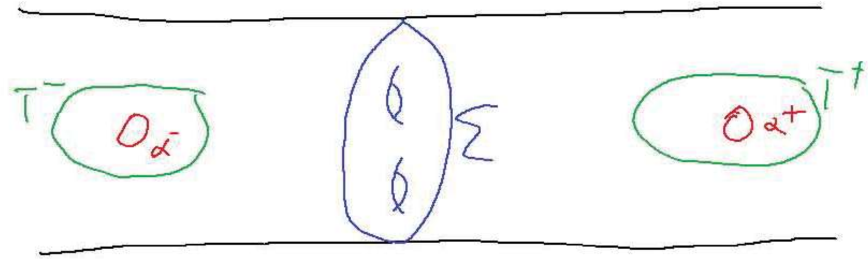

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Σ is away from α^- and α^+

by short geodesics lemma

+ Bounded Diameter Lemma

⇨ Σ is a smooth minimal surface

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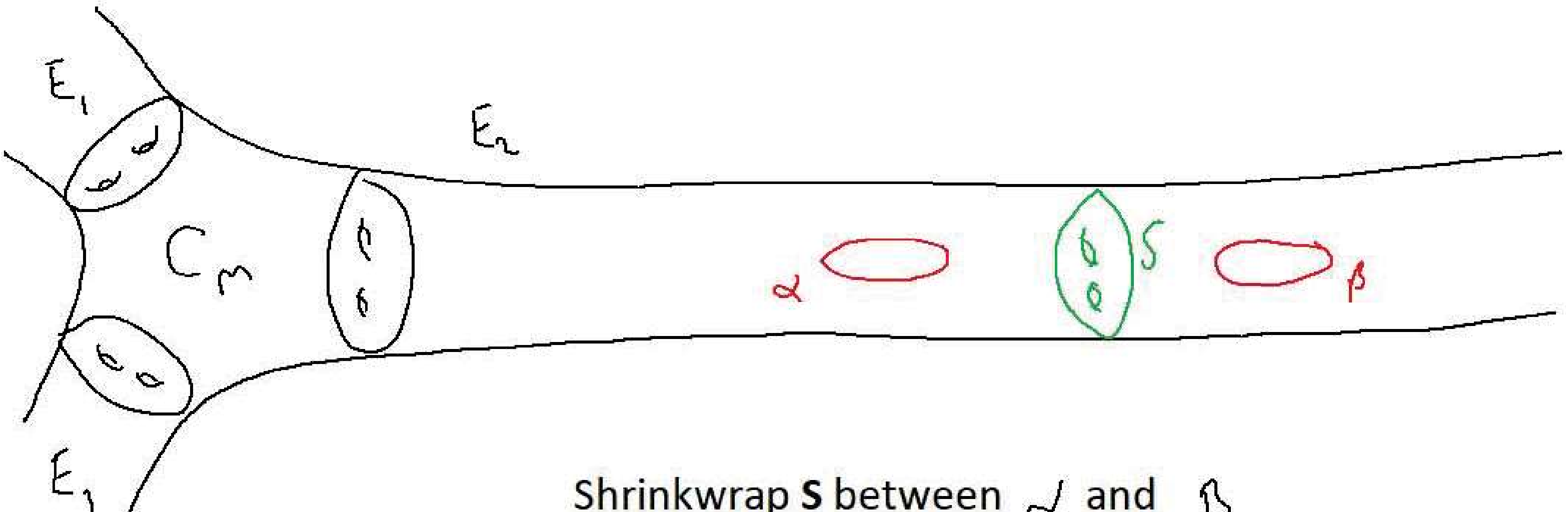

