Outer Automorphisms of Free Coxeter Groups

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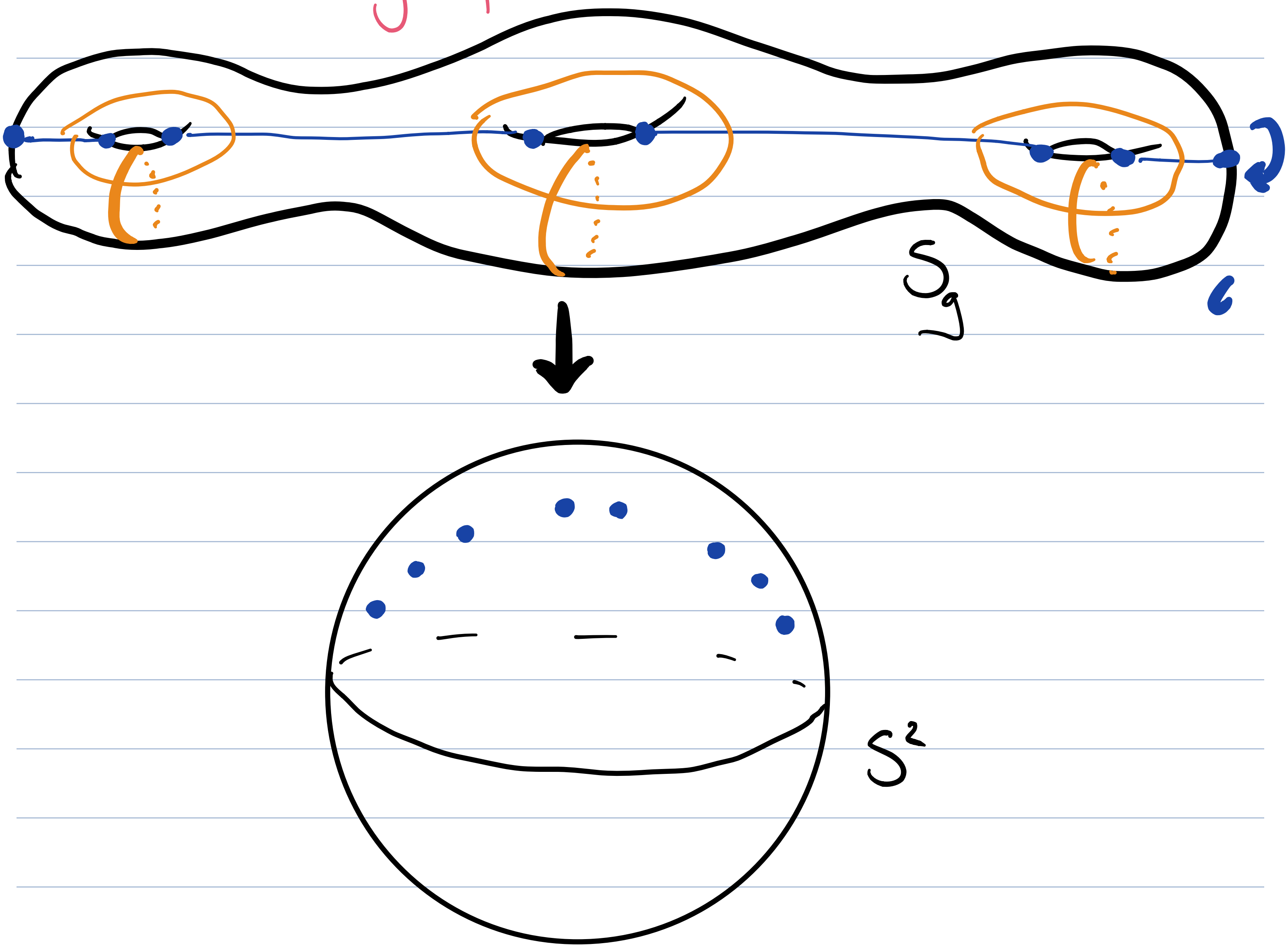
A **hyperelliptic involution** of a surface of genus g is an involution ι defining a degree-2 branched cover

$$S_g \longrightarrow S^2$$

with $2g+2$ branch points.

