How helpful is hyperbolic geometry?

Eric Samperton
University of Illinois at Urbana-Champaign
Joint w/ Chris Leininger

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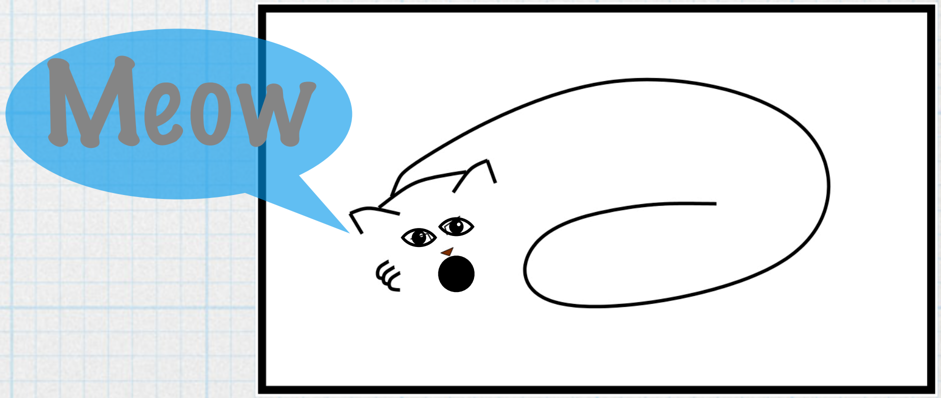
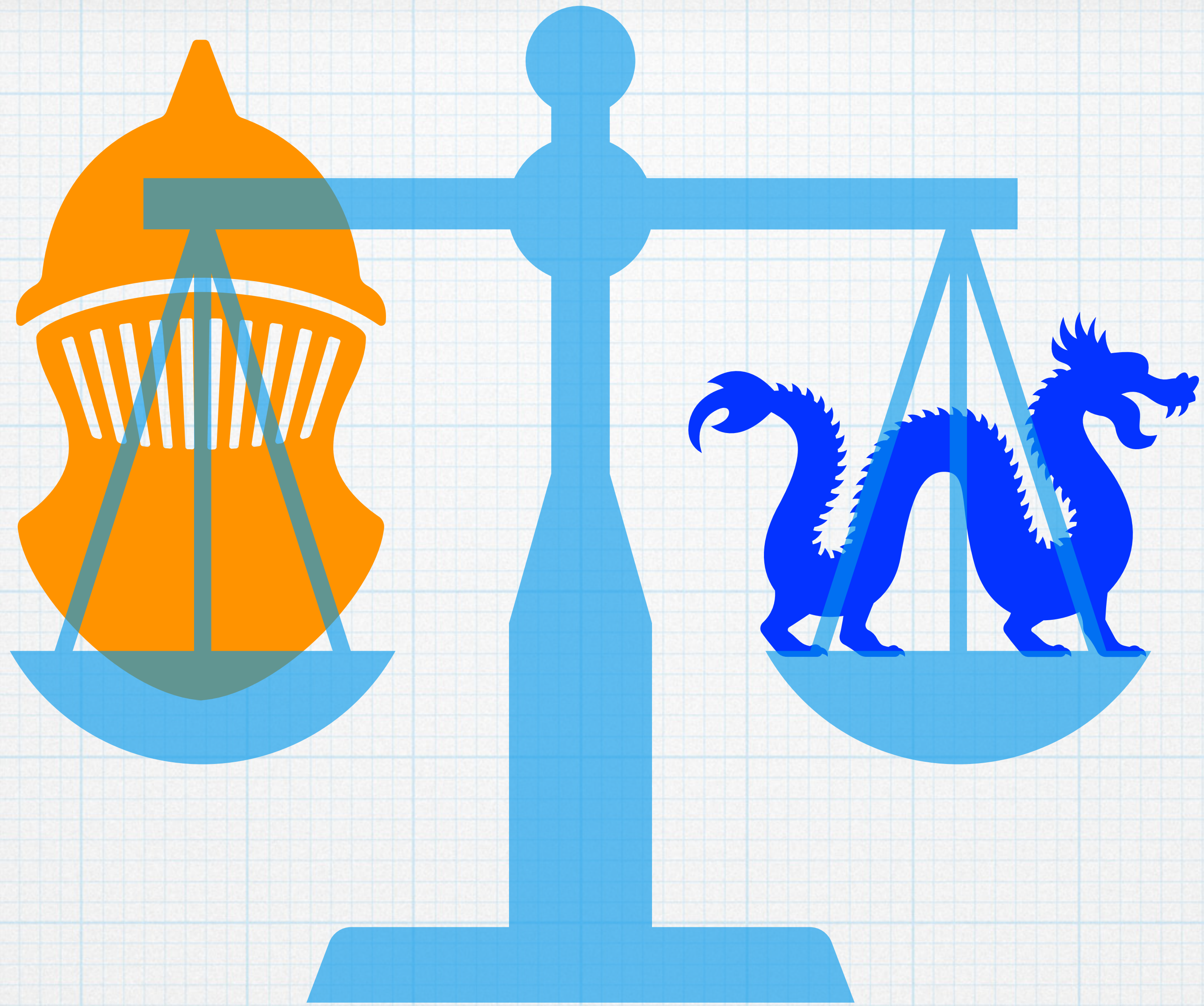
GaTO Seminar



