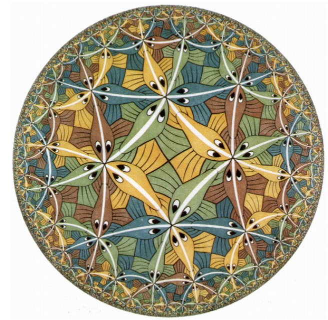
Sublinearly Morse Boundary

Yulan Qing

joint projects with Ilya Gekhtman, Kasra Rafi and Giulio Tiozzo

June 2020

Gromov boundary of a δ–hyperbolic space

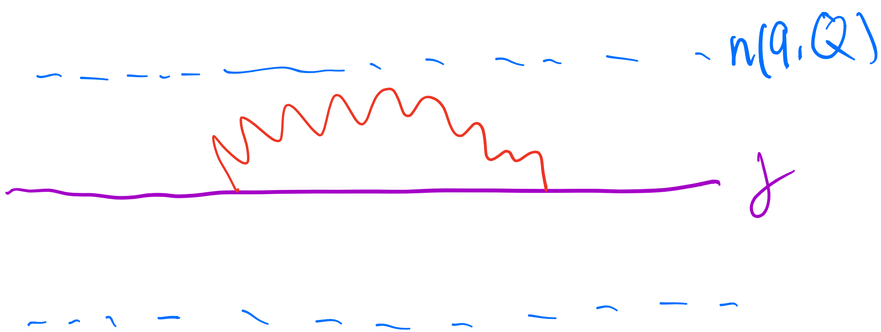


- A point in the boundary is a geodesic ray or a family of quasi-geodesic rays up to fellow traveling.
- cone topology

Gromov boundary of a hyperbolic space is QI-invariant.

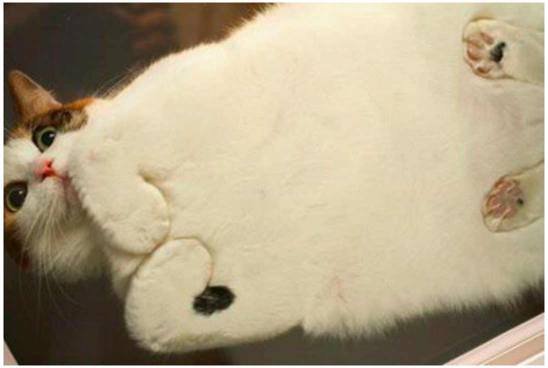
Key: geodesics are Morse in a Gromov hyperbolic space.

A quasi-geodesic ray γ is Morse if given any pair (q, Q), there exists constant $n(q, Q)$ such that all (q, Q)–quasi-geodesics whose endpoints are on γ stays inside the $n(q, Q)$-neighbourhood of γ.



Visual boundary of CAT(0) spaces

- geodesics, up to fellow travel.
- cone topology



–Croke-Kleiner: the visual boundary is not QI-invariant.

Morse boundary(Charney-Sultan, Cordes): Morse geodesics.

–Not large enough from the point of view of random walk.

κ-Morse boundary

Space: $(X, \mathfrak{o})$ is a proper, geodesic space, with a fixed base-point $\mathfrak{o}$.

Points in the boundary: families of quasi-geodesic rays starting at $\mathfrak{o}$.

Fix a sublinear function $\kappa(t)$. Let $\|x\| = d(\mathfrak{o}, x)$. A κ–neighbourhood around a quasi-geodesic γ is a set of point x

$$\mathcal{N}_\kappa(\gamma, n) := \{x \mid d(x, \gamma) \leq n \cdot \kappa(\|x\|)\}$$