It turns out $\iota_*: H_1(S_g;\mathbb{Z}) \to H_1(S_g;\mathbb{Z})$ is -1 (finger proof!)



A famous theorem of Birman–Hilden implies the existence of a short exact sequence

$$1 \longrightarrow \langle\iota\rangle \longrightarrow C_{\mathrm{Mod}(S_g)}(\iota) \longrightarrow \mathrm{Mod}(S_{0,2g+2}) \longrightarrow 1.$$

$$\langle\iota\rangle \cong C_2, \qquad C_{\mathrm{Mod}(S_g)}(\iota) = \mathrm{HMod}(S_g)$$

NB: An index-$(2g+2)$ subgroup of $\mathrm{Mod}(S_{0,2g+2})$ is isomorphic to

$$B_{2g+1}/Z(B_{2g+1}).$$

NB: $\mathrm{HMod}(S_2) = \mathrm{Mod}(S_2)$, so this gave the first finite presentation of $\mathrm{Mod}(S_2)$!

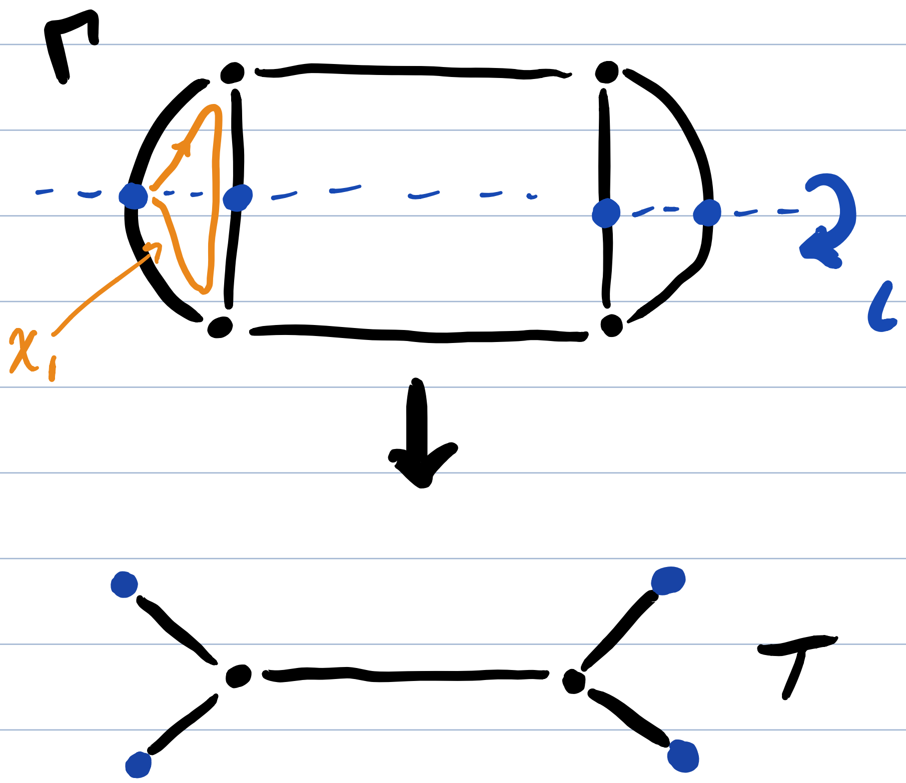
Braid groups have some nice properties that remain unknown for $\mathrm{Mod}(S_g)$:

What about for $\mathrm{Out}(F_n)$?

