Geometry in non-discrete groups of hyperbolic isometries: Primitive stability and the Bowditch BQ-conditions are equivalent.

THE UNIVERSITY OF WARWICK

Caroline Series

$F_2 =< a, b| >$ free group on 2 generators. Representation $\rho : F_2 \to SL(2, \mathbb{C})$.

Write $A = \rho(a), W = \rho(w), w \in F_2$ etc.

The *character variety* χ is the set of all such representations up to conjugation, identified with $\mathbb{C}^3$.

PS and BQ are conditions on the images of primitive elements in F_2

$u \in F_2$ is primitive if it is one of a generating pair.

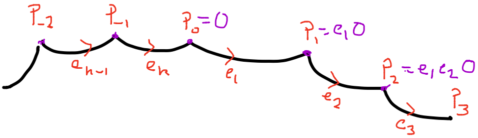
$\mathcal{P} := \{\text{primitive elements in } F_2\}$.

Up to conjugation and inverse, primitive elements can be identified with $\mathbb{Q} \cup \infty$.

Primitive stability

Fix $O \in \mathbb{H}^3$. Let $u = e_1 \dots e_k \in \mathcal{P}$, $e_i \in \{a^{\pm}, b^{\pm}\}$.

The broken geodesic $\mathbf{br}(u; a, b)$ associated to u.



$P_0 = O, P_1 = e_1 O, P_2 = e_1 e_2 O, \dots$ and $P_{-1} = e_k^{-1} O, P_{-2} = e_k^{-1} e_{k-1}^{-1} O, \dots$.

Definition ρ is Primitive stable (PS) if $\{\mathbf{br}(u; a, b) : u \in \mathcal{P}\}$ is uniformly quasigeodesic.
In other words, there exist $c, c', \epsilon > 0$ such that

$$c'|n-m| - \epsilon \leq d(P_n, P_m) \leq c|n-m| + \epsilon \quad \forall n, m \in \mathbb{Z}.$$

Proposition [Minsky 2013] The set of primitive stable ρ is open in the character variety χ.
Remark Minsky and later Lupi showed that there are primitive stable ρ which are not discrete.

Bowditch BQ-conditions

These are two conditions on the representation ρ:

- (BQI) $\operatorname{Tr}\rho(u) \notin [-2,2]$ for all $u \in \mathcal{P}$.
- (BQII) $\{u \in \mathcal{P} : |\operatorname{Tr}\rho(u)| \leq 2\}$ is finite.

Definition $\mathcal{B}$ is the set of ρ which satisfy the above two conditions (the BQ conditions).

Proposition [Bowditch 1998, Tan-Wong-Zhang 2008]
$\mathcal{B}$ is open in the character variety χ. $\rho \in \mathcal{B}$ implies a MacShane identity.

Remarks
- Bowditch assumed that $\operatorname{Tr}[A,B] = -2$. His results were generalised to arbitrary values of $\operatorname{Tr}[A,B]$ by TWZ.
- Bowditch conjectured that if $\rho \in \mathcal{B}$ and $\operatorname{Tr}[A,B] = -2$ then ρ is quasifuchsian and hence discrete. This is still open.
- S.-Tan-Yamashita (2017) showed that there are $\rho \in \mathcal{B}$ which are not discrete.

The theorem

THEOREM [BinBin Xu & JaeJong Lee; S.] Primitive stability and the Bowditch conditions are equivalent.

Proved by Xu-Lee (Trans. AMS 2020) and S. (arxiv) independently. Xu-Lee introduced some nice ideas which greatly simplify the proofs. This talk is an amalgam of the two methods. (Proved by Lupi for real representations.)

WARM UP PROPOSITION PS implies BQ

PROOF Let $u \in \mathcal{P}$. If the broken geodesic **br**$(u; a, b)$ is quasigeodesic then it is not parabolic or elliptic. (BQI)

If $\{$**br**$(u; a, b) : u \in \mathcal{P}\}$ are uniformly quasigeodesic then all the broken geodesics are at uniformly bounded distance from their respective axes.

