Large scale geometry of big mapping class groups

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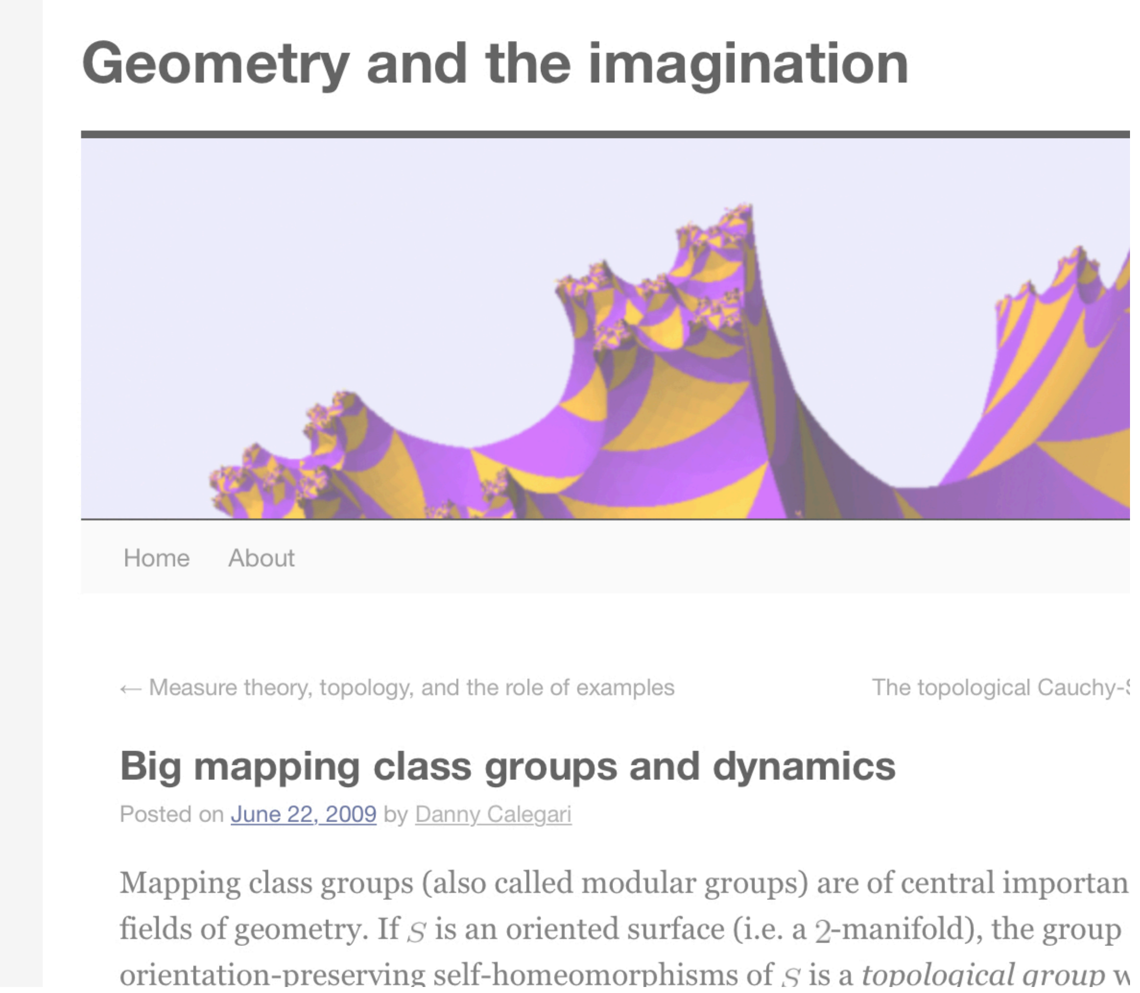
joint work with Kasra Rafi

(ultra brief) History


Geometry and the imagination

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Big mapping class groups and dynamics

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Mapping class groups (also called modular groups) are of central importan
fields of geometry. If S is an oriented surface (i.e. a 2-manifold), the group
orientation-preserving self-homeomorphisms of S is a topological group w

cantor set

- **2009** Calegari: $\mathrm{MCG}(S^2 - C)$ has <u>bounded commutator length</u>

  asks: same for $\mathrm{MCG}(\mathbb{R}^2 - C)$?

- **2016** Bavard: no, $\mathrm{MCG}(\mathbb{R}^2 - C)$ acts on hyperbolic graph...

  ...can build nontrivial quasi-morphism

- **Since then:** Many attempts to answer
  - Which MCG's act on hyperbolic spaces?
  - Which admit unbounded *length functions*?