A **hyperelliptic involution** of a graph is an involution ι defining a degree-2 branched cover

$$\Gamma \longrightarrow T$$

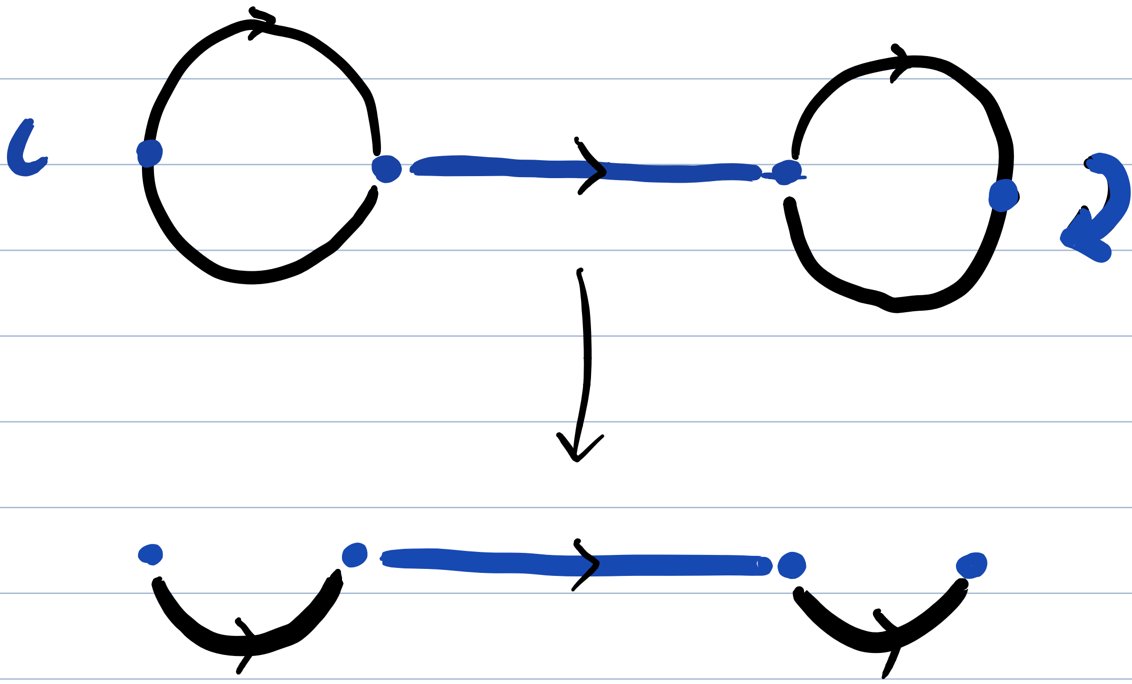
with $n+1$ branch points* $(b_1(\Gamma)=n)$.



It turns out there is a generating set $x_1, \dots, x_n$ for $\pi_1(\Gamma) \cong F_n$ s.t.

$$\iota_\# : \pi_1(\Gamma) \longrightarrow \pi(\Gamma) \qquad \iota_\#(x_i) = x_i^{-1}$$

A hyperelliptic involution of a graph may have $n+1$ **components** of fixed points:



Lemma: having fixed components that are not a single point happens $\iff$ Γ has separating edges.

Work of Krstić implies a group-theoretic Birman–Hilden result:

$$1 \longrightarrow \langle \iota \rangle \longrightarrow C_{\mathrm{Out}(F_n)}(\iota) \longrightarrow \mathrm{Out}(W_{n+1}) \longrightarrow 1$$

$$\langle \iota \rangle \cong C_2, \qquad C_{\mathrm{Out}(F_n)}(\iota) = \mathrm{HOut}(F_n)$$

Here $W_n = \underbrace{C_2 * \cdots * C_2}_{n \text{ copies}}$ is the **free** (universal) **Coxeter group** of rank n.

NB: $\mathrm{HOut}(F_2) = \mathrm{Out}(F_2) \cong GL_2(\mathbb{Z})$, so $\mathrm{Out}(W_3) \cong PGL_2(\mathbb{Z})$.