Shrinkwrap **S** between α and β

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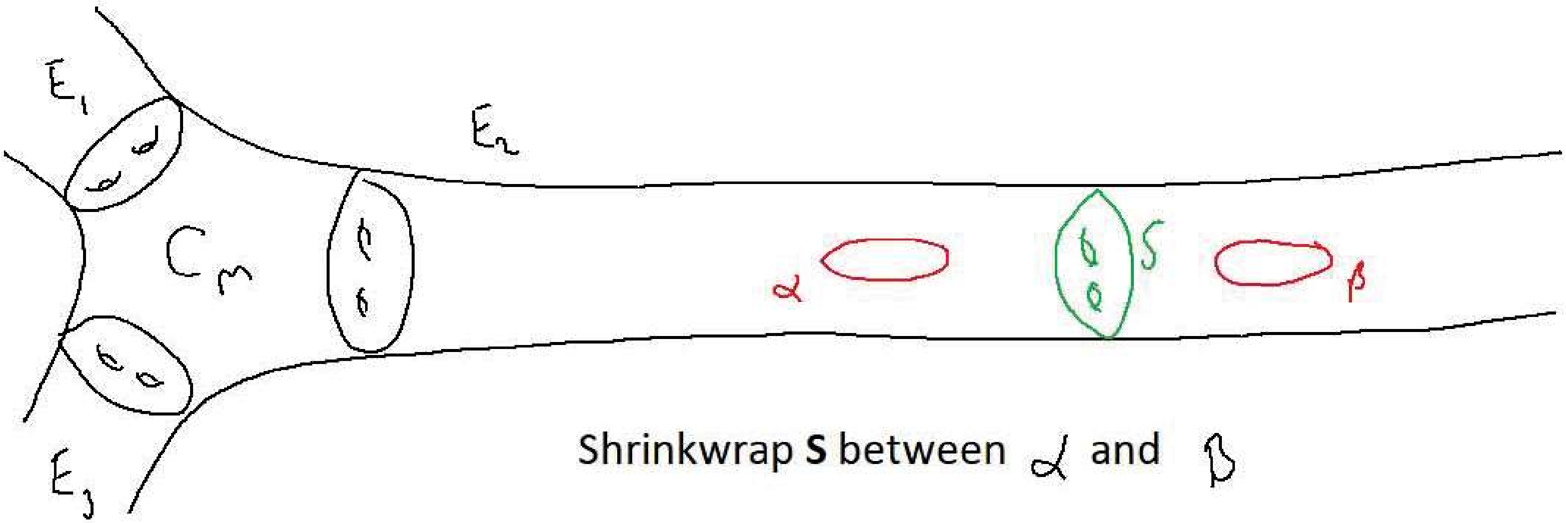

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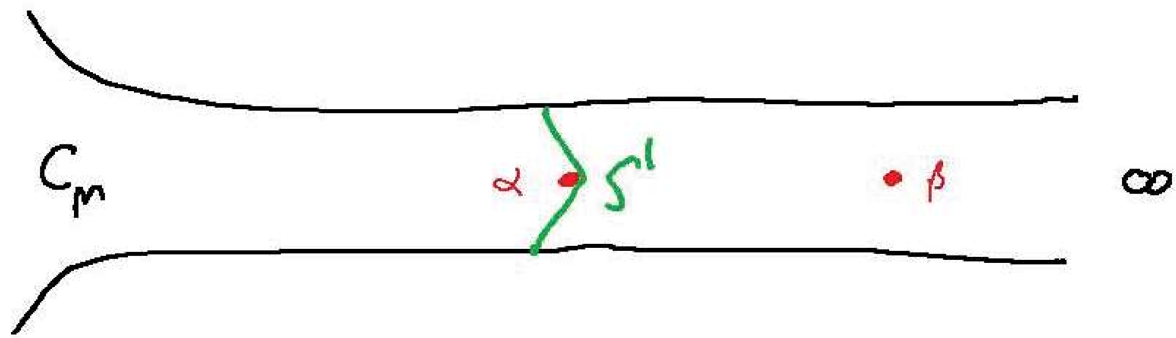
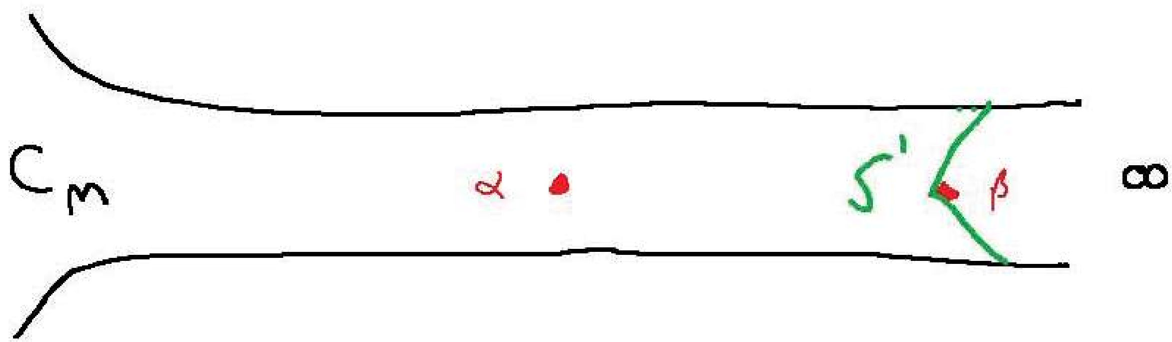
Convex Shrinkwrapping Surface:



Concave Shrinkwrapping Surface:



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Shrinkwrapping: Defective Minimal Surfaces In the Ends

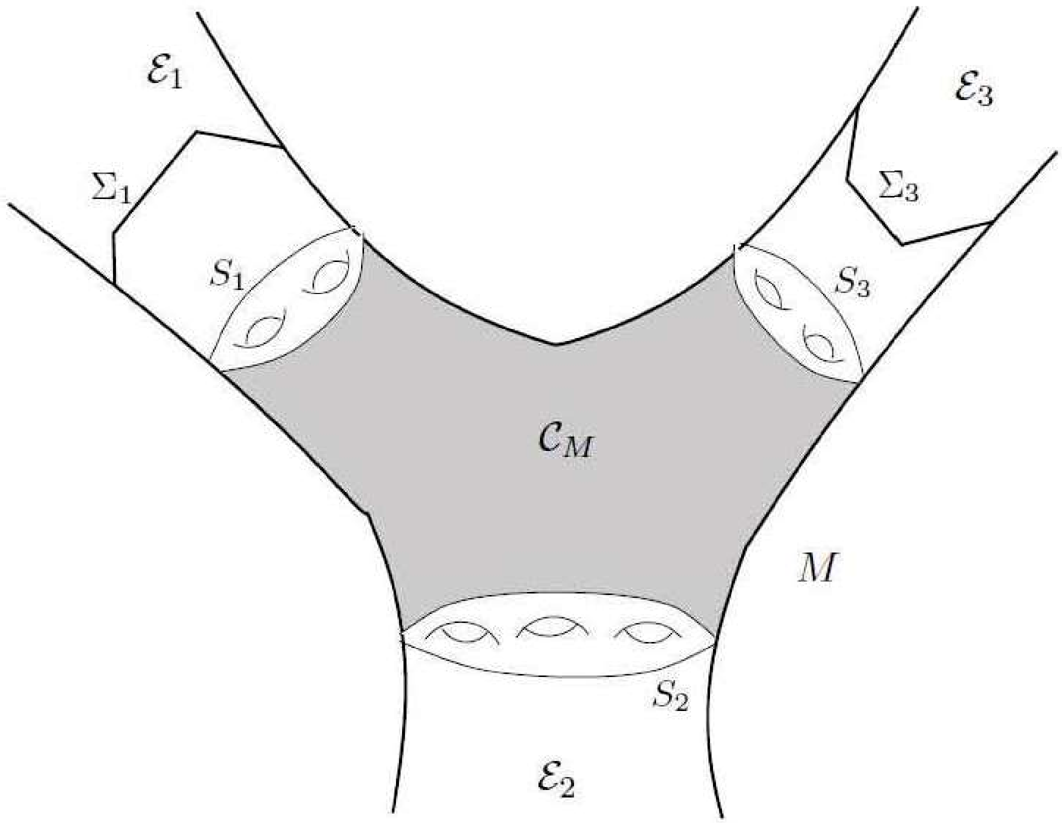

FIGURE 5. In the figure above, $\mathcal{E}_1$ has a convex shrinkwrapping surface