$F: G \to \mathbb{R}$
- $F(ab) \leq F(a) + F(b)$
- $F(id) = 0$
- $F(a^{-1}) = F(a)$

Unifying question

Which MCG’s have some intrinsic, nontrivial geometry?

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Which MCG's have some intrinsic, nontrivial geometry?

Theorem [M— Rafi]: *an answer to this question*

Challenges:
1. hard to say anything about <u>all</u> surfaces
2. what does intrinsic geometry even mean?
3. (actual details of actual proof)

1 Describing all surfaces [I. Richards]

$$\begin{cases} \text{Genus} \quad (\mathbb{N} \cup \{\infty\}) \\ \text{Space of ends} \quad (\cong \text{closed subset of cantor set}) \\ \text{Space of ends accumulated by genus} \quad (\text{further closed subset}) \end{cases}$$

1 Describing all surfaces [I. Richards]

- Genus ($\mathbb{N} \cup \{\infty\}$)
- Space of ends ($\cong$ closed subset of cantor set)
- Space of ends accumulated by genus (further closed subset)

Up to homeomorphism $\leftrightarrow$ surface up to homeomorphism

* Classifying such objects is <u>hard</u> (in the complexity theory sense)

[Ketonen '78] - Gives a description of closed subsets of Cantor up to homeomorphism

[Camerlo-Gao '00] - Show that this is optimal / "simplest possible"

2 Intrinsic geometry: boundedness

(easier to say what "trivial geometry" is than what "geometry" is.)

Def: G is *coarsely bounded* if every continuous length function on G is bounded

topological group

(equivalently, every continuous action on a metric space has bounded orbits)

Theorem: If Σ is *self-similar*, or *telescoping*, then $\mathrm{MCG}(\Sigma)$ is coarsely bounded

- genus 0 or ∞
- If you partition $\mathrm{Ends}(\Sigma)$ into 2 pieces, one contains a copy of $\mathrm{Ends}(\Sigma)$

example:

Ends = Cantor set $\cup$ Cantor set accumulated by genus

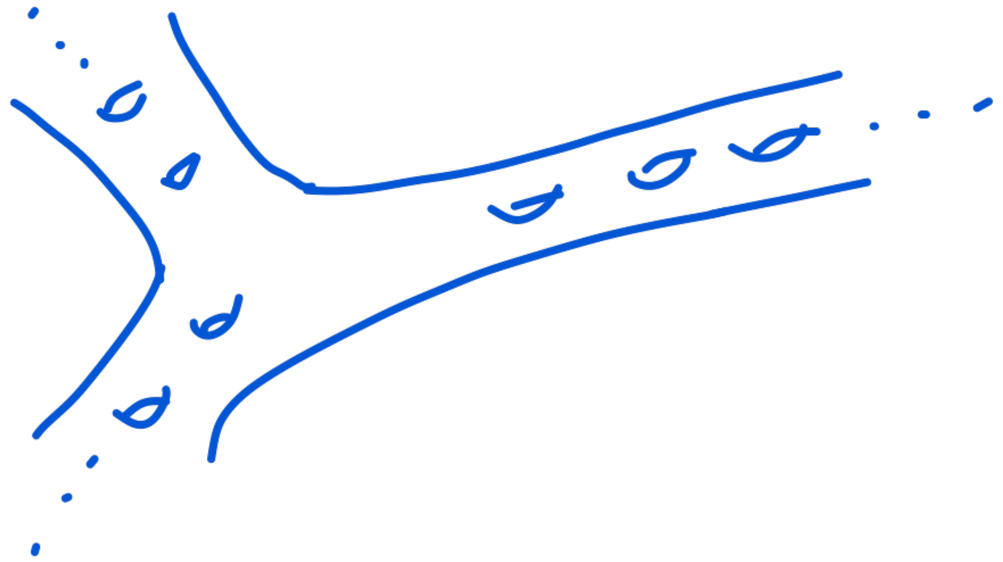
Theorem: For surfaces whose end space is *tame*, this is an iff

Intrinsic geometry: non-boundedness

Def: A subsurface $S \subset \Sigma$ is *displaceable* if $f(S) \cap S = \emptyset$ for some f

Theorem: If Σ has a non-displaceable finite-type subsurface, then MCG(Σ) is not coarsely bounded

Examples:

- Σ finite genus (put all the genus in S)
-  $= \Sigma$
- Ends = 

Idea of proof:

- WLOG S has unbounded $C(S)$
- Take $\mu \subset C(S)$ filling
- Use subsurface projection of $\varphi(\mu)$ to S to cook up a length function
- If φ restricts to p.A. on S, easy to see unbounded on φ^n.