NB: $\mathrm{Aut}(F_2) \cong \mathrm{Aut}(W_3) \cong \mathrm{Aut}(B_4)$

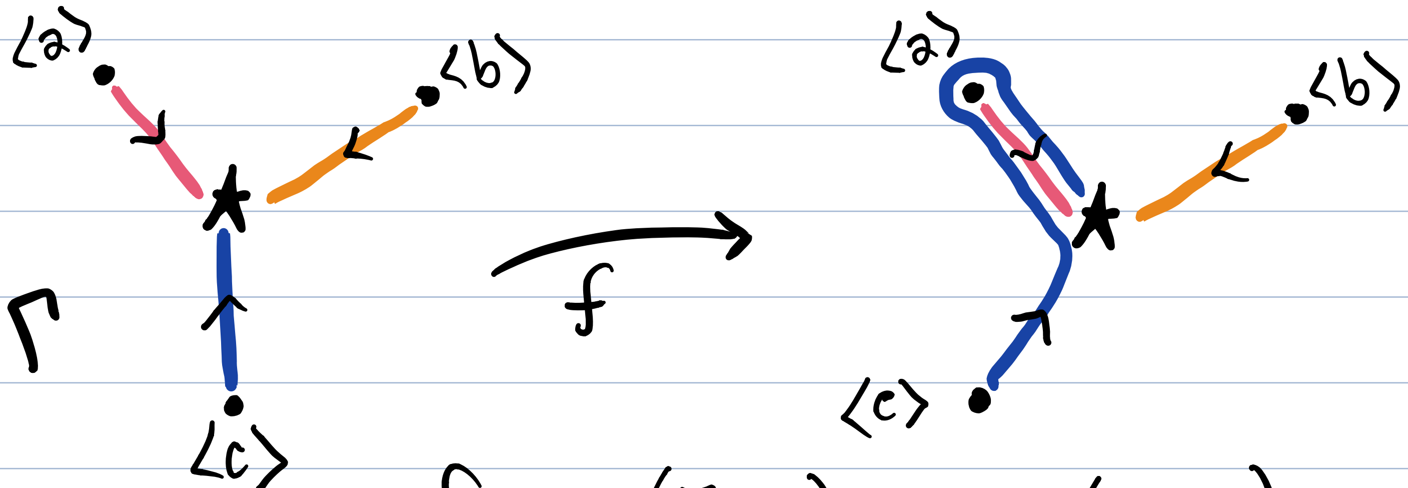
In the analogy between $\mathrm{Out}(F_{2g})$ and $\mathrm{Mod}(S_g)$, $\mathrm{Out}(W_{2g+1})$ plays a similar role to $\mathrm{Mod}(S_{0,2g+2})$ or $B_{2g+1}/Z(B_{2g+1})$

Just as every $\varphi \in Out(F_n)$ can be represented as a homotopy equivalence of a graph, so too can every $\varphi \in Out(W_n)$ be represented by a homotopy equivalence* of a **graph of groups** as below:

$$W_3 = \langle a,b,c : a^2 = b^2 = c^2 = 1 \rangle = \pi_1(\Gamma, *)$$

Ex.