Σ_1, and $\mathcal{E}_3$ has a concave shrinkwrapping surface Σ_3. $\mathcal{E}_2$ is geometrically finite, and $\Sigma_2 = S_2$.

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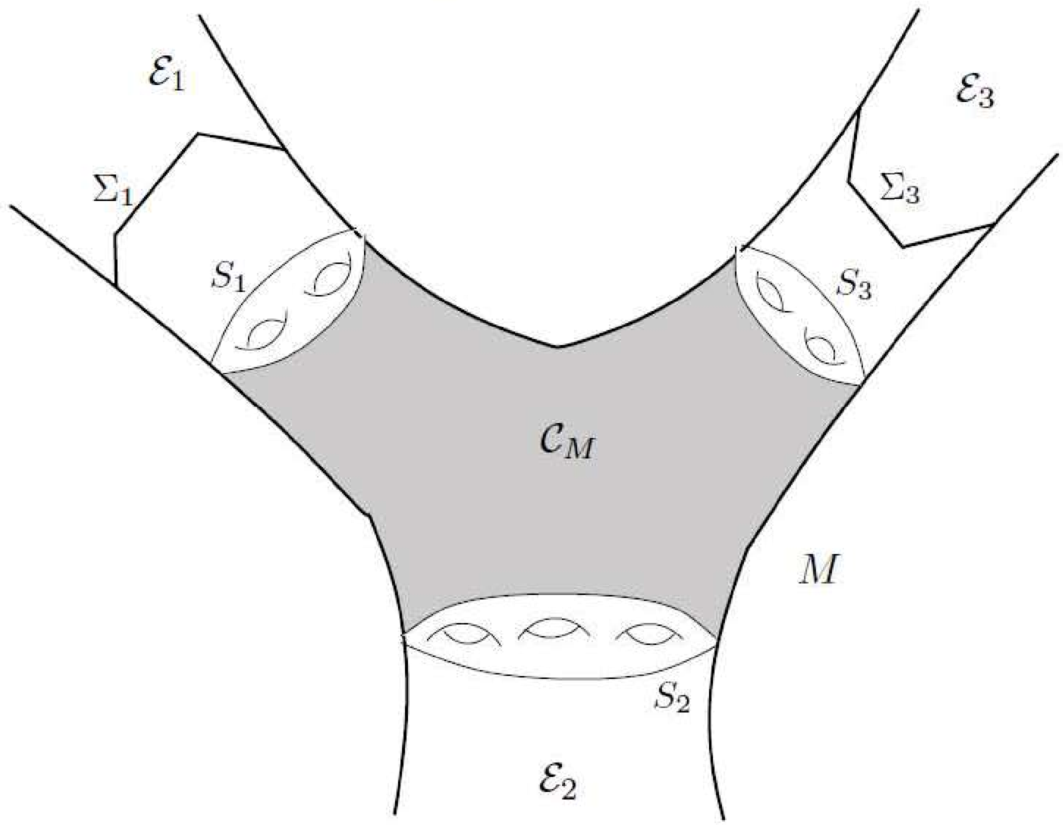
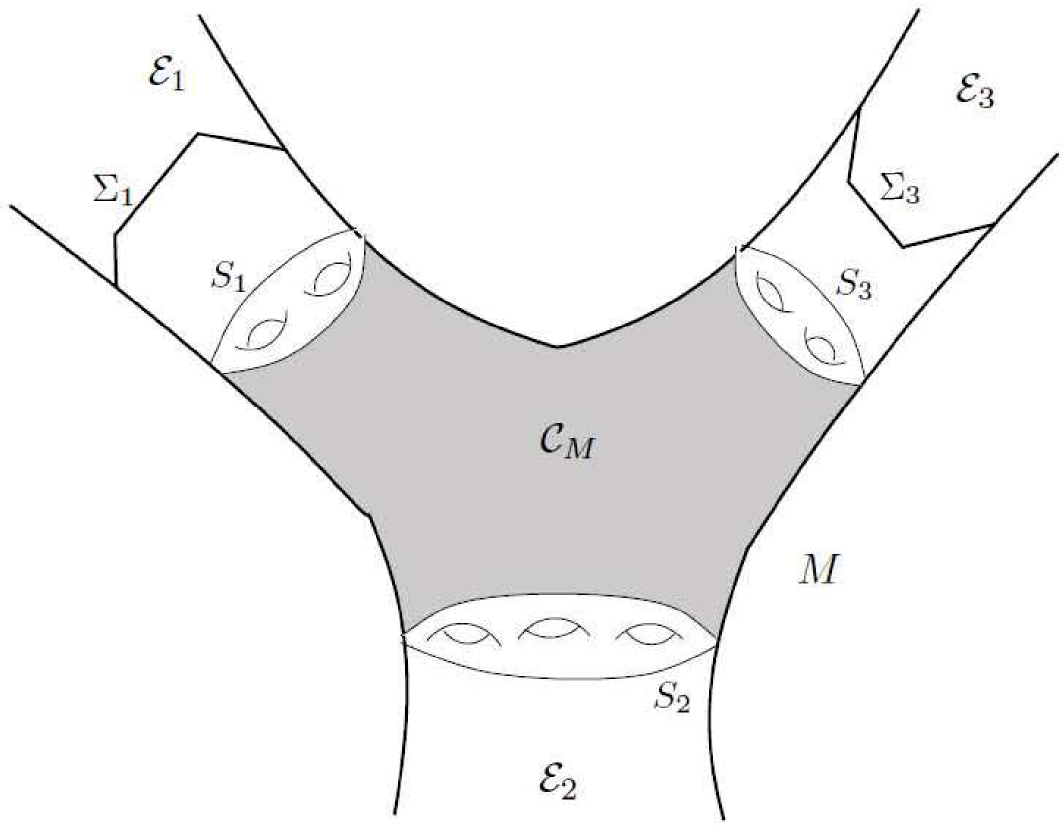


