Faces of the Thurston norm ball up to isotopy

GaTO

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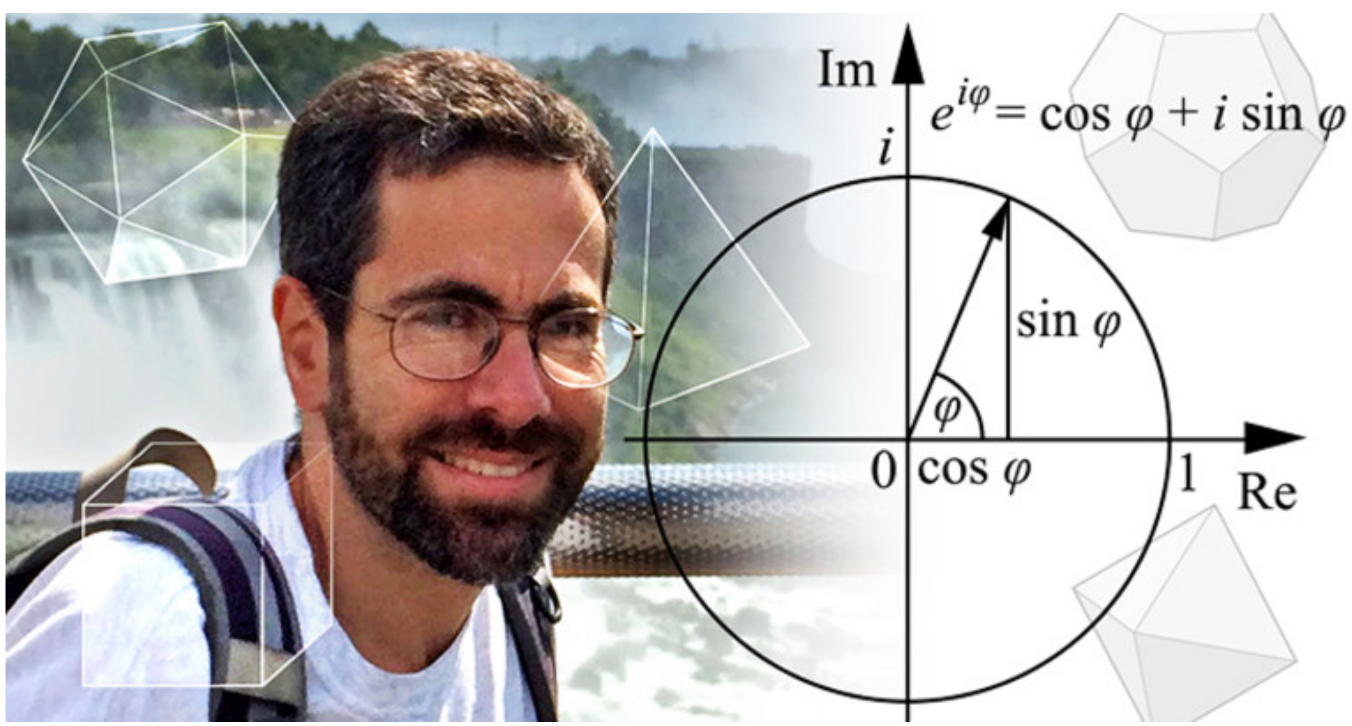
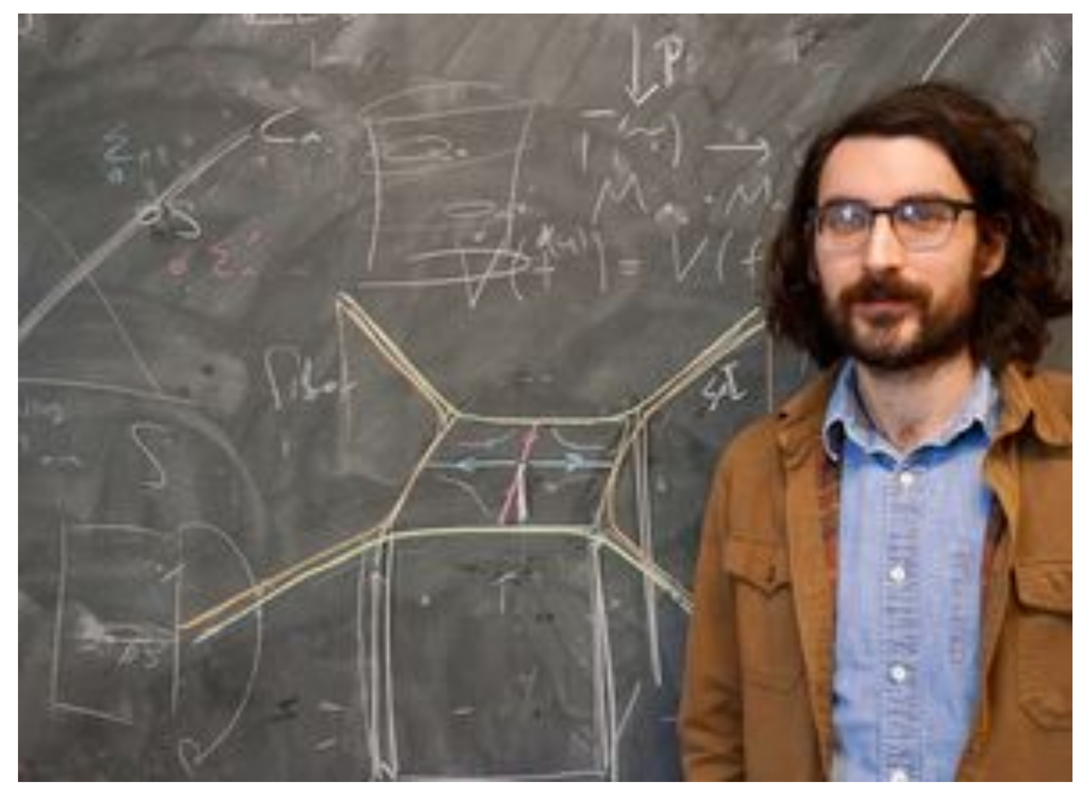
Advertisements

- [Lan20] *Veering triangulations and the Thurston norm: homology to isotopy*.
  arxiv:2006.16328 or math.wustl.edu/~landry

- [LMT20] *A polynomial invariant for veering triangulations* (joint w/ Yair Minsky and Samuel Taylor).
  Available soon.





M is a:
connected, oriented, closed, irreducible, atoroidal 3-manifold

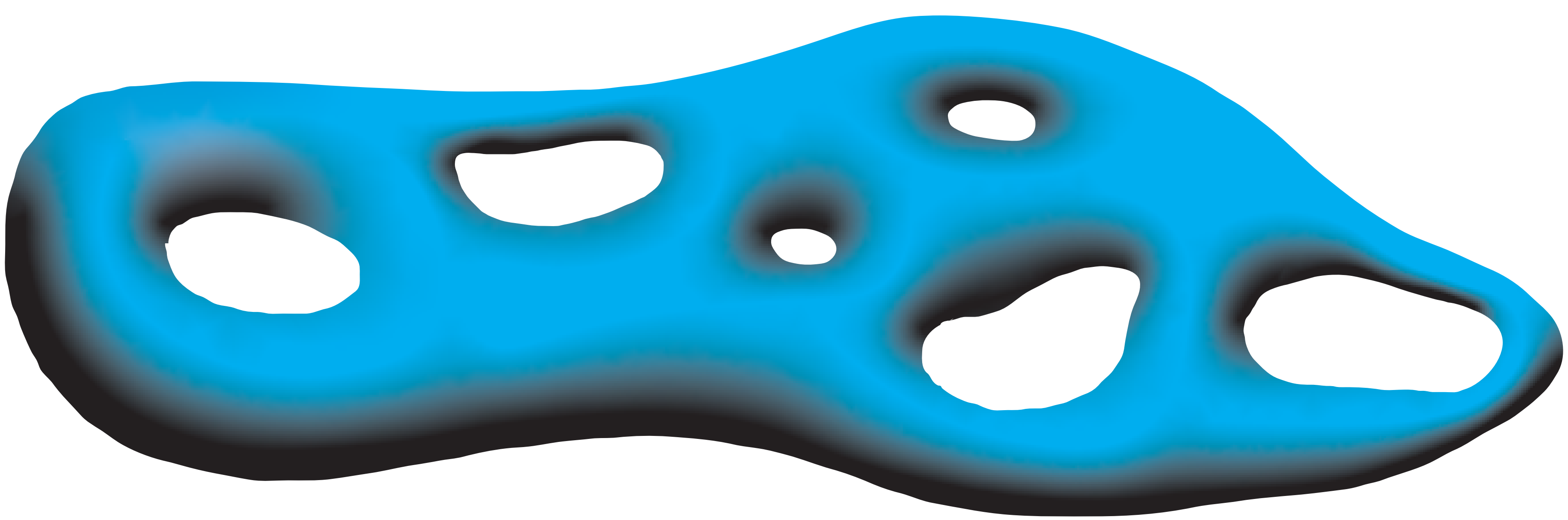
Some motivating questions (background to come)

1. Can we organize all essential surfaces in M?

2. What does a face of the Thurston norm ball mean?

   2a. Given an object associated to fibered faces (flow, veering triangulation, Teichmüller polynomial…), is there a generalization for non-fibered faces?