Hyperbolicity plays dual roles in group theory and low-dimensional topology

* On one hand, Gromov hyperbolicity rescues group theory from a morass of uncomputability.
* Consider word, conjugacy and isomorphism problems
* "Hyperbolic promise" saves the day
* Even better: from numerous perspectives, hyperbolic groups are "generic."

* On the other hand, hyperbolic 3-manifolds are seemingly responsible for much of the complexity of the 3-manifold topology.
* Not a precise statement; more of an observation about the history of 3-manifold topology.
* Perhaps worth noting: as in group theory, hyperbolic 3-manifolds are "generic" in various precise formulations of the word.
* What implications from a hyperbolic promise?



Where “in the middle” do these dual roles of hyperbolicity meet?

After all, fundamental groups of hyperbolic 3-manifolds are Gromov hyperbolic.

More precisely: can we find a problem that is no easier for hyperbolic things than it is for general things?

* One answer from the Rips construction:
    * subgroup membership problem is unsolvable for hyperbolic groups (and, hence, finitely presented groups in general). (h/t to Paul Schupp)
* So let's refine our question: can we find an intrinsically solvable problem whose complexity is not improved in the presence of a hyperbolic promise? Even better if we can restrict to hyperbolic 3-manifolds, not just hyperbolic groups.

A proposed problem

* Fix a nonabelian, finite, simple group G.
* Given a triangulated 3-manifold M, decide if there exists a non-trivial homomorphism from its fundamental group to G.

$$\exists ?\, \pi_1(M) \to G$$

Previous theorem (joint w/ Greg Kuperberg)

* For a fixed finite, nonabelian simple group G, the problem of deciding when a 3-manifold admits a nontrivial homomorphism from its fundamental group to G is NP-complete.

* Moreover, the problem remains NP-complete even if we promise that the 3-manifold is a homology sphere, and that whenever a nontrivial homomorphism exists, it is surjecrtive.

Goal theorem (work in progress w/ Leininger)

* Add a hyperbolic promise to the second bullet point on the left .

Remarks

* Greg and I also have a related theorem about complements of knots in the three-sphere. Can Chris and I promise the knots are hyperbolic?

* The goal theorem is still a work in progress, although we are currently at least able to guarantee hardness with the promise that the manifold has a Heegaard splitting with a pseudo-Anosov, Torelli gluing map that does not factor over any handlebody. More on this shortly.