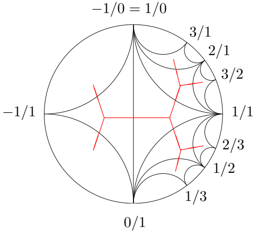
Recall $\mathrm{Tr}\, U = 2 \cosh \lambda(U)$ where $2\lambda(U) = \ell(U) + i\theta(U)$ is the complex translation length. So

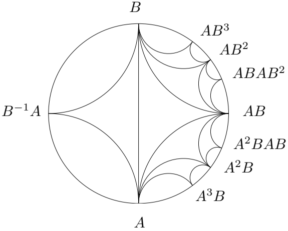
$$c' ||u|| - \epsilon \leq d_{\mathbb{H}}(O, UO) \leq c + \ell(U) \leq const + \log^{+} |\mathrm{Tr}\, U)|$$

where $||u||$ is the word length of u wrt (a, b).

Hence since only finitely many elements have word length less than a given bound the same is true of the traces. (BQII)

BQ implies PS I: Organising primitive elements





Assume all words are reduced and cyclically reduced (shortest). Say $u \sim u'$ if u' is a cyclic permutation of $u^{\pm}$. Primitive elements up to conjugation and inverse are identified with $\mathbb{Q} \cup \infty$.

The dual tree (in red) is trivalent. Its complementary regions correspond to $\mathbb{Q} \cup \infty$ hence to primitive elements up to conjugation and inverse.

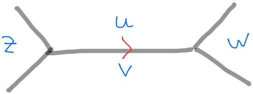
So as to easily distinguish between a word and its inverse, say a word w is *positive* if all the exponents of a in w are positive. This will be important later.

II. Arrows

We are going to define two different kinds of arrows on the edges of the tree $\mathcal{T}$.

- Trace arrows (following Bowditch)

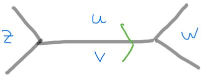
Put a T-arrow on an edge if $|\operatorname{Tr} Z| > |\operatorname{Tr} W|$. (Note: $z = uv$ and $w = uv^{-1}$ or vice versa.)

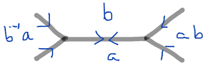


THEOREM [BOWDITCH, TWZ] If $\rho \in \mathcal{B}$ then $\mathcal{T}$ has a finite connected attracting subtree $\mathcal{T}_T$.

- Word arrows

Put a W-arrow on an edge if $||z|| > ||w||$. If u, v are positive this implies $z = uv, ||z|| = ||u|| + ||v||$.





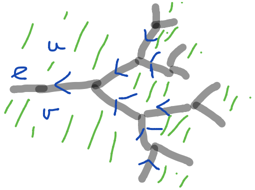
OBSERVATION The W-tree has a finite connected attracting subtree $\mathcal{T}_W$.

By enlarging $\mathcal{T}_T, \mathcal{T}_W$ if necessary:

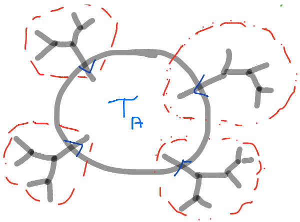
PROPOSITION There is a finite attracting tree $\mathcal{T}_A$ so that for all edges not in $\mathcal{T}_A$, the T- and W- arrows agree. Moreover given $M > 0$ we can assume that every edge not in $\mathcal{T}_A$ is adjacent to at least region u with $|\operatorname{Tr} U| > M$.

III. Reducing the proof

Let $\mathcal{T}_A$ be a finite attracting tree so that outside $\mathcal{T}_A$ the T- and W-arrows agree. The *wake* $\mathcal{W}(\vec{e})$ of a directed edge $\vec{e}$ outside $\mathcal{T}_A$ is the collection of all complementary regions which are adjacent to an edge whose arrow points into $\vec{e}$ (including $\vec{e}$).