3. Given a face F of the Thurston norm ball, can we organize all the essential surfaces in M whose homology classes lie over F?



First goal today: explain statement of, and give context for, the main result from [Lan20].

Main Theorem. *Let τ be a veering triangulation of a compact 3-manifold $\mathring{M}$. If M is obtained by Dehn filling each component of $\partial\mathring{M}$ along slopes with ≥ 3 prongs then M is irreducible and atoroidal. Let σ_τ be the face of the Thurston norm ball $B_x(M)$ determined by the Euler class e_τ. Then the following hold:*

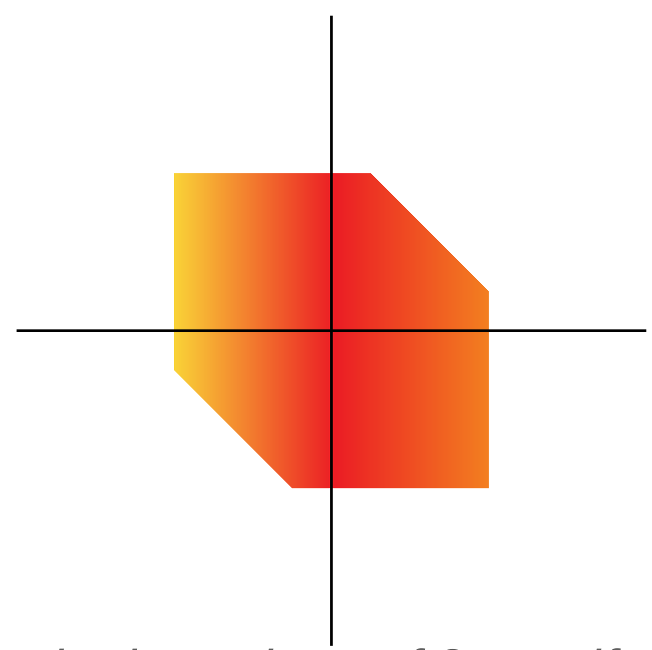
(i) $\text{cone}(\sigma_\tau) = \mathcal{C}_\tau^\vee$*, and the codimension of σ_τ in $\partial B_x(M)$ is equal to the dimension of the largest linear subspace contained in $\mathcal{C}_\tau$.*

(ii) If $S \subset M$ is a surface, then S is taut and $[S] \in \text{cone}(\sigma_\tau)$ if and only if S is carried by $\tau^{(2)}$ up to isotopy.

Second goal: tell you some of what is in [LMT20]

Thurston norm on $H_2(M;\mathbb{R})$

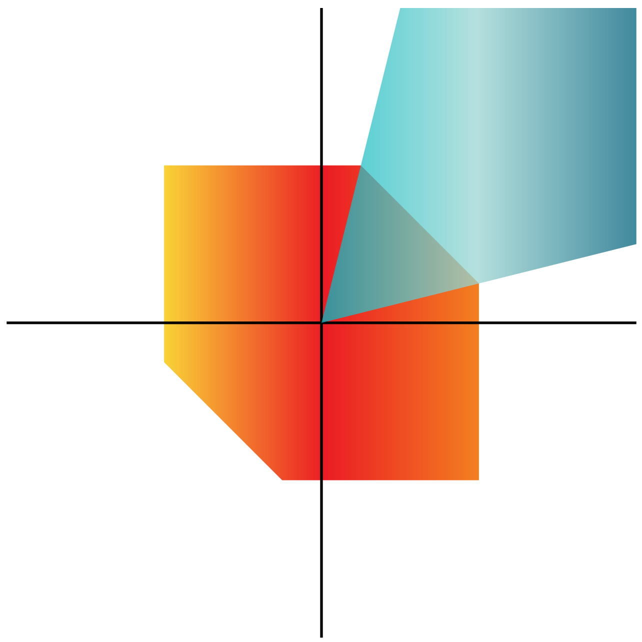
- Let $\alpha \in H_2(M;\mathbb{R})$ be a $\mathbb{Z}$-lattice point
- define $x(\alpha) = \min\left\{ -\chi(S) \mid S \hookrightarrow M \text{ sphereless}, [S] = \alpha \right\}$
- Thurston: x extends to a norm on H_2
- unit ball B_x is a finite sided polyhedron w/ rational vertices



[William Thurston, *A norm for the homology of 3-manifolds*, Memoirs AMS, 1986]

Thurston norm ctd.

- If M fibers over the circle with fiber S, then $[S] \in \text{int}(\text{cone}(F))$ for a top dimensional face F of B_x
  - further, *any* lattice point in $\text{int}(\text{cone}(F))$ is represented by a fiber of some fibration $M \to S^1$
- such an F is called a **fibered face** of B_x

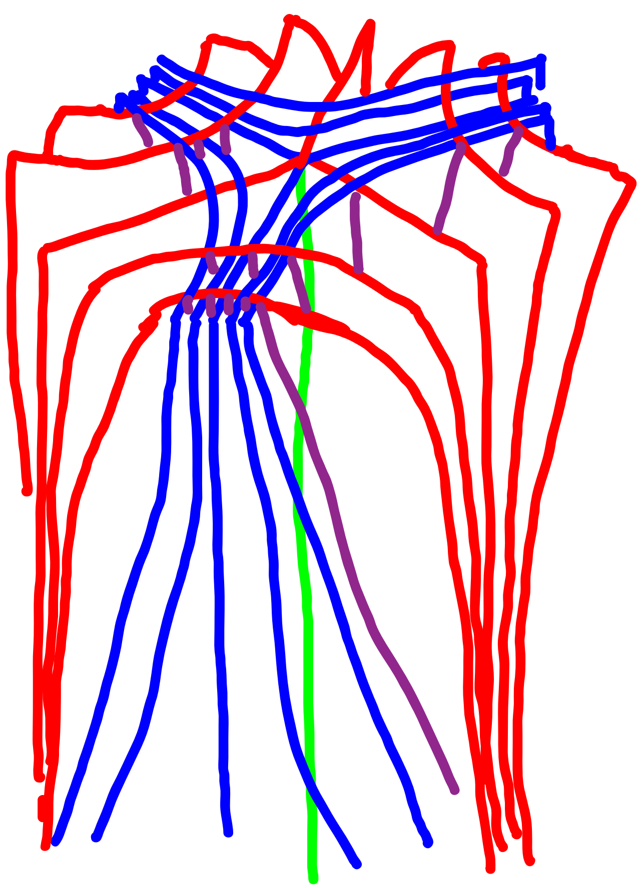


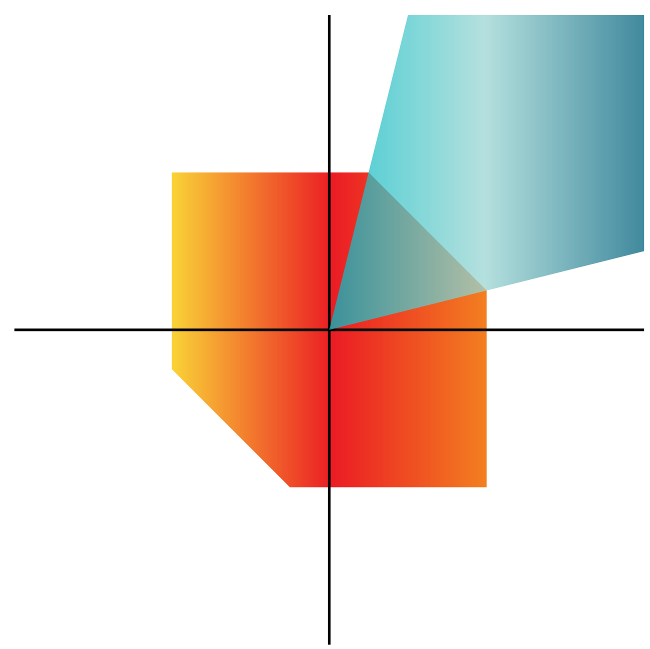
[William Thurston, *A norm for the homology of 3-manifolds*, Memoirs AMS, 1986]