* For manifolds with the promises provided, finding a nontrivial homomorphism to the alternating group Alt(5) is equivalent to finding a connected 5-sheeted covering space. Compare this to recently proved virtual properties of hyperbolic 3-manifolds...

* Hyperbolic quantum computing? Key point: every quantum representation of MCG factors through a p-power subgroup, which contains lots of pseudo-Anosovs.

Reversible circuits

* Fix the following:
  * Finite set A
  * Subsets I and F of A
  * Some bijections of A x A (called binary reversible gates)
* A reversible circuit C (with these parameters) of width n is a factorization of a bijection $C : A^n \to A^n$ into gates

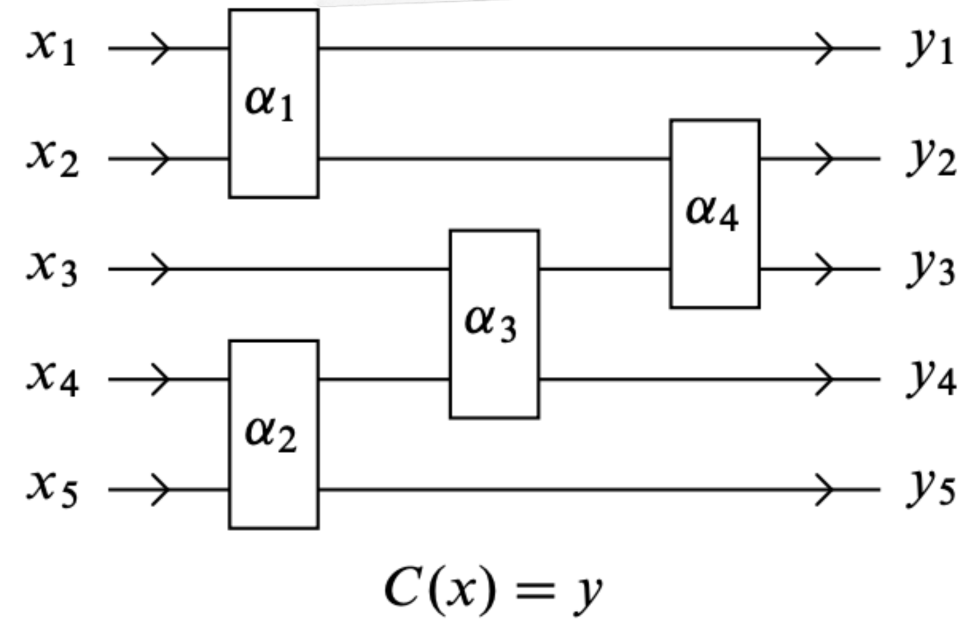


Figure 3: A planar, reversible circuit

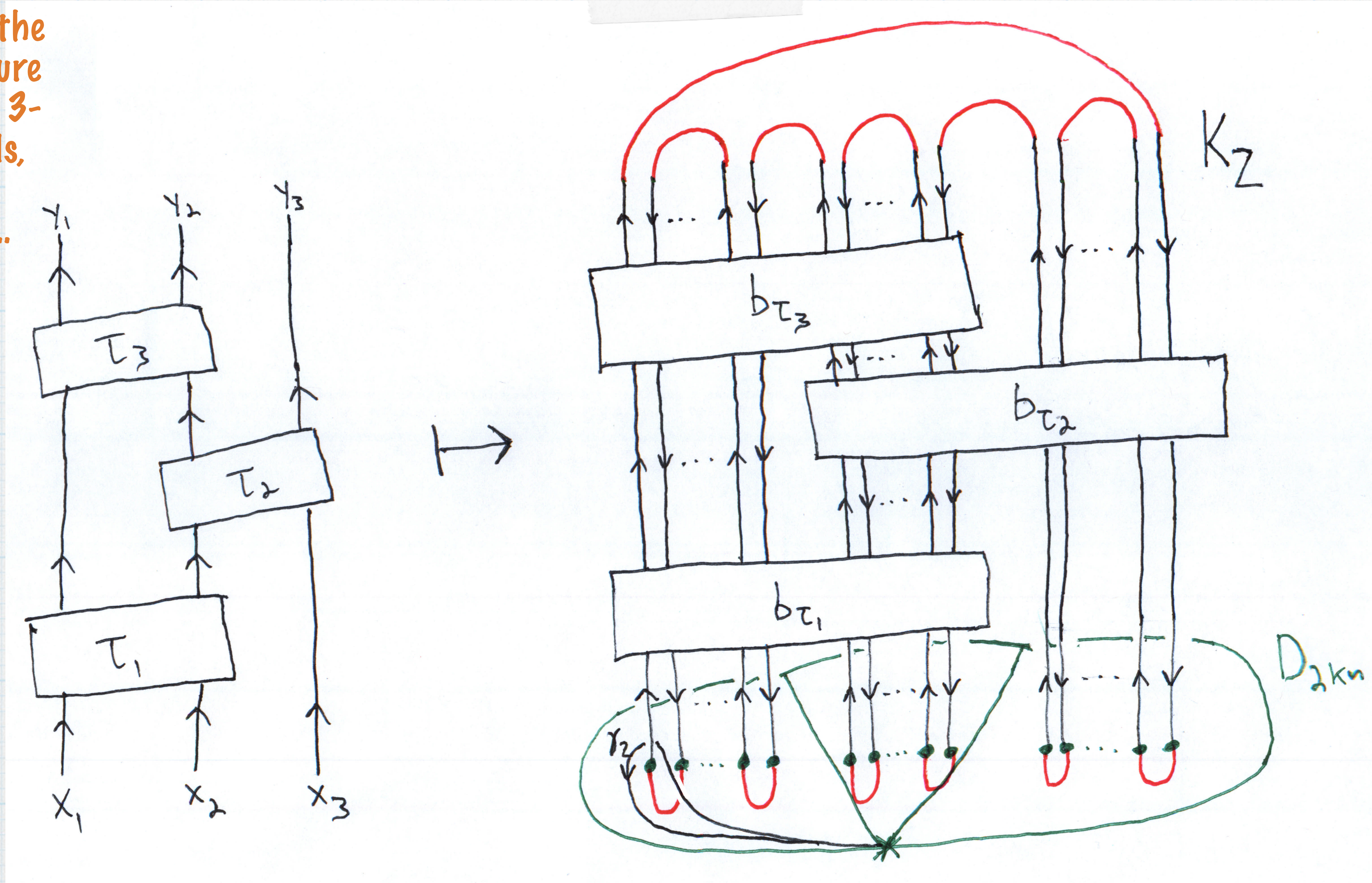
FRESH INGREDIENT ON THE MENU: Circuits of width n form a RAAM w.r.t. stacking.

A decision problem:

$$\exists ? x \in I^n : C(x) \in F^n$$

Often NP-complete, depends on choices though

Not quite the right picture for closed 3-manifolds, but it's coming...



The circuit reduction

Greg and I showed that counting homomorphisms to G is hard by converting reversible circuits to 3-manifolds so the circuit is satisfiable if and only if the 3-manifold's fundamental group has a nontrivial homomorphism to G.

Outline of strategy for new theorem

* Using techniques of Clay-Leininger-Mangahas, improve the reduction so that it is a quasi-isometric embedding from the monoid of reversible circuits to the mapping class group.
* Pay careful attention to quasi-isometry constants, especially as a function of the width of the circuits.
* Consider the action of this monoid on the complex of curves, and in particular the disk sets of the standard Heegaard splitting of 3-sphere.
* Use padding tricks for reversible circuits to ensure we only have to consider circuits that act so that one of the disk sets gets moved very far from the original two.
* Use Hempel's corollary of geometrization that Heegaard splittings whose disk sets have distance at least 5 yield hyperbolic manifold. (Even better, use Scharlemann-Tomova to ensure unique, minimal genus splittings...)