Nielsen-Thurston Classification, Revisited

w/ Camille Horbez

[Horbez-T] Give a new proof of the Nielsen-Thurston classification of mapping classes & new representatives for pseudo-Anosovs.

Tool: Bers' approach using Thurston metric on Teichmüller Space

Geodesic Laminations

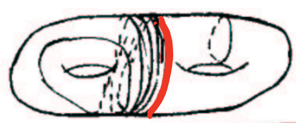
X hyperbolic surface of genus $g \geq 2$

<u>Geodesic lamination on X</u>:

A closed subset λ of X foliated by complete simple geodesics

Example: Multicurve

multicurve + spiraling leaves

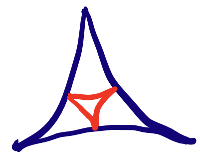


- uncountably many leaves
- cross sections are cantor sets

Each component of $X - \lambda$ is a hyperbolic surface w/ geodesic boundary

Defn: λ is filling if $X - \lambda$ is a union of ideal polygons.

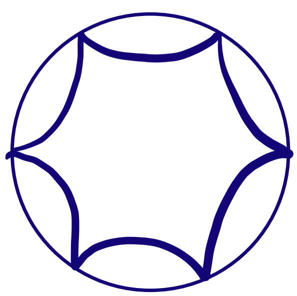
λ is **maximal** if $X - \lambda$ is a union of ideal triangles



Defn: Suppose λ is filling on X.

Say X is **λ-symmetric** if each ideal polygon P of $X - \lambda$ is <u>regular</u>.

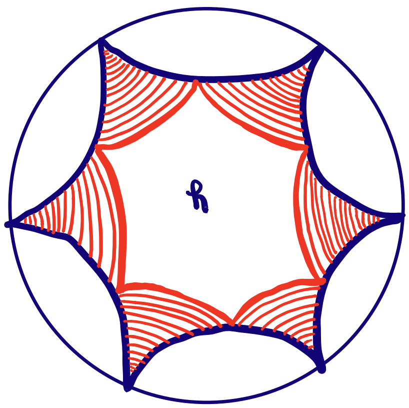
isometric



P of $X \setminus \lambda$

Note: When λ is maximal X is always λ-symmetric.

Dual Horocyclic Foliation



P regular polygon

P circumscribes an equilateral <u>horogon</u> h

F_P - Dual Horocyclic Foliation

- support on $P - \mathrm{int}(h)$
- leaves are horocyclic segment
- Measure coincides w/ arclength along sides of P.

λ filling lamination on X

X λ-symmetric : each P of $X - \lambda$ is regular

$F_X(\lambda)$ dual horocyclic foliation

- Support of $F_X(\lambda)$ is $X - \bigcup \mathrm{int}(h)$

  h inscribed horogon in P

• Restriction of $F_X(\lambda)$ to each P of $X-\lambda$ is $F_P(h)$.

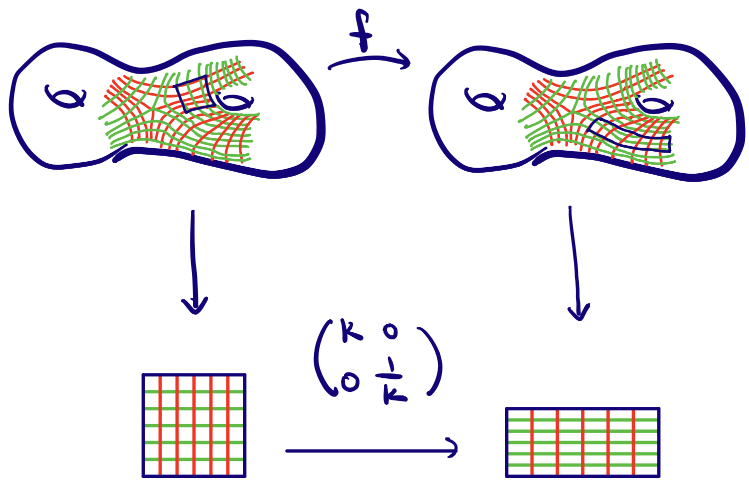
<u>Mapping Class Group</u>: $\Gamma(S) = \mathrm{Homeo}^+(S) / \text{isotopy}$