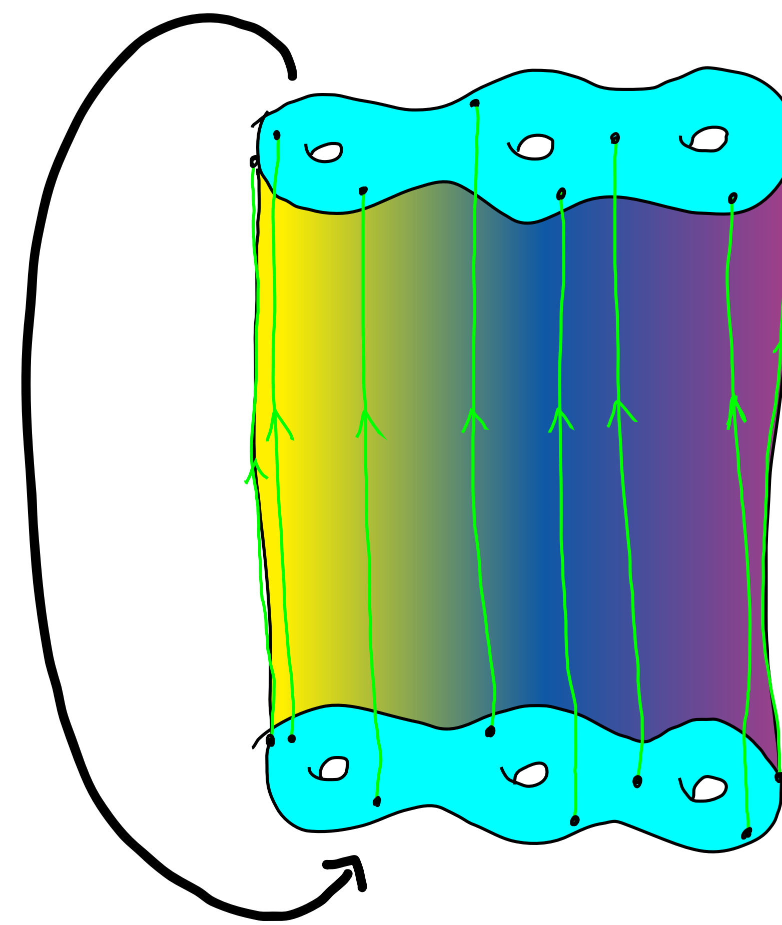
Fried: Let F be a fibered face. There is a pseudo-Anosov flow φ on M such that every lattice point in int(cone(F)) is represented by a cross section to φ. (*cross section*: transverse, intersects every orbit)

The flow φ can be constructed as the suspension flow of any of the fibers corresponding to F. Unique up to isotopy, reparametrization.

Let $e_\varphi \in H^2(M)$ be the Euler class of φ. Then $x = -e_\varphi$ on cone(F).



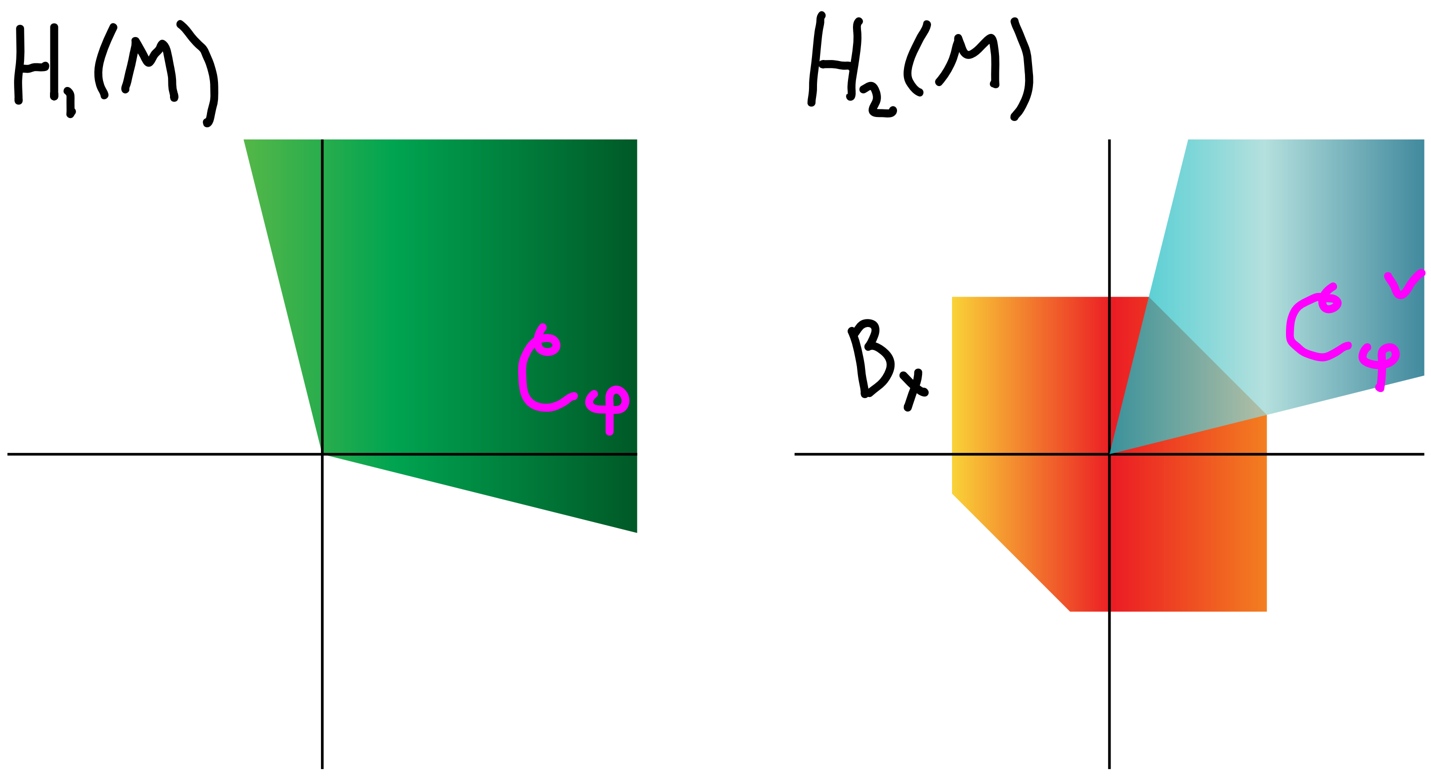




[David Fried, "Fibrations over S^1 with pseudo-Anosov monodromy," Thurston's work on Surfaces (FLP) Ch. 14]

Fried's picture:

- let $\mathscr{C}_\varphi$= cone in $H_1(M)$ generated by periodic orbits of φ
- let $\mathscr{C}_\varphi^\vee \subset H_2(M)$ be classes pairing nonnegatively with $\mathscr{C}_\varphi$
- then $\mathscr{C}_\varphi^\vee = \text{cone}(F)$.