Intrinsic geometry: general framework

Geometric group theory works for *locally compact, compactly generated* groups

Rosendal showed this can be extended to topological groups that are

locally coarsely bounded and generated by a *coarsely bounded set*

(neighborhood of id.) (analytic)

(the word metric for such a generating set gives a well-defined coarse geometric structure)

Q: Which big mapping class groups fit this framework?

C.B. neighborhood of identity

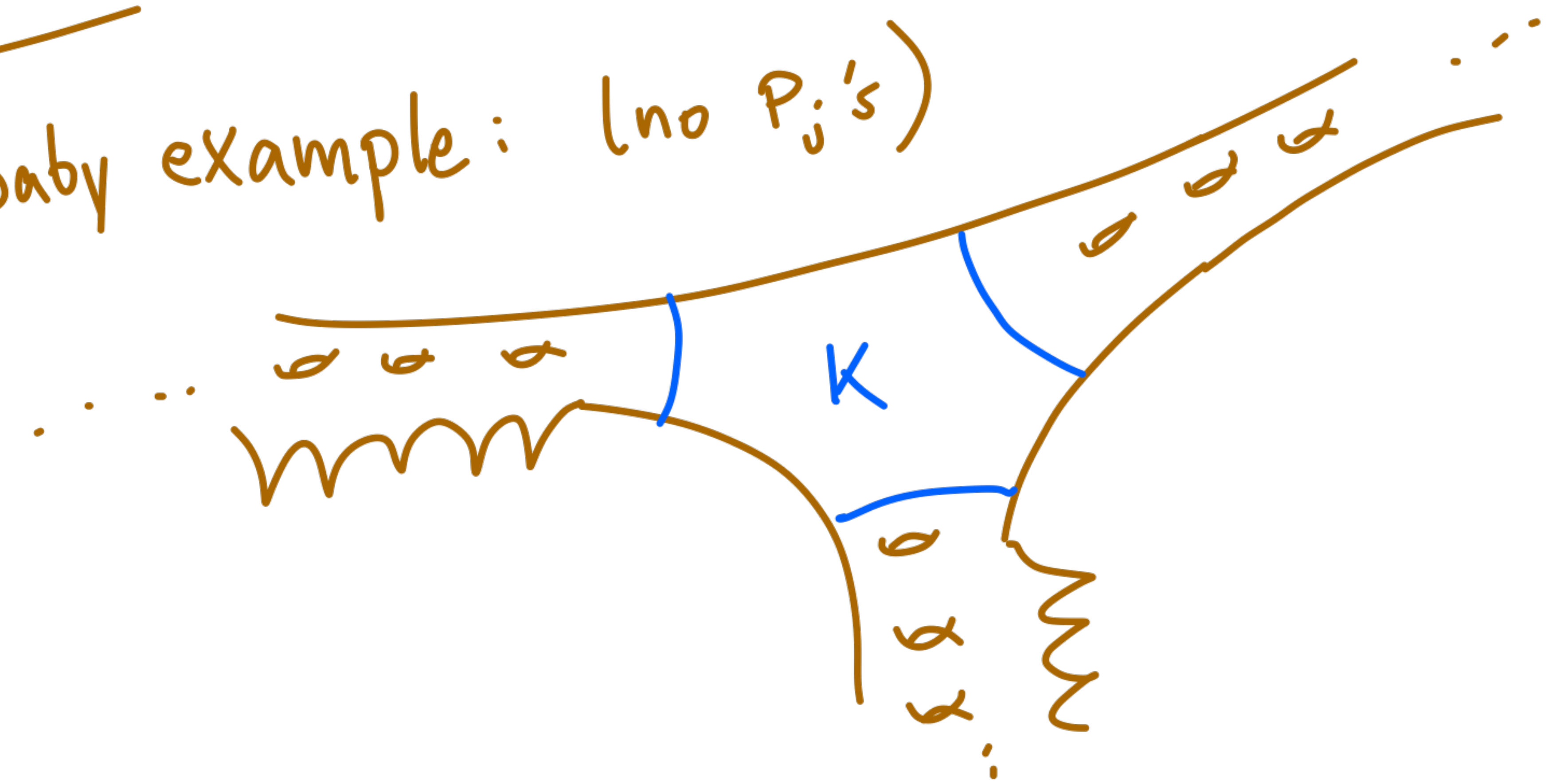
Theorem: $MCG(\Sigma)$ has a CB neighborhood of id. $\Leftrightarrow$ $\exists$ finite type $K \subset \Sigma$ partitioning Ends s.t.