Case A: If one end has CONVEX shrinkwrapping surface

⇨ Area minimizing representative cannot escape to infinity.

◇ **Case 2b:** *M* has bounded geometry



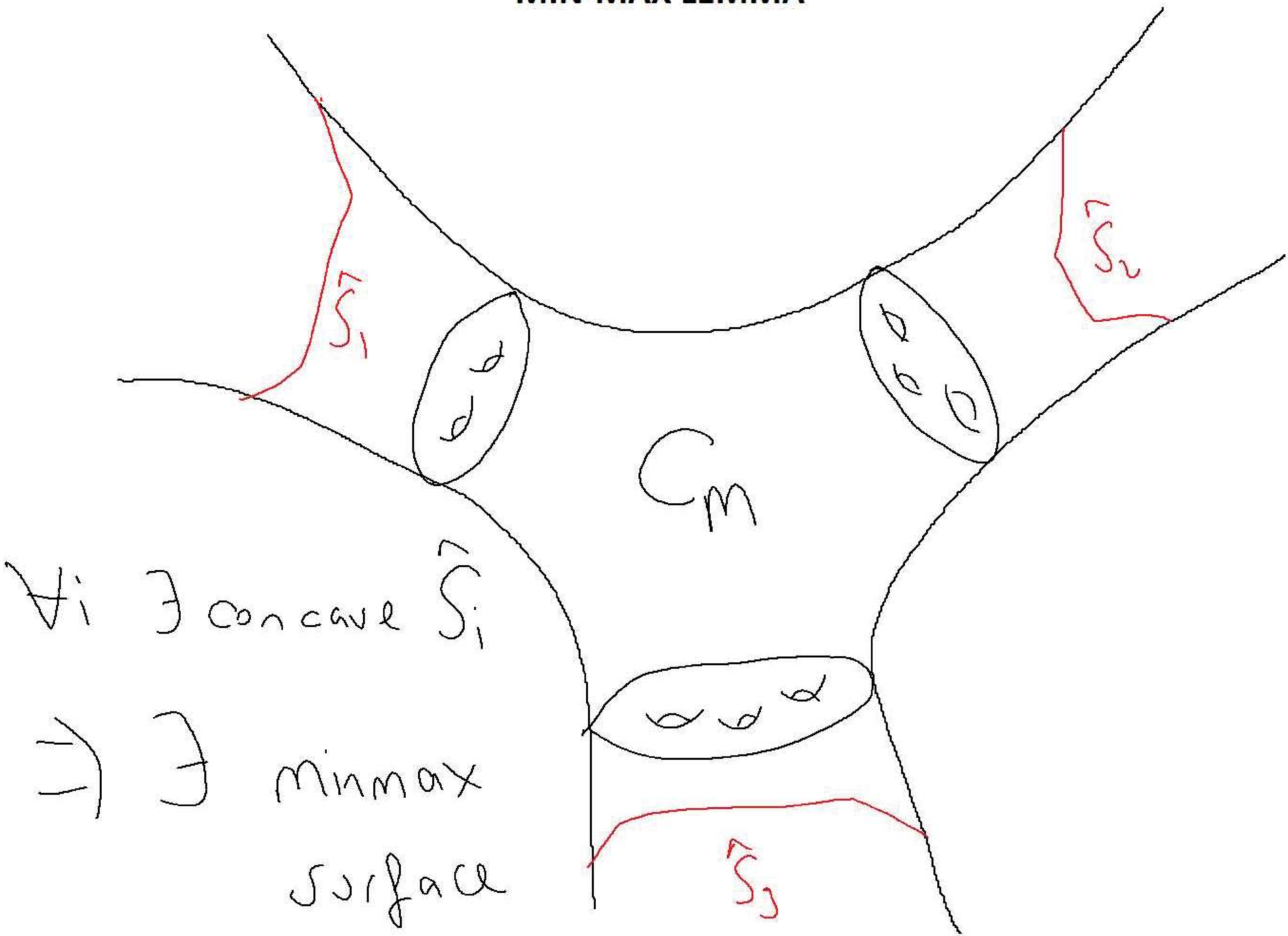
Case A: If one end has CONVEX shrinkwrapping surface

⇨ Area minimizing representative cannot escape to infinity.

Case B: If all ends have CONCAVE shrinkwrapping surface

⇨ There exists a minmax minimal surface in M.