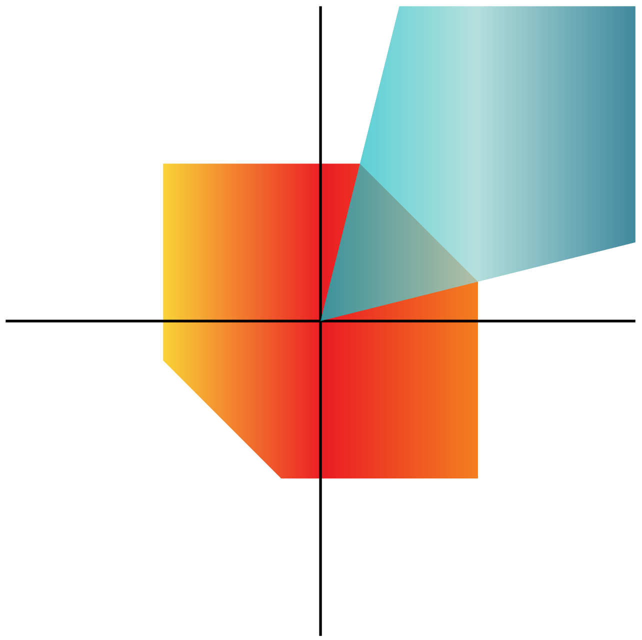
[David Fried, "Fibrations over S^1 with pseudo-Anosov monodromy," Thurston's work on Surfaces (FLP) Ch. 14]

Another perspective (McMullen):

- any point $\alpha \in \mathrm{int}(\mathrm{cone}(F))$ assigns a positive "length" to each periodic orbit.

- set $h(\alpha)$=exponential growth rate of periodic orbits w.r.t this length. Then h is an analytic function on $\mathrm{int}(\mathrm{cone}(F))$, tends to infinity at boundary of cone.

- Set $G = H_1(M; \mathbb{Z})$/torsion. There exists an element $\Theta_F \in \mathbb{Z}[G]$ called the **Teichmüller polynomial** that packages all these growth rates.



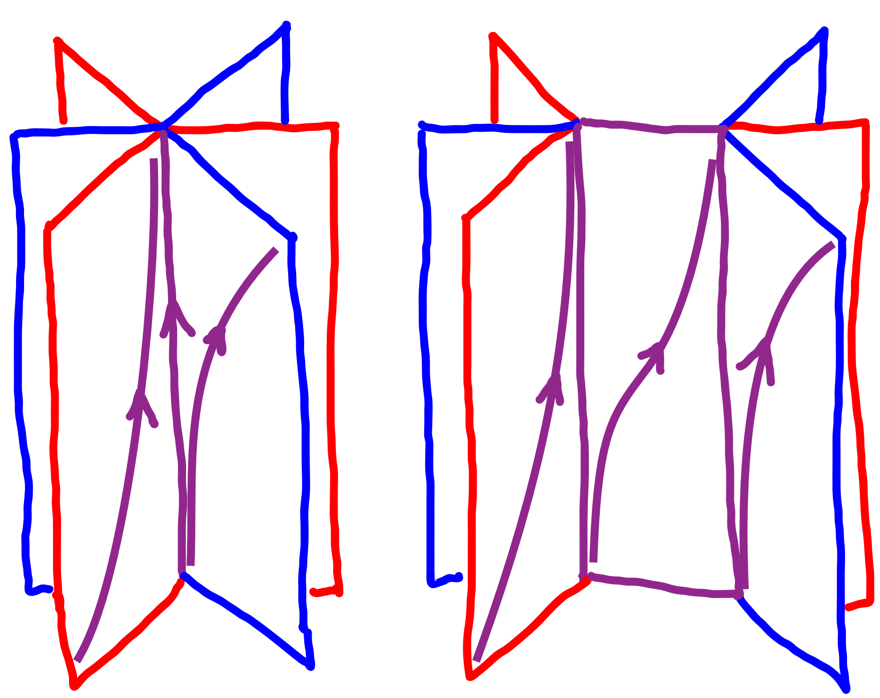
[Curtis McMullen, *Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations*, Ann. Sci. E.N.S. 1999]

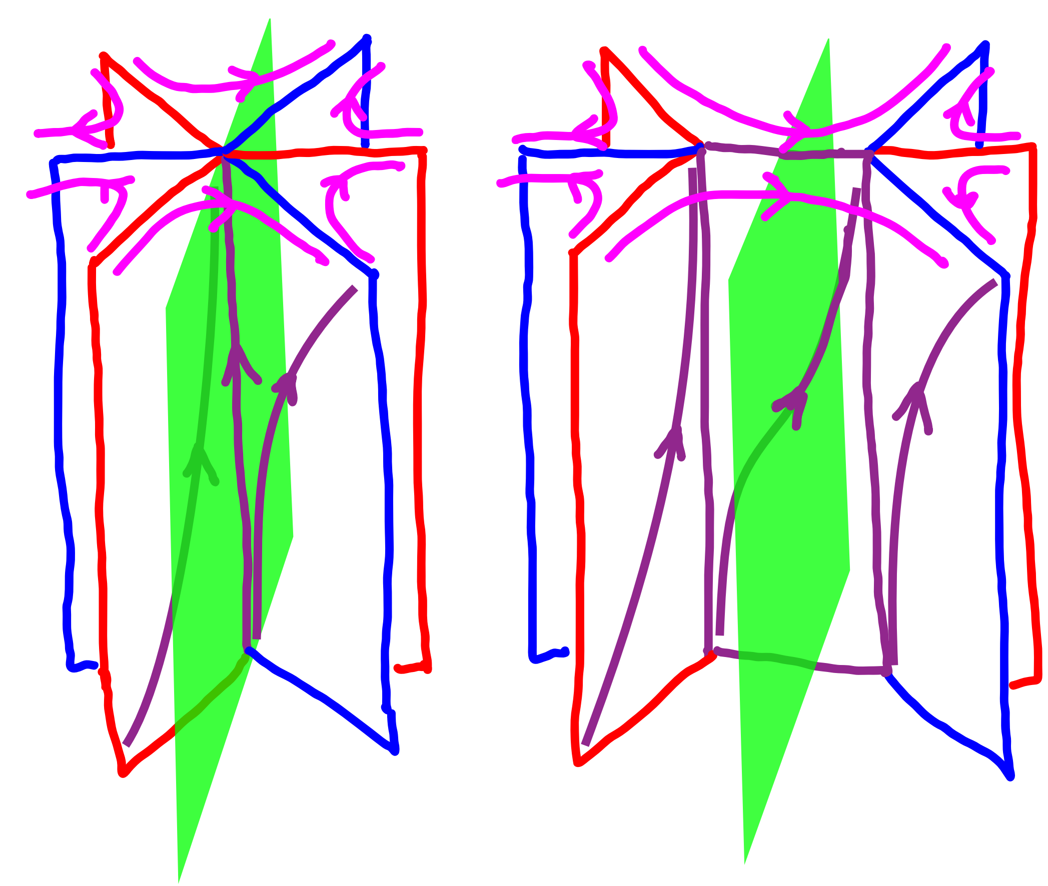
Q: what about classes in $\partial\mathrm{cone}(F)$?

- each one pairs trivially with some closed orbit of φ
- <u>Mosher</u>: any lattice point $\alpha \in \partial\mathrm{cone}(F)$ is represented by a surface which is <u>almost transverse</u> to φ.

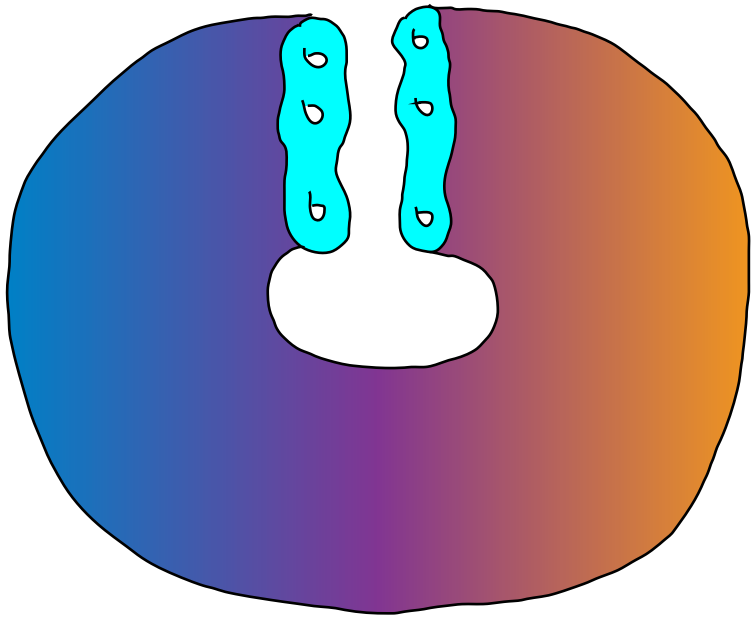
(almost transverse: there is a "dynamic blowup" of φ to which the surface is transverse)

<u>dynamic blowup:</u>





[Lee Mosher, *Surfaces and branched surfaces transverse to pseudo-Anosov flows on 3-manifolds*, J. Diff. Geom.1991]



<u>Definition</u>: an oriented surface $S \hookrightarrow M$ is **taut** if:

- no components are nulhomologous, and
- $x([S]) = -\chi(S)$

<u>Example 1</u>: fiber of a fibration $M \to S^1$

<u>Example 2</u>: more generally, compact leaf of a taut foliation (Thurston)

<u>Example 3</u>: surface almost transverse to a pseudo-Anosov flow (Mosher)

<u>Fact</u>: If $S \hookrightarrow M$ is a fiber, then S is the unique taut representative of its homology class up to isotopy (Thurston).