Then all but finitely many regions lie in $\mathcal{W}(\vec{e}_i)$ for one of the finitely many directed edges $\vec{e}_i, i = 1, \ldots, m$ whose heads meet $\mathcal{T}_A$.

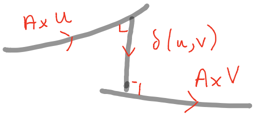


Let (u_i, v_i) be the generators adjacent to $\vec{e}_i$. It is sufficient to show that $\{\mathbf{br}(w; u_i, v_i) : w \in \mathcal{W}(\vec{e}_i)\}$ is uniformly quasigeodesic for each such $\vec{e}_i$. We will do this using the methods of Lee & Xu.

The Key Estimate

Focus on the the broken geodesics corresponding to words in $\mathcal{W}(\vec{e}_i)$. Let u_i, v_i be the regions adjacent to $\vec{e}_i$, chosen to be positive. Since the W- and T-arrows agree, we may assume $|\operatorname{Tr} UV| > |\operatorname{Tr} UV^{-1}|$ and $||uv|| > ||uv^{-1}||$ so $||uv|| = ||u|| + ||v||$. By the construction of $\mathcal{T}_A$ we can also assume that at least one of $|\operatorname{Tr} U|, |\operatorname{Tr} V|$, say $|\operatorname{Tr} U|$ is large.

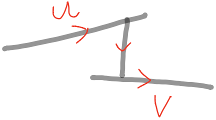
Let $\mathcal{D}$ be the common perpendicular to $\operatorname{Ax} U, \operatorname{Ax} V$ and let $\delta_{U,V}$ be the complex distance between them. $\delta_{U,V} = d_{U,V} + i\theta_{U,V}$.

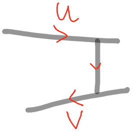


KEY ESTIMATE Fix $0 < \alpha < \pi/2$. For large enough $\ell(U)$ we have $|\theta_{U,V}| \leq \alpha$.

This means the twist angle $\theta_{U,V}$ between $\operatorname{Ax} U$ and $\operatorname{Ax} V$ along their common perpendicular $\mathcal{D}$ is 'small'. *The orientations of U, V are crucial.*

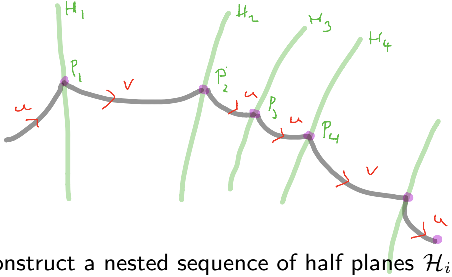
In other words, as long as $|\operatorname{Tr} U|$, and hence $\ell(U)$, is large, roughly speaking

we have 

and not 

Assuming the key estimate: proving quasigeodesicity

Let $\mathcal{C}(u,v)$ be the set of words w which are a product of positive powers of u, v. For $w \in \mathcal{C}(u,v)$ the broken geodesic $\mathbf{br}(w;u,v)$ is made of arcs connecting points P_i so that $P_{i+1} = g_i P_i$ for $g_i \in \{U, V\}$. ($P_0 = O, i \in \mathbb{Z}$)

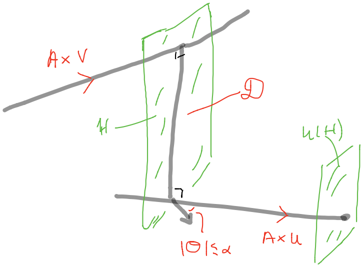


We are going to construct a nested sequence of half planes $\mathcal{H}_i$ with $P_i \in \mathcal{H}_i$. There are only 4 possible relative arrangements of triples of consecutive half planes $\mathcal{H}_{i-1}, \mathcal{H}_i, \mathcal{H}_{i+1}$ depending on g_{i-1}, g_i. So the distance between any two bend points P_n, P_m of $\mathbf{br}(w;u,v)$ is bounded below $d|m-n|$ for some fixed $d > 0$. Hence $\{\mathbf{br}(w;u,v) : w \in \mathcal{C}(u,v)\}$ is uniformly quasigeodesic.