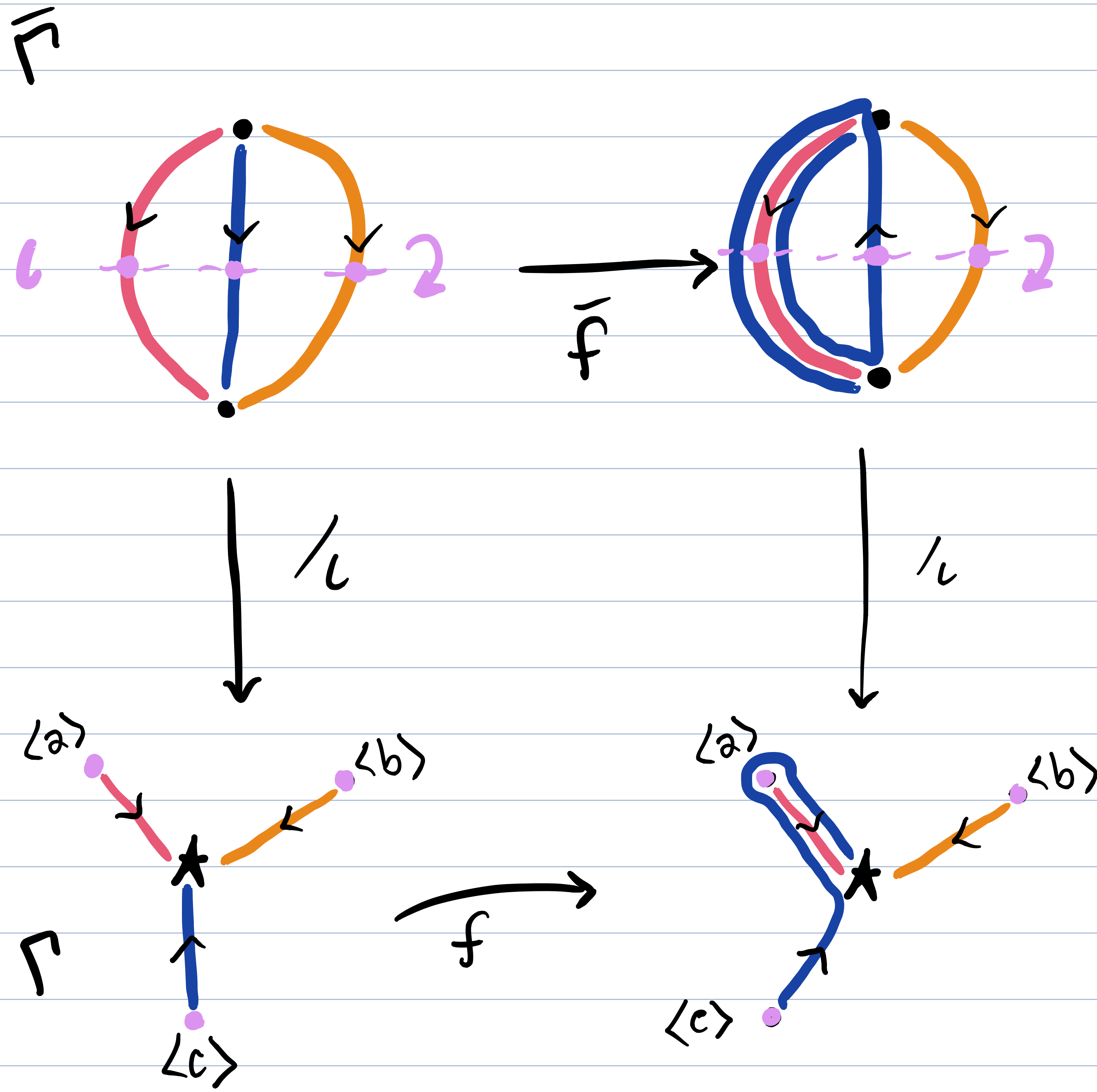
$$\Phi \begin{cases} a \mapsto a \\ b \mapsto b \\ c \mapsto a^{-1}ca \end{cases}$$



$$f_\# : \pi_1(\Gamma, *) \longrightarrow \pi_1(\Gamma, *)$$
$$f_\# = \Phi$$

* or an equivariant map of Bass–Serre trees

Ex. lifting homotopy equivalences



NB: Every (core) graph in rank 2 admits a hyperelliptic involution

A few similarities between $B_n / Z(B_n)$ and $\mathrm{Out}(W_n)$:

write $\quad \mathrm{POut}(W_n) = \ker\left(\mathrm{Out}(W_n) \to S_n\right)$

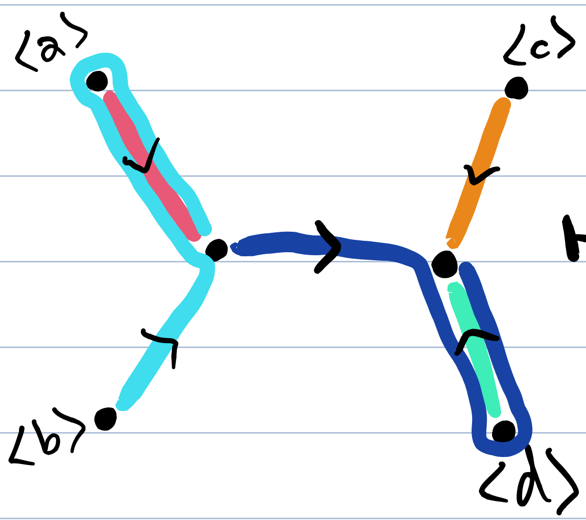
<u>**Thm**</u> (essentially Varghese '19) $\exists$ "forgetful map"

$$\mathrm{POut}(W_n) \twoheadrightarrow \mathrm{POut}(W_{n-1}).$$

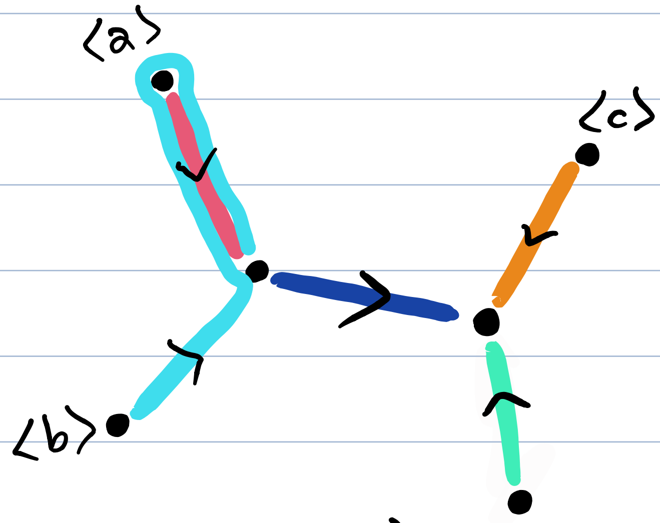
$\Rightarrow \mathrm{POut}(W_n)$ does not have property FA

$\Rightarrow \mathrm{Out}(W_n)$ does not have property (T)

Ex. "forget a cone point"



$\xmapsto{\text{forget}}$



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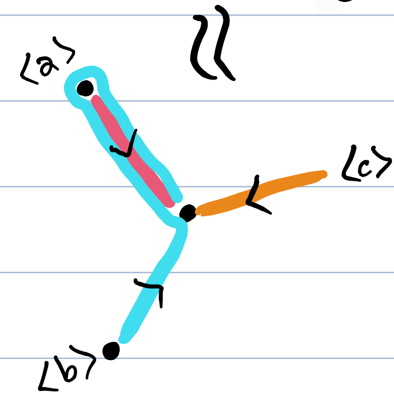


The first two statements are true of $B_n / Z(B_n)$

Thm (Guerch '20)

$$\mathrm{Out}(\mathrm{Out}(W_n)) = 1 \qquad n \geq 5$$
$$\mathrm{Out}(\mathrm{Out}(W_4)) \cong C_2 \qquad n = 4$$

Thm (Krstić–Vogtmann '93, McCullough–Miller '96)

$$\mathrm{vcd}(\mathrm{Out}(W_n)) = n-2$$

Thm (Birman–Hilden '73) if $W_n = \langle a_1, \dots, a_n \rangle$, the subgroup of $\mathrm{Out}(W_n)$ preserving the conjugacy class $[a_1 a_2 \cdots a_n]$ is isomorphic to $B_n / Z(B_n)$