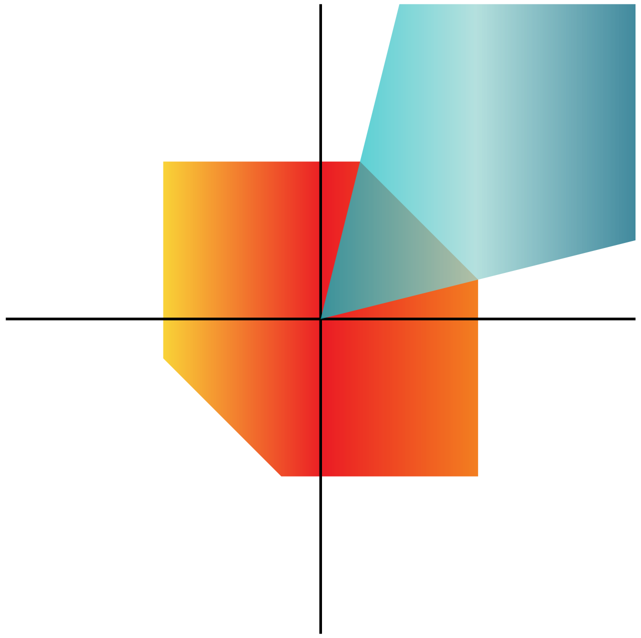
<u>However</u>: taut surfaces are not necessarily unique up to isotopy in their homology classes.

Combining Fried, Mosher, Thurston:

Given a fibered face F of B_x, the flow φ sees **every** isotopy class of taut surface lying over $\mathrm{int}(F)$, and sees **one** taut representative of every class lying over ∂F.

<u>Questions</u>

- What about the missing isotopy classes of taut surfaces over ∂F?
- What about other faces? (non-fibered and/or lower dimensional)

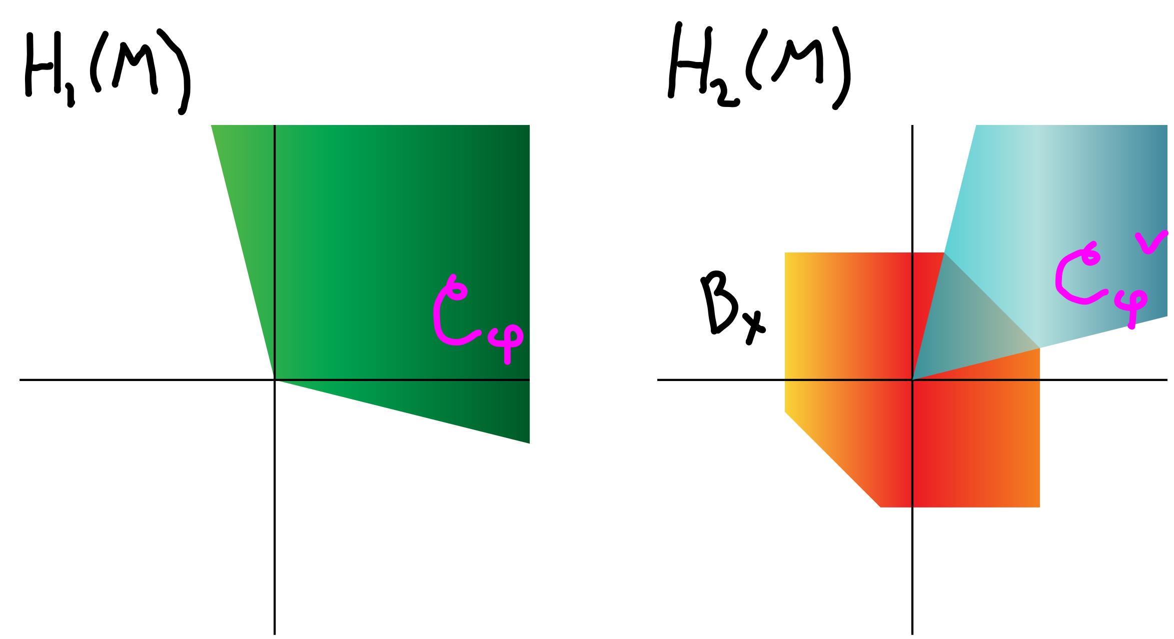


<u>Theorem (Mosher)</u>. Let φ be a pseudo-Anosov flow on M with *no dynamically parallel closed orbits*. Then φ *dynamically represents* a face F of B_x, and every integral class in cone(F) is represented by a surface almost transverse to φ.

we will gloss over "no dynamically parallel closed orbits"

"dynamically represents" means cone(F) is equal to both:

1. set on which $x = -e_\varphi$
2. set of all classes pairing nonnegatively with closed orbits of φ



[Lee Mosher, *Dynamical systems and the homology norm of a 3-manifold II, Invent. Math.* 1992] back to our main theorem:

Main Theorem. *Let τ be a veering triangulation of a compact 3-manifold $\mathring{M}$. If M is obtained by Dehn filling each component of $\partial \mathring{M}$ along slopes with ≥ 3 prongs then M is irreducible and atoroidal. Let σ_τ be the face of the Thurston norm ball $B_x(M)$ determined by the Euler class e_τ. Then the following hold:*

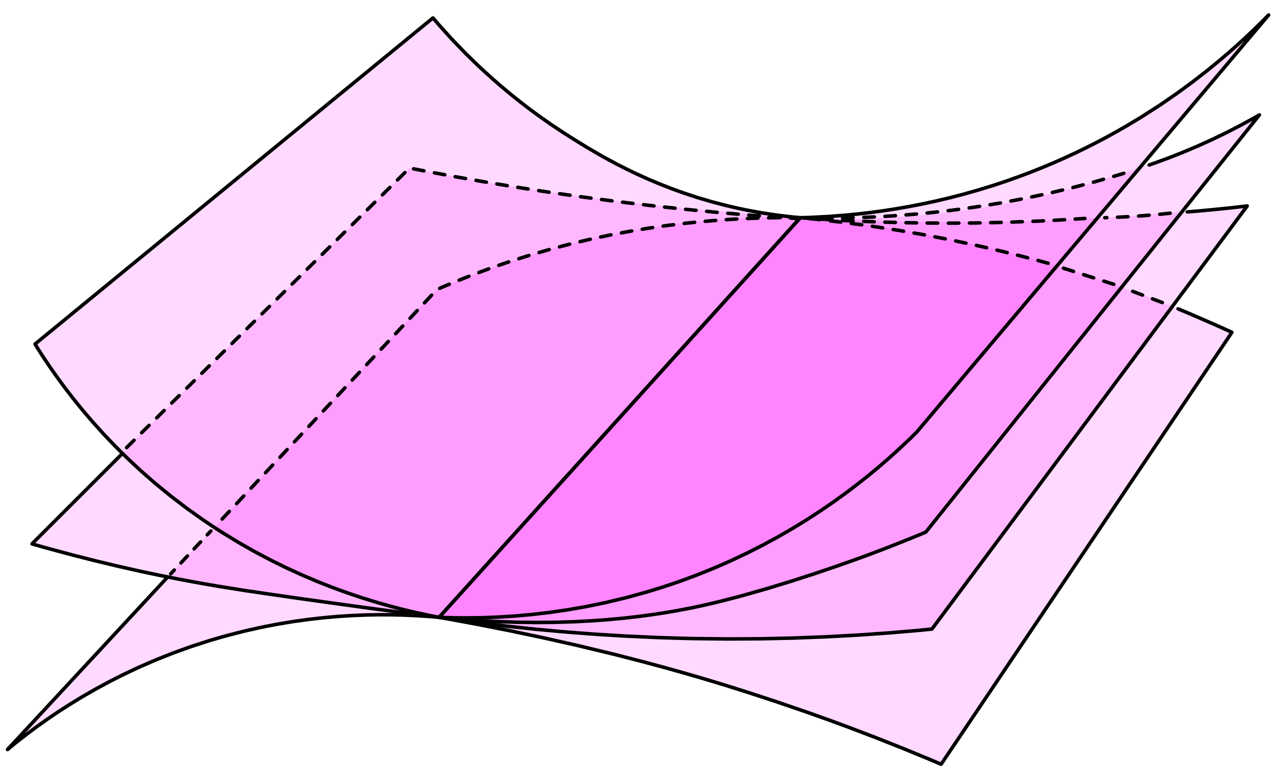
(i) $\operatorname{cone}(\sigma_\tau) = \mathcal{C}_\tau^\vee$, *and the codimension of σ_τ in $\partial B_x(M)$ is equal to the dimension of the largest linear subspace contained in $\mathcal{C}_\tau$.*