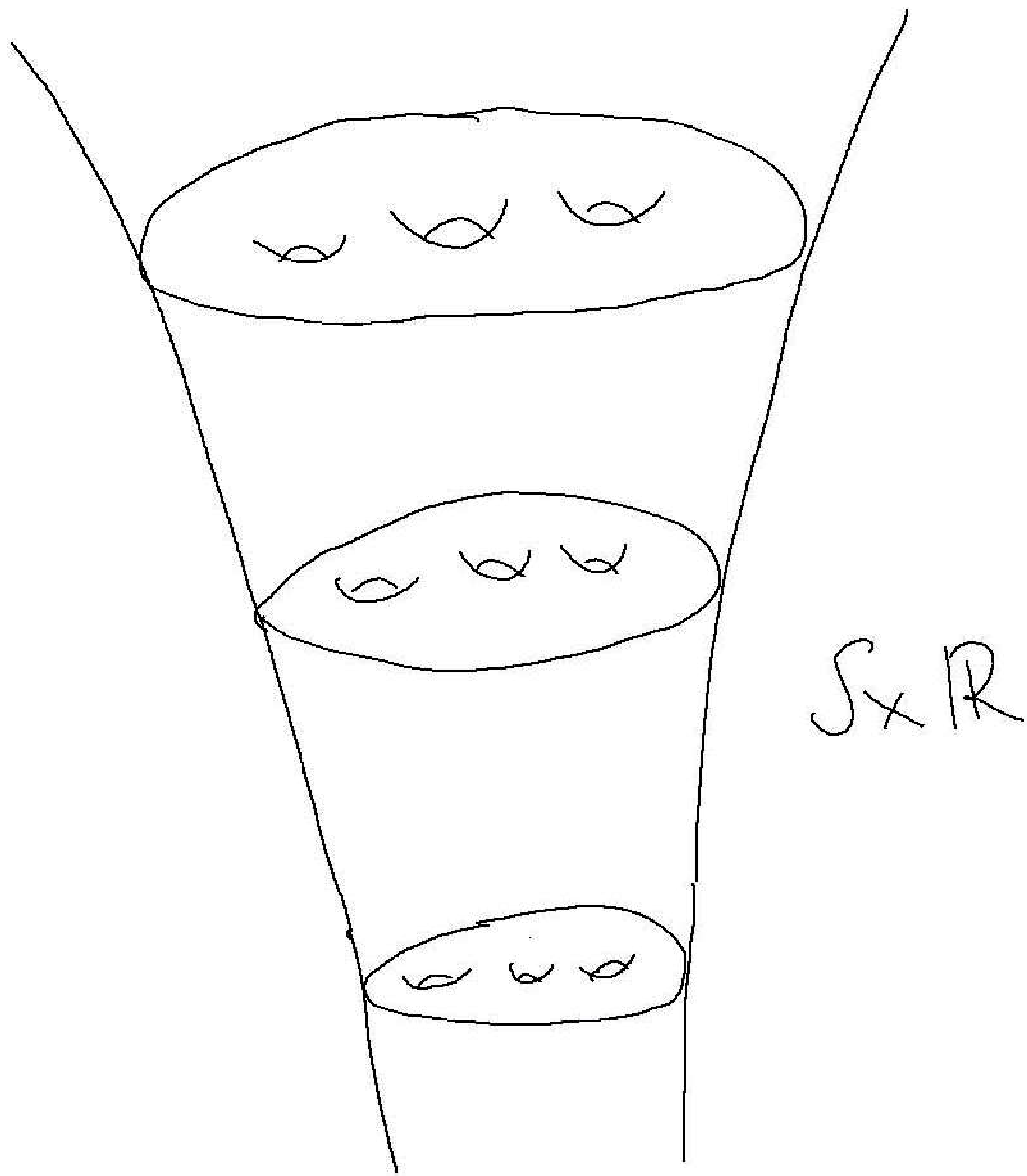
MIN-MAX LEMMA



Exceptional Cases and Final Remarks

- **Exceptional Case I:** $M = S \times \mathbf{R}$ and bounded geometry.