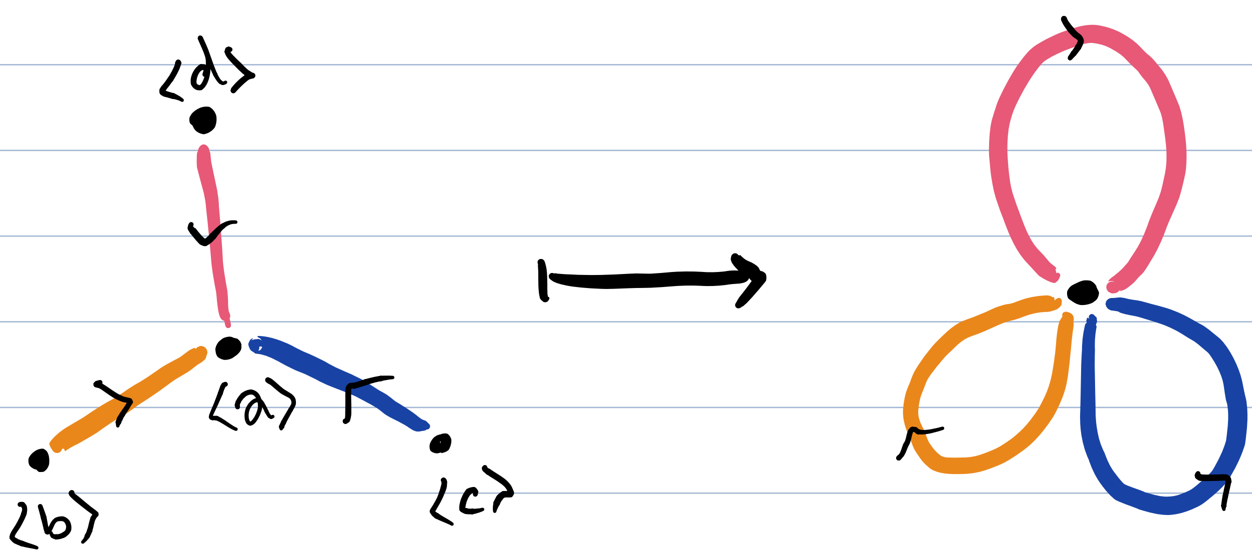
NB: This subgroup has infinite index when $n \geq 4$.

Thm (L '20) There is a natural map from Guirardel-Levitt's Outer Space for W_n to Culler-Vogtmann's Outer Space for F_{n-1}.

The map is an isometric embedding in the (asymmetric) Lipschitz metric

and its image is the intersection of the fixed-point set of a hyperelliptic involution with **reduced** Outer Space

The map sends a marked, metric W_n graph of groups to its characteristic double cover.



Culler–Vogtmann's **Outer Space CV_n** plays the role of Teichmüller space for $Out(F_n)$. Points of CV_n are equiv. classes of pairs (Γ, τ)

- Γ is a metric graph w/ vertices of valence ≥ 3 ("core" graph)

- $\tau : R_n \longrightarrow \Gamma$ is a homotopy equivalence

  rose w/ n petals

  "marking"

