Generation of unbounded derived categories of modules over groups in Kropholler's hierarchy

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Definitions and Preliminaries

We will be dealing with triangulated categories mostly in the first few minutes of the presentation. Here are some important definitions for a triangulated category T with arbitrary coproducts and products.

- We say a full triangulated subcategory of T is localising (resp. colocalising) if it is closed under coproducts (resp. products).
- For any class of objects $U \subseteq T$, we denote the smallest full triangulated subcategory of T containing U closed under coproducts (resp. products) by $Loc_T(U)$ (resp. $Coloc_T(U)$.
- We say U “generates" (resp. “cogenerates") T iff $Loc_T(U) = T$ (resp. $Coloc_T(U) = T$).
- If the smallest full triangulated subcategory of T containing U is T itself, then we say U "strongly generates" T.

What is Kropholler's hierarchy?

This hierarchy is named after Peter Kropholler who introduced it in the nineties. To start the definition, we need to first fix a class of groups X.

- Define $H_0X := X$.
- For any successor ordinal α, A group $G \in H_\alpha X$ iff there is a finite dimensional contractible CW-complex on which G acts by permuting its cells with all cell stabilisers in $H_{\alpha-1}X$.
- If α is a limit ordinal, then $H_\alpha X := \bigcup_{\beta<\alpha} H_\beta X$.
- HX is the union over $H_\alpha X$ for all ordinals α.

How big is this hierarchy?

Here, take $X = F$, where F is the class of all finite groups.

- It is easy to show that for any two ordinals $\alpha < \beta$, $H_\alpha F \subseteq H_\beta X$.
- For a long time, it was not known if there were groups at each level of this hierarchy which were not in the level immediately below.
- In 2010, Januszkiewicz, Kropholler and Leary showed that for all integers n, $H_{n+1}F$ is strictly bigger than H_nF.
- For eg, the free abelian group of rank t with $\aleph_0 \leq t < \aleph_\omega$, where ω is the first infinite ordinal, is in H_2F but not in H_1F.

More concrete examples of groups in the hierarchy

- All groups of finite virtual cohomological dimension, all groups of finite Bredon cohomological dimension are in $H_1 F$.
- All known examples of groups admitting complete resolutions are in $H_1 F$.
- Any finitely generated soluble group of derived length n is in $H_n F$.

Why do we want to study these groups?

The short answer is that these groups admit some interesting generation properties in their module categories, and those results in turn help us derive useful generation properties for derived unbounded, bounded, bounded above, and bounded below categories and even stable categories where stable categories can be defined.
Before introducing some crucial generation concepts that will let us proceed further, we take a small detour to discuss some interesting questions on derived categories.

Two relevant results of Amnon Neeman

- Neeman '92 - if A is Noetherian, then there is a bijection betwene the localising subcategories of $D(Mod\text{-}A)$ and subsets of $Spec(A)$.
- Neeman '11 - if A is Noetherian, then there is a bijection between the localising subcategories of $D(Mod\text{-}A)$ and the colocalising subcategories of the same.

Both of Neeman's results that we quote above have an interesting history with the Noetherian-ness assumption in their statements.

- Before stating our new results, I'd like to draw people's attention to the fact that it is not that difficult to make a group algebra RG non-Noetherian where G is an unruly infinite group from Kropholler's hierarchy and R can be any commutative ring.

Neeman-like results in stable categories

- Benson, Iyengar and Krause have achieved similar results in classifying all localising and colocalising subcategories of the stable module category of finite groups over fields.
- (*)Stable module categories of finite groups are known to be compactly generated, as in they are generated by their compact objects. An object C in a triangulated category T with arbitrary coproducts is called compact if the functor $Hom_T(C, ?)$ commutes with coproducts.

Our results

Take a group $G \in H_n X$ and fix a commutative ring R, and for any i, denote by $\Lambda_i(G, X)$ be the class of all RG-modules induced up from $H_i X$-subgroups of G. We put the superscript $^\oplus$ on a class to denote we are closing the class under arbitrary direct sums.
Two of our main results are the following:

- $\Lambda_{n-1}(G, X)^\oplus$ both generates and cogenerates $D(Mod\text{-}RG)$.
- If we work in the derived bounded category, $D^b(Mod\text{-}RG)$, then $\Lambda_{n-1}(G, X)^\oplus$ strongly generates the whole $D^b(Mod\text{-}RG)$.

Our results (contd)

We therefore get a coincidence of generated localising and colocalising subcategories of the derived unbounded category. Now, an intriguing question to look into will be whether one really gets any "new" colocalising subcategories other than the ones obtained "from" localising subcategories there. And again it is important to remember that in our cases, the rings are seldom Noetherian.

Some new open questions

- Rouquier defines a dimension for generation in triangulated categories which is basically the number of steps taken to generate the whole category from a single object. It will be interesting to investigate whether our derived unbounded categories have finite Rouquier dimension although it seems unlikely.
- Rouquier also defines a notion of Krull dimension for a triangulated category T which he defines as the maximal n such that there is a chain of thick irreducible subcategories $0 \neq I_0 \subset I_1 \subset \ldots \subset I_n = T$.
  For $\alpha = \omega.n$, and $G \in H_\alpha F \setminus H_{<\alpha} F$ (such groups exist), we get a filtration of $D(Mod\text{-}RG)$ that takes us very close to getting a lower bound on the Krull dimension of $D(Mod\text{-}RG)$.

A generation result in stable categories

First, we provide a little background on stable categories for finite groups.

- Stable module categories, for finite groups where it is usually studied over fields, is comprised of the standard module objects as the objects and for the arrows, we quotient out the standard set of arrows with those arrows that factor through a projective.
- Since finite groups admit complete resolutions, one gets to use the inverse syzygy functor as the triangulation functor in the stable category.
- Not all infinite groups admit complete resolutions, so stable module categories cannot be defined in the same way for all infinite groups.

A generation result in stable categories

First, we provide a little background on stable categories for finite groups (continued).

- However, Mazza and Symonds have recently done a construction of a stable module category of infinite groups which do admit complete resolutions, and their construction coincides with the usual stable category when one takes the group to be finite.

Well-generated stable categories

There's a conjecture that's around a decade old that a group admits complete resolutions if and only if it is in H_1F. So, now we can use our "strong generation" result for the derived bounded category from earlier and prove the following.

Theorem

Take G to be a group in Kropholler's hierarchy HF that admits complete resolutions. Take R to be a commutative ring of finite global dimension. Then, the Mazza-Symonds stable module category, denoted $Stab(RG)$, is "well-generated", as a triangulated category admitting arbitrary coproducts, in the sense of Neeman.

I am not going to define "well-generation"here but I'll just leave with the comment that it is a generalization of compact generation that we have for stable module categories of finite groups.