Covers and Curves

Marissa Kawehi Loving

{ joint with Tarik Aougab, Max Lahn, Yang (Sunny) Xiao

Q: Given a covering map $\pi: S' \to S$, how (if at all) can we determine π by looking at simple closed curves on S' and S?

Thm (Aougab - Lahn - L. - X) If $p: X \to S$ and $q: Y \to S$ are regular covers s.t. given any closed curve $\gamma \subset S$, $\exists$ a simple elevation of γ to X iff $\exists$ a simple elevation of γ to Y, then p & q are equivalent covers.

Why did we care?

Thm (Sunada) There exist hyperbolic surfaces which have the same *unmarked length spectrum* but which are not isometric.

Marked vs. unmarked?

Marked: curve + its length → [Fricke, Yes]

unmarked: multi-set of lengths, no curve info → [Sunada, No]

Q: What if we replace *unmarked length spectrum* with unmarked simple length spectrum ?

i.e. are hyperbolic metrics determined by their unmarked simple length spectrum?

We conjecture that the answer is YES!

Thm (Maungchang, 2018) Sunada's construction does not generically produce non-isometric surfaces with the same unmarked simple length spectrum.

Ex: $G = (\mathbb{Z}/8\mathbb{Z})^{\times} \ltimes \mathbb{Z}/8\mathbb{Z}$
$H = \{(1,0),(3,0),(5,0),(7,0)\}$ and
$K = \{(1,0),(3,4),(5,4),(7,0)\}$ are
almost conjugate, but not conjugate.

if for each conjugacy class C in G, $|C \cap H| = |C \cap K|$.

Thm (Maungchang, 2018) Let M_0 be a closed surface of genus 2. $\exists \rho : \pi_1(M_0) \twoheadrightarrow G$ s.t. for almost every $[m] \in T(M_0)$, M_H and M_K are not unmarked simple length iso-spectral.

How is this proved?

Find a curve $\gamma \subset M_0$ s.t. lifts to $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ on M_K

lifts to $\beta_1, \beta_2, \beta_3, \beta_4$ on M_H

$\boxed{\text{Show that } \ell_{M_K}(\alpha_i) = \ell_{M_H}(\beta_i).}$

But α_i are simple and β_i are not...

Remember our question? Are hyperbolic metrics determined by their unmarked simple length spectrum?

A first step: Generalize Maungchang's construction (i.e. establish our conjecture for pairs of surfaces arising from Sunada's construction).

Want to show: If two covers of a surface aren't equivalent then there is a curve on the base surface that admits a different number of simple elevations to the two covers.

This turns out to be pretty tough...

- Assuming regularity helps a lot!

Thm (Aougab - Lahn - L. - X) If $p: X \to S$ and $q: Y \to S$ are regular covers s.t. given any closed curve $\gamma \subset S$, $\exists$ a simple elevation of γ to X iff $\exists$ a simple elevation of γ to Y, then p & q are equivalent covers.

- But regularity also doesn't help at all...

$\downarrow$

Covers arising from Sunada's construction are never regular!

Currently working with Tarik Aougab, Max Lahn, and Nick Miller to remove assumption of regularity.