Nielsen – Thurston Classification:

Thm: If $\phi \in \Gamma(S)$ is not reducible or finite order, then ϕ has a representative $f \in \mathrm{Homeo}^+(S)$ which is $\boxed{\text{pseudo-Anosov}}$:

$\Updownarrow$

$\exists$ a pair of transverse (singular) measured foliations F_+ & F_- on S & $K>1$ s.t. $f(F_\pm) = K^{\pm 1} F_\pm$.



[Bers]: If $\phi \in \Gamma(S)$ is not reducible or finite-order, then $\exists$ a complex structure X on S, a quadratic differential q on X, $K > 1$, and a rep $f \in \phi$ s.t.:

1) $f: X \to X$ has quasi-conformal constant K^2, which is minimal among all maps $X \circlearrowleft$ rep ϕ.

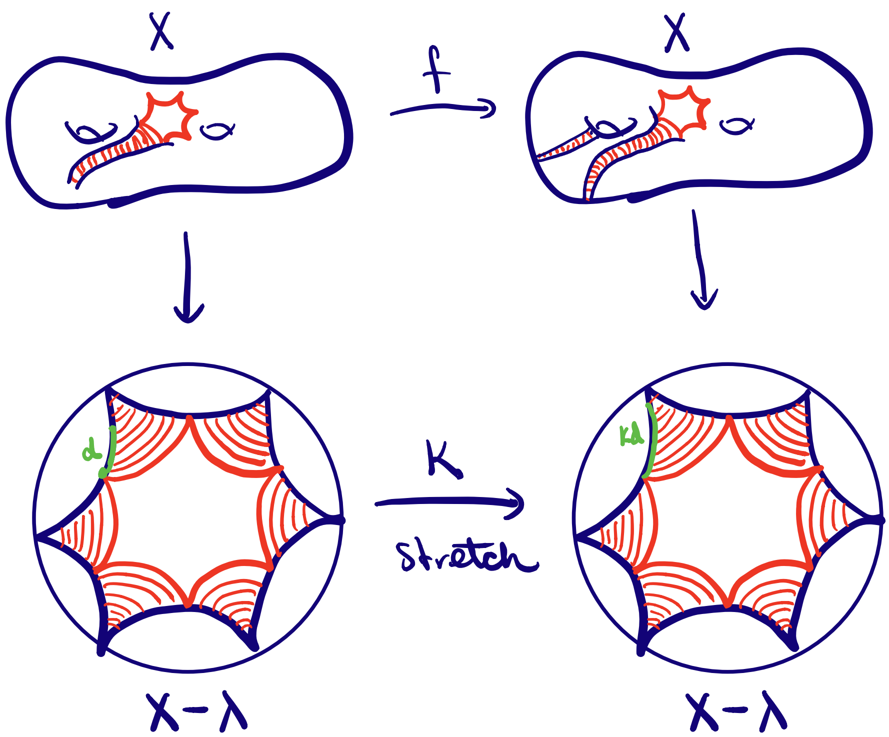
2) f preserves the leaves of the vertical & horizontal foliations V_q & H_q of q, acting by $f(V_q) = K V_q$, $f(H_q) = \frac{1}{K} H_q$

New Proof + New representative.

Thm [H-T]: If $\phi \in \Gamma(S)$ is not reducible or finite-order, then $\exists$ a hyp structure X on S, a filling λ for which X is λ-symmetric, $K > 1$, and a rep $f \simeq \phi$ s.t :

1) $f : X \to X$ has Lipschitz constant K, which is minimal among all maps $X \circlearrowleft$ rep ϕ.

2) f preserves the leaves of λ & $F = F_X(\lambda)$ acting by $f(F) = KF$.