Exceptional Case I: SxR with bounded geometry



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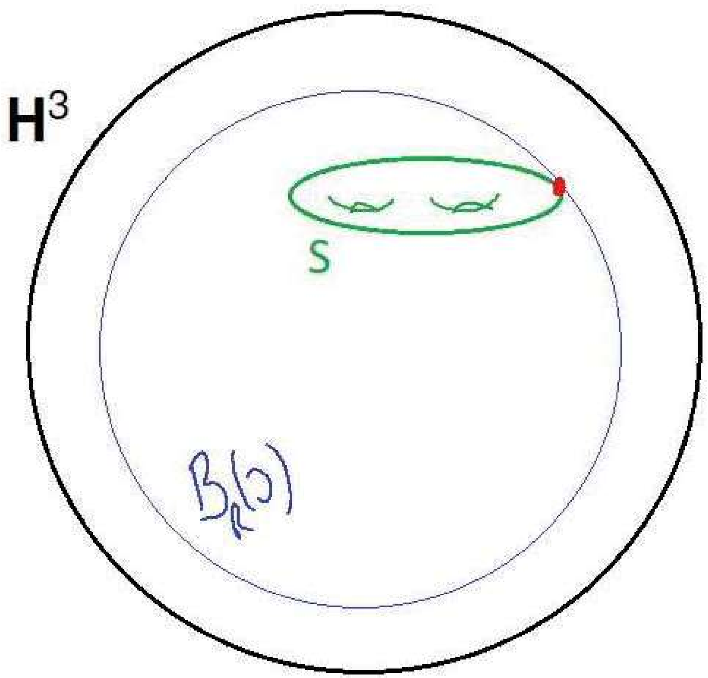