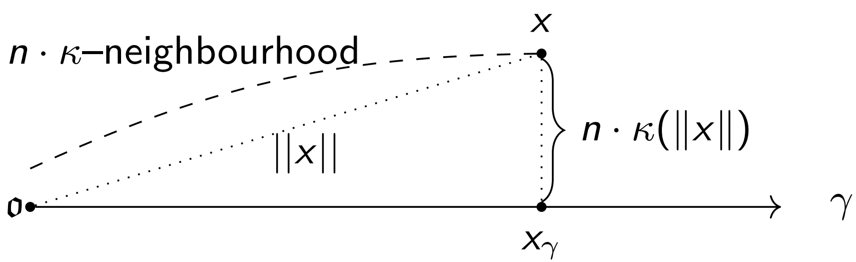
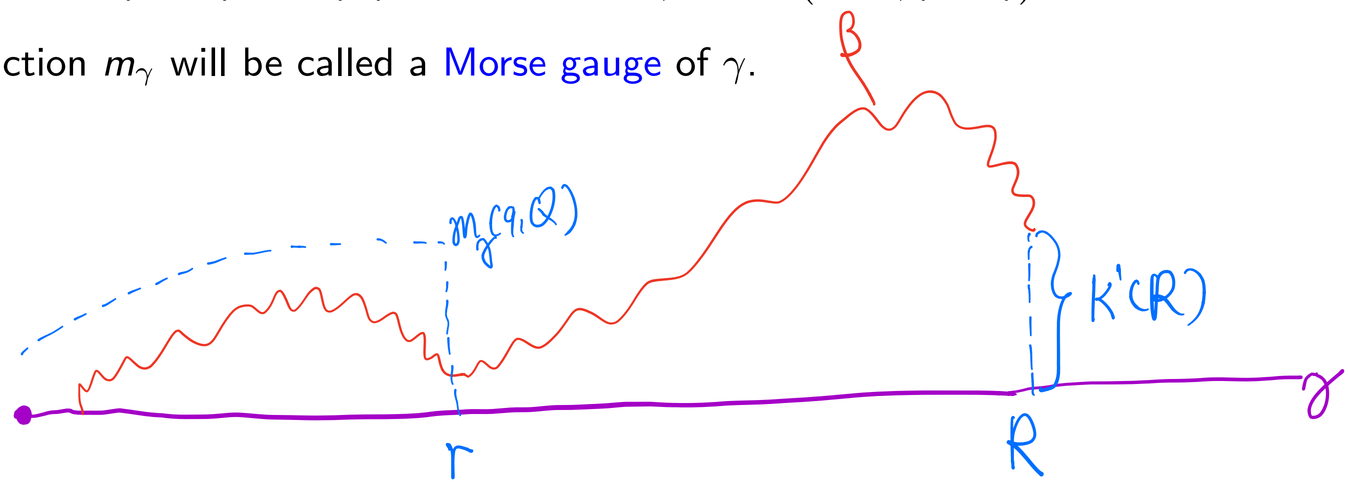


Figure: A κ–neighbourhood of γ

A quasi-geodesic ray γ is κ–Morse if there exists a proper function $m_\gamma : \mathbb{R}^2 \to \mathbb{R}$ such that for any sublinear function κ' and for any $r > 0$, there exists R such that for any (q, Q)–quasi-geodesic β with $m_\gamma(q, Q)$ small compared to r, if

$$d_X(\beta_R, \gamma) \le \kappa'(R) \qquad \text{then} \qquad \beta|_r \subset \mathcal{N}_\kappa(\gamma, m_\gamma(q, Q))$$

The function m_γ will be called a Morse gauge of γ.



Equivalence class: given two quasi-geodesics α, β based at $\mathfrak{o}$, we say that $\beta \sim \alpha$ if they sublinearly track each other: i.e. if

$$\lim_{r \to \infty} \frac{d(\alpha_r, \beta_r)}{r} = 0.$$

Let $\partial_\kappa X$ denote the set of equivalence class of κ–Morse quasi-geodesic rays, equipped with coarse cone topology.