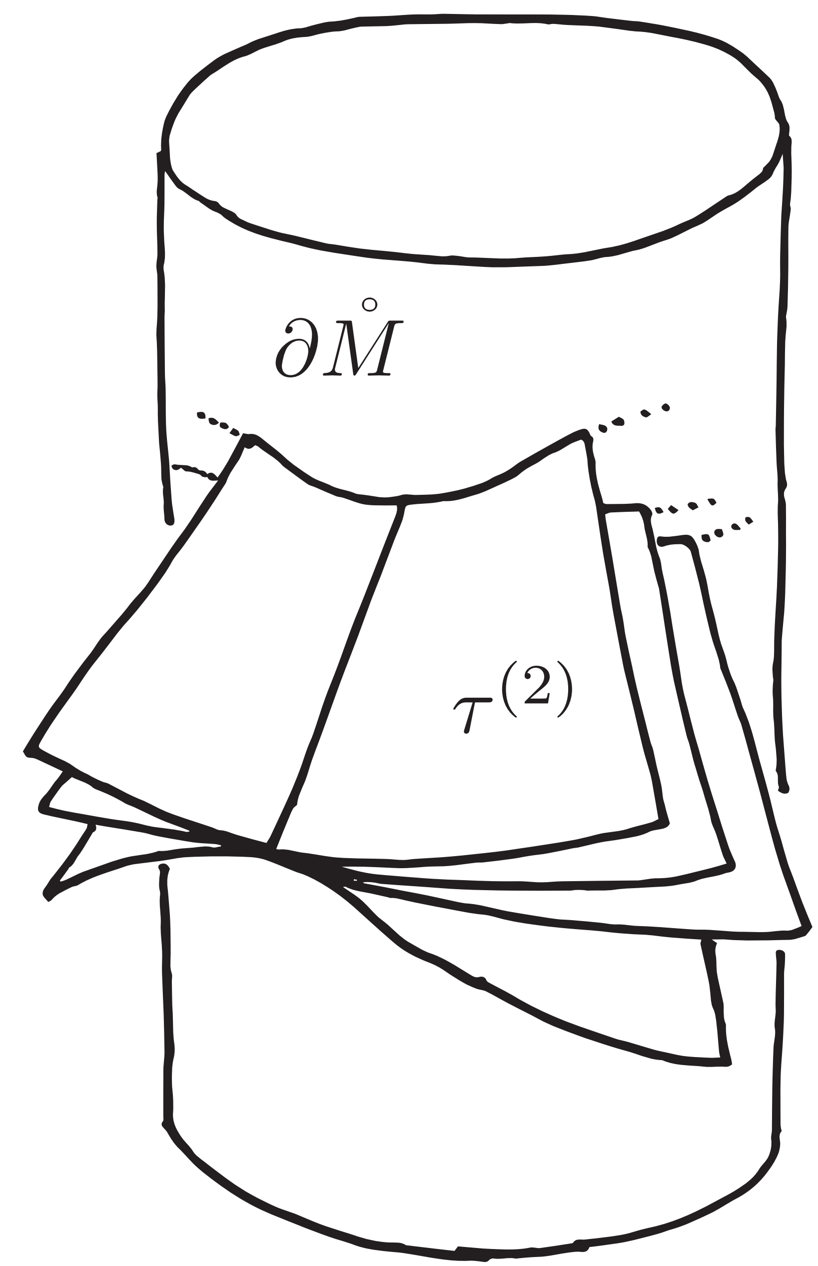
(ii) If $S \subset M$ is a surface, then S is taut and $[S] \in \operatorname{cone}(\sigma_\tau)$ *if and only if S is carried by $\tau^{(2)}$ up to isotopy.*

We will elide the ≥ 3 prongs condition (it's a mild restriction) and briefly explain the other terms.

A **veering triangulation** τ is a cellular decomposition of a torally bounded compact 3-manifold $\mathring{M}$ which satisfies a combinatorial condition called **veering**. (Defined by Agol).

The 2-skeleton $\tau^{(2)}$ is a cooriented branched surface. Its branch locus looks like this:





Let M be a closed Dehn filling of $\mathring{M}$. Then $\tau^{(2)}$ is not quite a branched surface in M (it stops at $\partial\mathring{M}$).

[Ian Agol, *Ideal triangulations of pseudo-Anosov mapping tori,* Top. and Geom. in Dim. 3, 2011]

There is an **Euler class**

$$e_\tau \in H_1(M)$$

naturally associated to τ.

- e_τ is a weighted sum of the cores of the filling tori
- the weights depend on the filling slopes.

Let $W \subset H_2(M)$ be the set on which $x = \langle -e_\tau, \cdot \rangle$. If this is nonempty then it equals cone(F_τ) for some face F_τ of B_x.

Let $\mathscr{C}_\tau \subset H_1(M)$ be the cone generated by closed positive transversals to $\tau^{(2)}$.

Let $\mathscr{C}_\tau^\vee \subset H_2(M)$ be the cone of classes intersecting everything in $\mathscr{C}_\tau$ nonnegatively.

Let $\mathscr{C}_\tau \subset H_1(M)$ be the cone generated by closed positive transversals to $\tau^{(2)}$.

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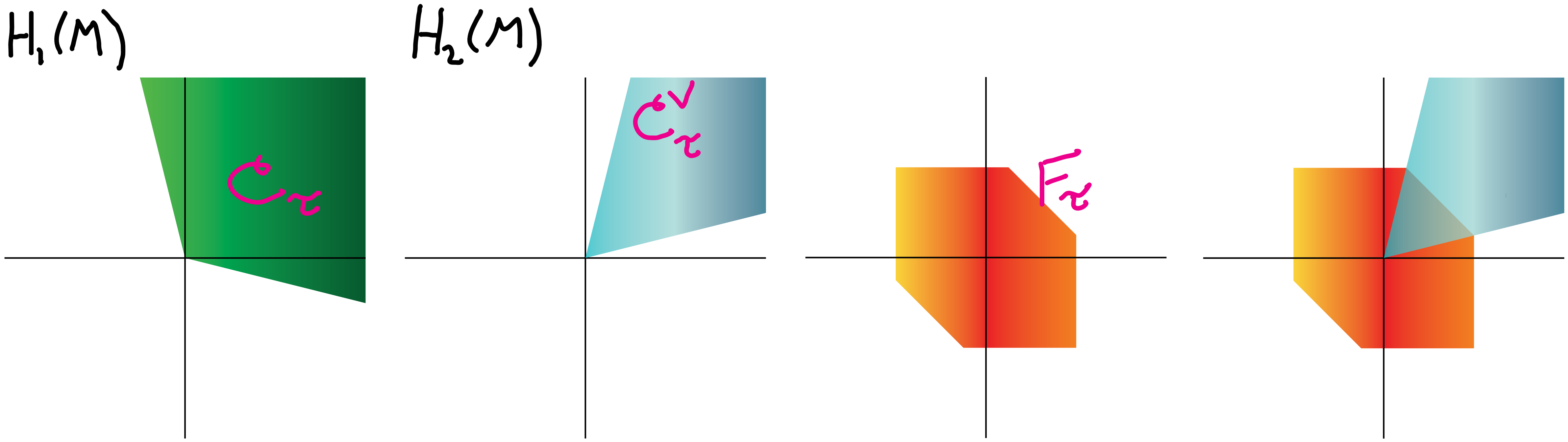
Theorem (L): $\mathscr{C}_\tau^\vee = \text{cone}(F_\tau)$