- $\text{Ends} = \underbrace{\bigsqcup A_i}_{\text{self similar sets}} \cup \underbrace{\bigsqcup P_j}_{\text{Each} \cong \text{ some piece of some } A_i \text{, with } A_i \cup P_j \cong A_i}$

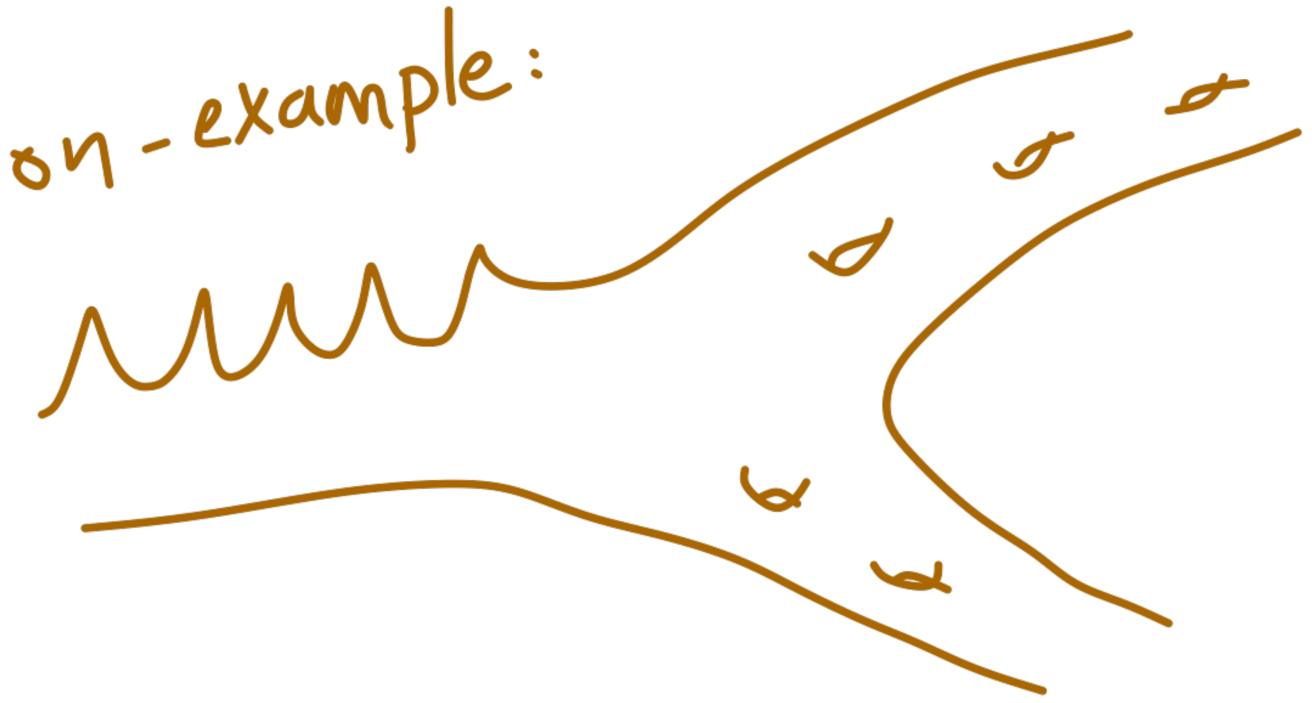
- "Most complicated" ends appear in the A_i's and for any such end $\varepsilon \in A_i$, and nbhd V of ε, $\exists f$ s.t. $f(V) \supset A_i$

In this case, {mapping classes that are trivial on K} is a CB nbhd of id

Baby example: (no P_j's)



Non-example:



Complexity of an end

KEY **Def**: for $x, y \in \text{Ends}$,

$x \preceq y$ if $\forall$ nbhd $\mathcal{U}$ of y, $\exists$ subset $V \subset \mathcal{U}$ homeomorphic to a nbhd of x.

⚠ $x \preceq y$ and $y \preceq x$ $\not\Rightarrow$ $\exists$ nbhd of y homeomorphic to nbhd of x

"TAME" is the requirement that this ↑ doesn't happen for maximal ends (wrt. $\preceq$) & their immediate predecessors.

(all concrete examples we have seen in other papers are tame, but we can painfully construct some non-tame ones!)

Classification theorem

Thm: Among tame surfaces,

$MCG(\Sigma)$ has a well-defined Q.I. type $\Longleftrightarrow$ • Either it is globally C.B. (trivial geometry)

or

• Locally CB + "Finite Rank"

&

not "Limit type"

Finite Rank: prohibits surjections to $\mathbb{Z}^N$ for arbitrarily large N

not Limit type: excludes cases where Ends $\cong \omega^\alpha + 1$

α limit ordinal.