X
f
X
α
K
Kα
stretch
X−λ
X−λ

Tools of the two Proofs

$T(S) = \{$ isotopy classes of complex / hyperbolic structure on $S \}$

$X, Y \in T(S)$

$d(X,Y) = \log \inf \{ K_f : f : X \to Y$ isotopic to $id_S \}$

- <u>Teichmüller metric</u> d_{Teich} : $\sqrt{K_f}$ QC constant
- <u>Thurston metric</u> d_{Th} : L_f Lipschitz constant

- Both are geodesic metric spaces
- $\Gamma(S) \curvearrowright T(S)$ by isometries.

<u>Lemma:</u> If $\phi \in \Gamma(S)$ is not reducible or finite-order, then in either metric on $T(S)$, ϕ is a <u>hyperbolic isometry</u>

$\Updownarrow$

1) $\tau_\phi = \inf_{Y \in T(S)} d(Y, \phi(Y)) > 0$

2) $\mathrm{MinSet}(\phi) = \{ X \in T(S) \mid \tau_\phi = d(X, \phi(X)) \} \neq \emptyset$

Lemma $\Rightarrow$ $K = e^{\tau_\phi} > 1$ & $X \in \mathrm{MinSet}(\phi)$

Rest of Proof Diverge.

[Bers] Show the quadractic differential q associated to $[X, \phi(X)]$ is ϕ-invariant.

Apply Teichmüller's Extremal Map Thm.

[H:T] 1) Find a geodesic lamination λ on X which is ϕ-invariant

Irreducible $\Rightarrow$ λ is filling

[λ = tension lam from X to $\phi(X)$]

2) Show X is λ-circumscribing

$\Updownarrow$

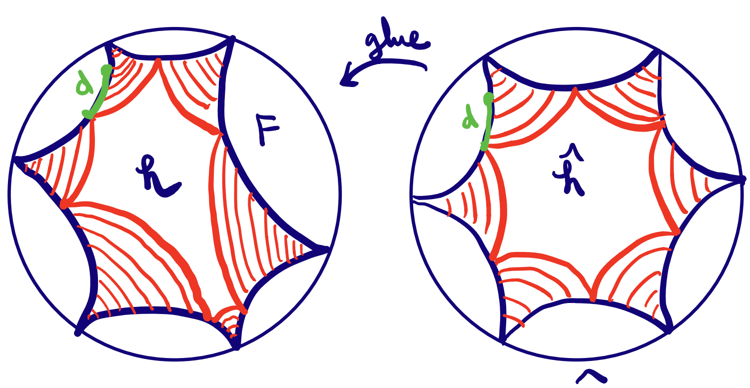
(Each P of $X - \lambda$ circumscribes a horogon.)

$\Leftrightarrow$ $\exists$ a choice of h in each P s.t.

dual horocyclic foliation F has $\phi(F) = kF$.

[Uses hyp version of Grötzsch's Thm]

3) Promote $X \rightsquigarrow \hat{X}$ λ-symmetric



P of $X \setminus \lambda$ $\qquad$ $\hat{P}$

$$F = F_{\hat{X}}(\lambda)$$

4) Build optimal Lipschitz map $\hat{X} \longrightarrow \phi(\hat{X})$ generalizing Thurston's stretch maps $\blacksquare$

Classification of Isometries of $T(S) \curvearrowright T(S)$.

[Bers] Teichmuller Metric	Thurston Metric [Horbez-T]
ϕ elliptic $\Leftrightarrow$ ϕ is finite order	" "
ϕ parabolic $\Leftrightarrow$ ϕ is reducible & ∞-order	ϕ parabolic $\Leftrightarrow$ some power of ϕ is a multi twist.
ϕ hyperbolic $\Leftrightarrow$ ϕ is pseudo-Anosov	ϕ is hyperbolic $\Leftrightarrow$ ϕ has a pseudo Anosov component.


THANK
YOU
GRACIAS
谢谢
ARIGATO
SHUKURIA
DANKSCHEEN
BIYAN
SHUKRIA
TINGKI
YAQHANYELAY
TASHAKKUR ATU
SUKSAMA
EKHMET
GOZAIMASHITA
EFCHARISTO
JUSPAXAR
KOMAPSUMNIDA
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MERCI