Theorem (Q-Rafi, Q-Rafi-Tiozzo)

Let X be a proper, geodesic metric space, then $\partial_\kappa X$ is a topological space that is quasi-isometrically invariant, and metrizable.

Examples:

- $\mathbb{Z}^2$
- $\mathbb{H}^2$
- $\mathbb{Z} \star \mathbb{Z}^2$

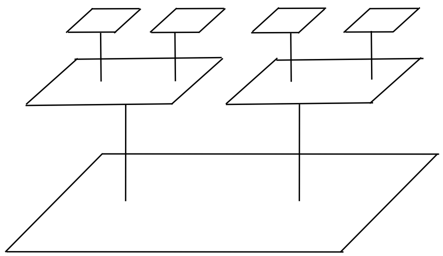


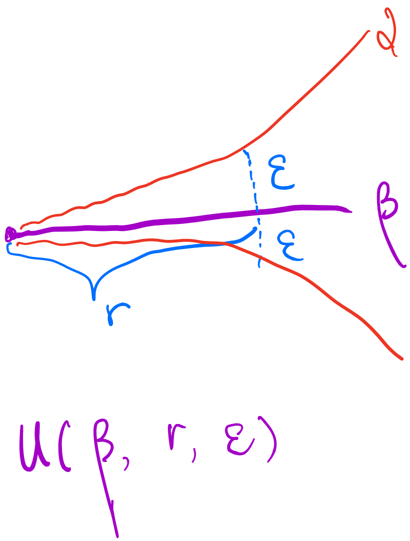
Figure: A tree of flats.

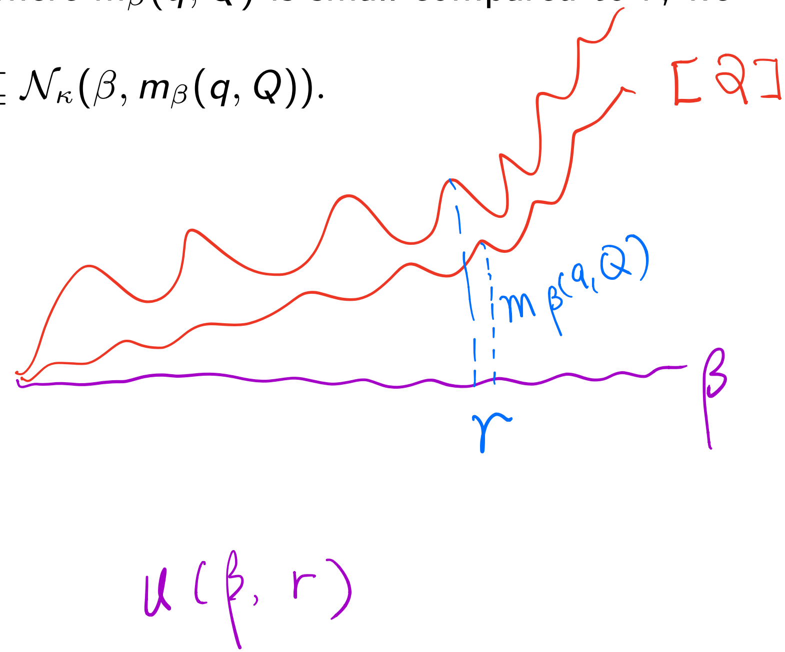
Definition of Coarse cone Topology

We define the set $\mathcal{U}(\beta, r) \subseteq X \cup \partial_\kappa X$ as follows.

- An equivalence class $\mathbf{a} \in \partial_\kappa X$ belongs to $\mathcal{U}(\beta, r)$ if for any (q, Q)-quasi-geodesic $\alpha \in \mathbf{a}$, where $m_\beta(q, Q)$ is small compared to r, we have the inclusion

$$\alpha|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(q, Q)).$$





Random walk and Poisson boundaries

Let $\langle S \rangle$ be a symmetric generating set with a probability distribution μ. A *random walk* is a process on a group G where sample paths are $s_{r_1} s_{r_2} s_{r_3} \ldots$, $s_{r_i} \in \langle S \rangle$.