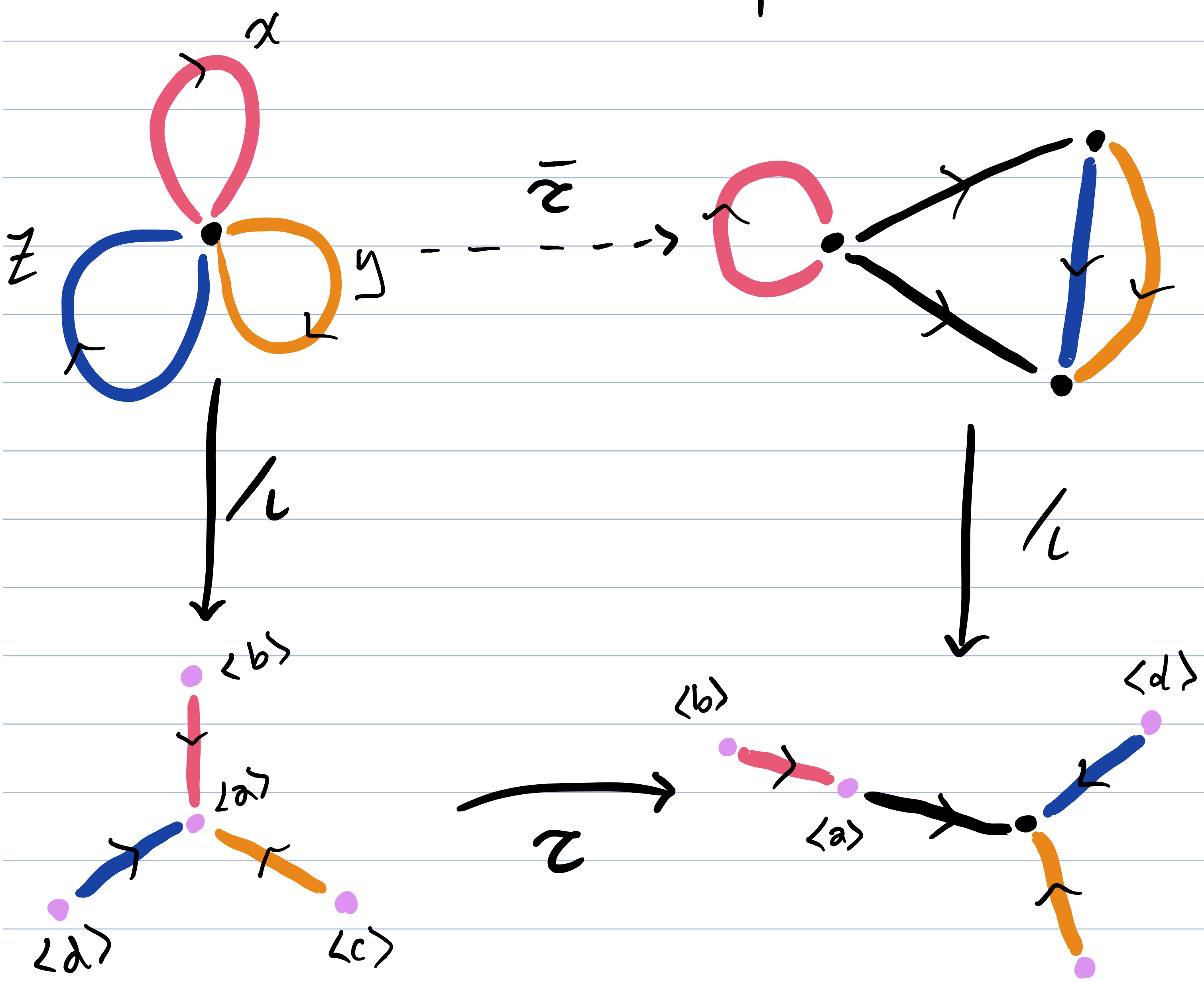
- $(\Gamma, \tau) \sim (\Gamma', \tau')$ when there exists h a homothety s.t. the diagram commutes up to homotopy

$$\begin{tikzcd} & \Gamma \arrow[dd, "h"] \\ R_n \arrow[ur, "\tau"] \arrow[dr, "\tau'"'] & \\ & \Gamma' \end{tikzcd}$$

Guirardel–Levitt's Outer Space for a free product is the same idea.

Reduced Outer Space only includes graphs without separating edges

The lemma $\Rightarrow$ the double cover of a pt in Guirardel-Levitt Outer Space lives in Reduced Outer Space.



markings!

The **Lipschitz distance** between graphs (Λ, σ), (Γ, τ) in Outer Space is

makes the quantity homothety-invariant

$$d(\Lambda, \Gamma) = \log \inf \left\{ L \cdot \frac{\mathrm{vol}(\Lambda)}{\mathrm{vol}(\Gamma)} \right\}$$

where there exists an L-Lipschitz homotopy equivalence $f : \Lambda \longrightarrow \Gamma$ s.t.

$$f\sigma \simeq \tau : R_n \longrightarrow \Gamma$$

NB: Typically $d(\Lambda, \Gamma) \neq d(\Gamma, \Lambda)$!