No closed minimal surface S because of the maximum principle

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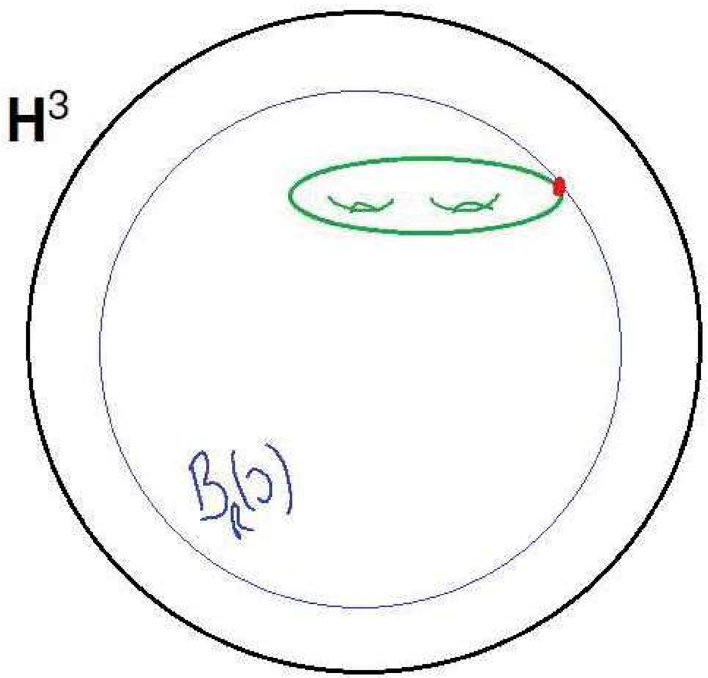

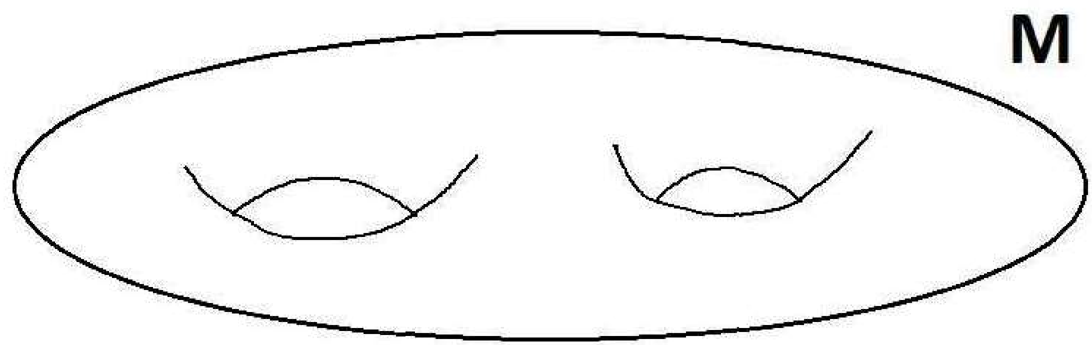

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Conjectures:

◇ There exists closed, embedded minimal surface in **Case I**.

◇ There is no closed, embedded minimal surface in **Case II**.