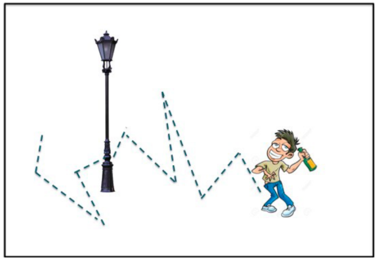


Figure: A random walk.

Definition

Given a finitely generated group and a probability measure μ with finite support, its *Poisson boundary* is the maximal measurable set to which almost all sample paths converge, with hitting measure ν arising from μ.

Kaimanovich: Let G be a hyperbolic group, then Gromov boundary is a model for it's associated Poisson boundary.

Theorem (Gekhtman-Q-Rafi)

Let X be a rank-1 CAT(0) space, and $G \curvearrowright X$ geometrically. Then there exists a κ such that the Poisson boundary can be identified with $\partial_\kappa G$.

Theorem (Q-Rafi-Tiozzo)

the Poisson boundary can be identified with $\partial_\kappa G$ for the following groups.

- *Right-angled Artin groups,* $\kappa(t) = \sqrt{t \log t}$.
- *Relative hyperbolic groups,* $\kappa(t) = \log t$
- *Mapping class groups,* $\kappa(t) = \log^d t$
- *Hierarchically hyperbolic groups,* $\kappa(t) = \log^d t$

Two ingredients.

1. Almost every sample path tracks a κ-Morse geodesic ray, we need sublinear excursion.
Sisto-Taylor: Projections systems.

- Relative hyperbolic groups
- Curve complex of subsurfaces in mapping class group.
- Hierarchically hyperbolic groups.

Let G be a group and let $(\mathcal{S}, Z_0, \{\pi_Z\}_{Z\in\mathcal{S}}, \pitchfork)$ be a projection system on G. Let (w_n) be a random walk on G. Then there exists $C \geq 1$ so that, as n goes to ∞,

$$\mathbb{P}\Big(\sup_{Z\in\mathcal{S}} d_Z(1, w_n) \in [C^{-1} \log n, C \log n]\Big) \to 1$$

2. Maximality: the tracking is sublinear. Sisto, Tiozzo, Maher-Tiozzo, Karlsson-Margulis, Q-Rafi-Tiozzo.

For CAT(0) spaces there are also two steps.

A unit speed, parametrized geodesic ray τ in X is said to be frequently contracting if there is a number $N > 0$ such that for each $R > 0$ and $\theta \in (0, 1)$ there is an $L_0 > 0$ such that for $L > L_0$ length θL subsegment of $\tau([0, L])$ contains N-(strongly) contracting subsegment of length at least R.

1. A generic sample path tracks a frequently contracting geodesic ray.
   - Stationary measure: follow the proof of Baik-Gekhtman-Hamstadt.
   - Patterson Sullivan measure (defined by Ricks): Birkhoff ergodic theorem.
2. A frequently contracting geodesic ray is sublinearly Morse.
(Gekhtman-Q-Rafi)

Other hyperbolic-like properties of the sublinearly Morse quasi-geodesics.

- $\partial_\kappa X$ is a visibility space.(Q-Zalloum)
- a κ-Morse geodesics ray has at least quadratic κ-lower-divergence. (Q-Murray-Zalloum)
- In CAT(0) spaces, κ-Morse is equivalent to κ-contracting. (Q-Rafi)

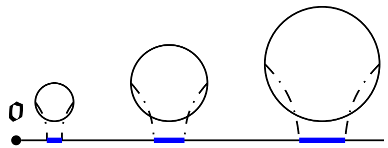


Figure: A sublinearly contracting geodesic ray

Question

- When does a group G has a $\partial_\kappa G$ that can be identified with the Poisson boundary?

Thank you!