Constructing the half planes

To construct the $\mathcal{H}_i$: Assume wlog that $\ell(U) \geq \ell(V)$. Choose O to be the intersection point of $\mathrm{Ax}\, V$ and the common perpendicular $\mathcal{D}$ of $\mathrm{Ax}\, V$ and $\mathrm{Ax}\, U$. Let $\mathcal{H}$ be the half plane perpendicular to $\mathrm{Ax}\, V$ through O and containing $\mathcal{D}$.

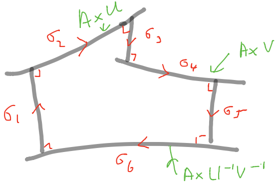
By the Key Estimate, the angle between $\mathrm{Ax}\, U$ and the normal to $\mathcal{H}$ is 'small', that is, uniformly bounded away from $\pi/2$. Likewise the angle between $\mathrm{Ax}\, U$ and the normal to $U(\mathcal{H})$.



It is an exercise in hyperbolic geometry to show that there exists $c > 0$ so that if $\ell(U) > c$ then the planes $\mathcal{H}$ and $U(\mathcal{H})$ are nested.

Proof of the Key Estimate: The amplitude of a hexagon

We use the *amplitude* of the hexagon formed by $\mathrm{Ax}\, U, \mathrm{Ax}\, V, \mathrm{Ax}\, U^{-1}V^{-1}$. Defined (see Fenchel) as $\mathrm{Amp}(\sigma_1, \sigma_3, \sigma_5) = \pm 1/2\, \mathrm{Tr}(S_5 S_3 S_1)$; S_i is π rotation round σ_i; use line matrices to fix signs. $(\mathrm{Amp}(\sigma_1, \sigma_3, \sigma_5) = -i \sinh \sigma_2 \sinh \sigma_3 \sinh \sigma_4.)$



PROPOSITION With the U, V hexagon as shown, and u and v positive, up to sign $\mathrm{Amp}(\sigma_1, \sigma_3, \sigma_5) = \mathrm{Amp}(U, V)$ is an invariant of generator pairs.
PROOF Formulae in Fenchel relate $\mathrm{Amp}(U, V)$ to $\mathrm{Tr}\, UVU^{-1}V^{-1}$.

COROLLARY Applying to the U, V hexagon shows $|\sinh \delta_{U,V} \sinh U \sinh V|$ is independent of (u, v). Thus if $\ell(U) \to \infty$ then $|\sinh \delta_{U,V}| \le ce^{-\ell(U)}$. With $\delta_{U,V} = d_{U,V} + i\theta_{U,V}$ this gives $d_{U,V} \to 0$ and $\theta_{U,V} \to 0$ or π as $\ell(U) \to \infty$. We want $\theta_{U,V} \to 0$ (axes align) as opposed to $\theta_{U,V} \to \pi$ (axes backtrack).

To distinguish the two cases, use the cosine formula in the hexagon together with $|\,\mathrm{Tr}\, UV| > |\,\mathrm{Tr}\, UV^{-1}|$ (equivalent to $\Re(\frac{\mathrm{Tr}\, UV}{\mathrm{Tr}\, U\, \mathrm{Tr}\, V}) > 1/2$) to deduce that $\theta_{U,V} \to 0$ as $\ell(U) \to \infty$.

Summary

- Bowditch conditions give a finite attracting tree $\mathcal{T}_A$ outside which word length and trace increase in the same direction.
- Amplitude of hexagon controls angle between positive directions of $\mathrm{Ax}\, U, \mathrm{Ax}\, V$ where u, v are both positive.
- Control of angle between positive directions gives uniform quasi-geodesity.

Thank you