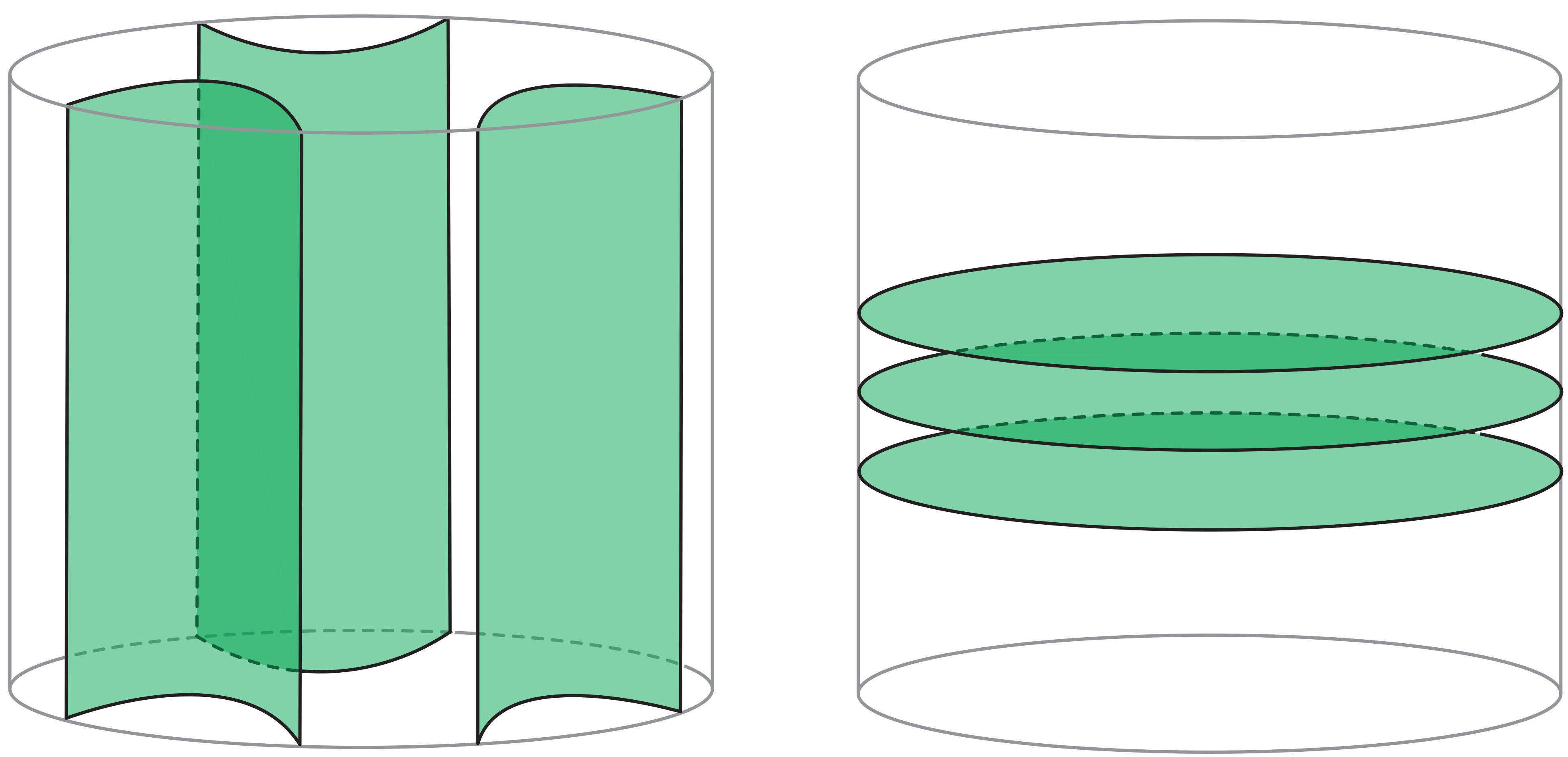


in words: the veering triangulation linear-algebraically determines the cone over a face of the norm ball, and computes x over that face.

Recall $\tau^{(2)}$ is not quite a branched surface in M.

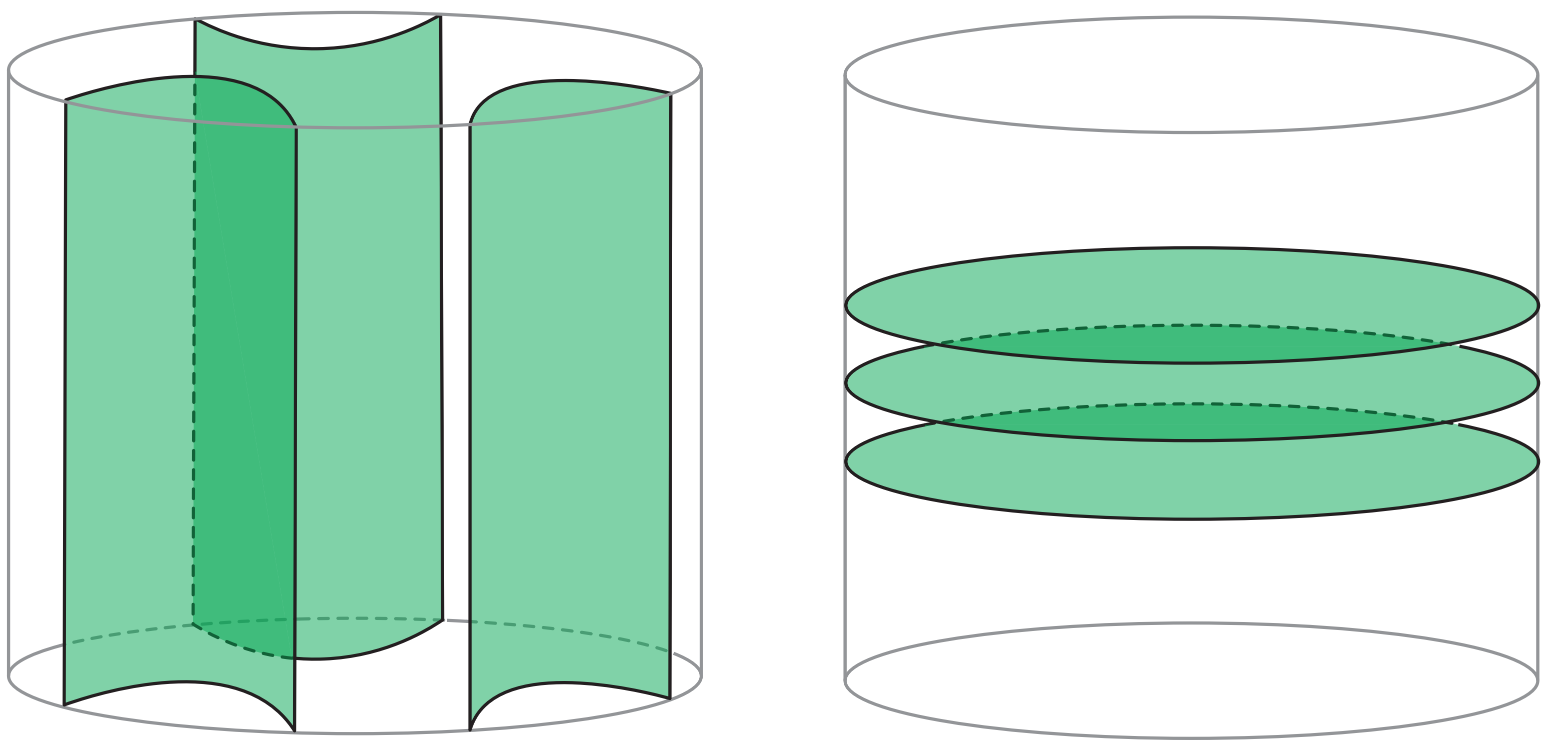
say $S \hookrightarrow M$ is **carried** by $\tau^{(2)}$ if

- $S \cap \mathring{M}$ is carried by $\tau^{(2)}$ in the normal sense
- each component of $S - \mathring{M}$ is π_1-injective annulus or meridional disk in a filling torus



<u>Theorem (L)</u>: Let S be an embedded surface in M. Then:
S is carried by $\tau^{(2)}$ up to isotopy **iff** S is taut and $[S] \in \text{cone}(F_\tau)$.

i.e. $\tau^{(2)}$ sees $\text{cone}(F_\tau)$ at the level of isotopy, not just homology.

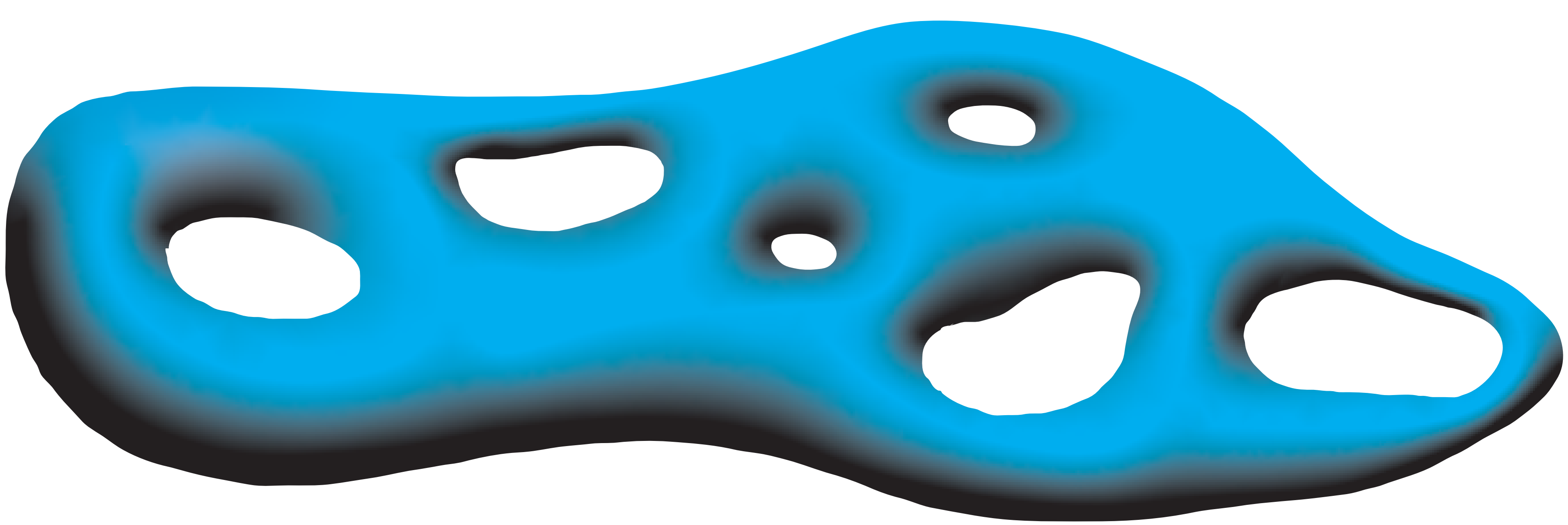


<u>technical point</u>: F_τ could be "the empty face." This happens when $\tau^{(2)}$ doesn't carry anything. Equivalently: $\mathscr{C}_\tau = H_1(M)$.

For which M are hypotheses of the theorem satisfied?

- when M fibers with pseudo-Anosov monodromy (Agol)
- when M admits a pseudo-Anosov flow with no perfect fits (Agol-Guéritaud)
- when M admits a taut $\mathbb{R}$-covered foliation, or more generally a taut foliation with one-sided branching (Calegari, Fenley)

Unresolved: given a taut surface in M, is it carried by the 2-skeleton of a veering triangulation?



Another perspective: given τ, suppose you are interested in $\mathring{M}$ (the unfilled manifold). In this case there is again a natural Euler class $e_\tau \in H_1(\mathring{M})$, giving a face F_τ of $B_x(\mathring{M})$.

Might want to know: is τ layered? non-measured? dimension of F_τ? is F_τ fibered face?

<u>**Theorem (LMT)**</u>: Let τ be a veering triangulation of $\mathring{M}$. Then:

- $\mathscr{C}_\tau^\vee = \text{cone}(F_\tau)$.
- The codimension of cone(F_τ) is the dimension of the largest linear subspace contained in $\mathscr{C}_\tau$.
- Moreover, the following are equivalent:
  1. the union of all closed transversals to $\tau^{(2)}$ lies in an open half-space in $H_1(\mathring{M})$
  2. τ is layered
  3. F_τ is fibered

Veering polynomial

Let $G = H_1(\mathring{M}; \mathbb{Z})/\text{torsion}$

Theorem (LMT). Given τ, there is an element $V_\tau \in \mathbb{Z}[G]$ called the *veering polynomial* that recovers the Teichmüller polynomial Θ_{F_τ} when τ is layered.
More explicitly, V_τ factors as

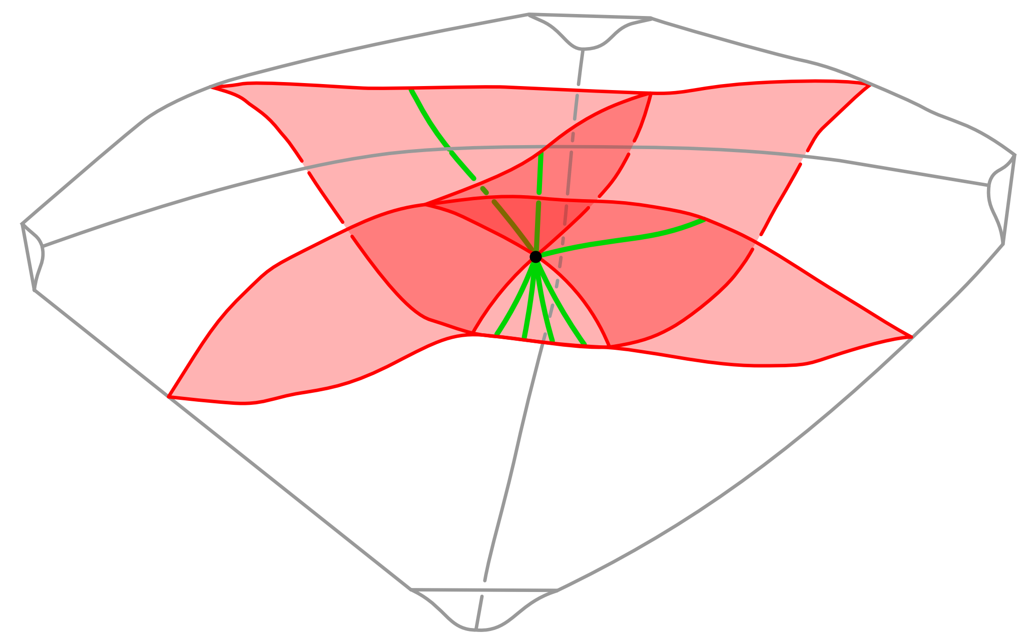
$$V_\tau = V^{AB} \cdot \Theta_\tau,$$

where Θ_τ equals the Teichmüller polynomial up to a unit and V^{AB} has a simple formula.

Remark 1. The veering polynomial behaves well under Dehn filling and recovers the Teichmüller polynomial in general. Combined with the earlier results about filled manifolds, it is a generalization of the Teichmüller polynomial to "veering faces" of the Thurston norm ball.

Remark 2. It can be constructed 2 ways:

A. as the determinant of a presentation matrix for a $\mathbb{Z}[G]$-module, or
B. as the Perron polynomial of a directed graph.



Remark 3. Anna Parlak wrote a computer program that computes these things and is writing a couple of papers about it.

[Anna Parlak, Computation of the taut, the veering and the Teichmüller polynomials, in preparation]
[Anna Parlak, The taut polynomial and the Alexander polynomial]

Thank you