Knots in the curve graph

of $\Sigma_{0,5} \hookrightarrow \mathcal{PML}(\Sigma_{0,5}) \cong \mathbb{S}^3$

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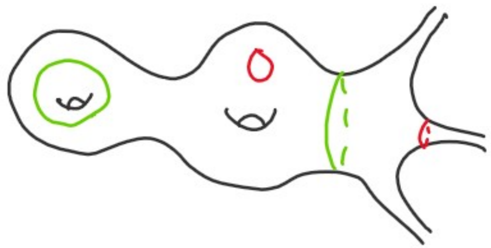
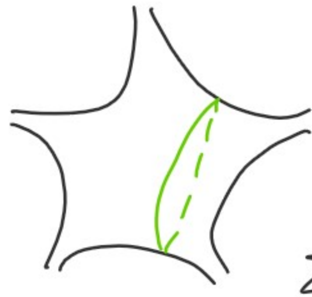
1. The curve graph [Harvey]

$\Sigma = \Sigma_{g,n}$ surface of genus g with n punctures
($g \geq 1$ or $n \geq 4$)



$\Sigma_{2,3}$



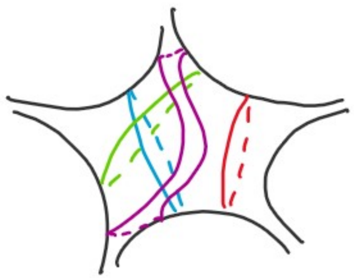
$\Sigma_{0,5}$

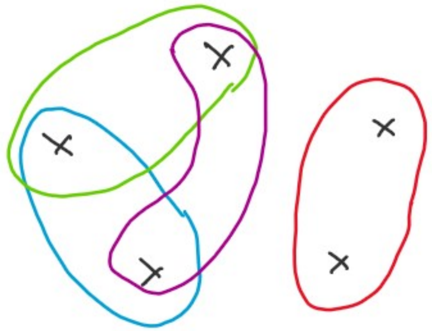
curve on Σ = isotopy class of simple closed curves
that do not bound a disc or a
once punctured disc

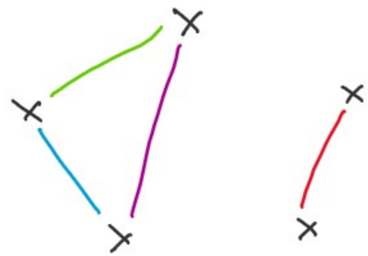
Def. The curve graph $\mathcal{C}(\Sigma)$ is given by

* vertices $\Leftrightarrow$ curves on Σ
* edge $\{\alpha, \beta\}$ $\Leftrightarrow$ α and β have disjoint representatives

Example. 4 curves on $\Sigma_{0,5}$:







$\Sigma_{0,5}$

corresponding subgraph of $\mathcal{C}(\Sigma_{0,5})$:

2. $\mathcal{PML}(\Sigma)$ [Thurston]

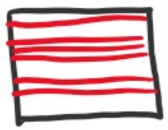
Fix a hyperbolic structure on Σ.

$\mathcal{PML}(\Sigma)$ = the space of projective measured laminations on Σ

"limits of curves"

lamination λ on Σ = compact subset of Σ that is foliated by geodesics

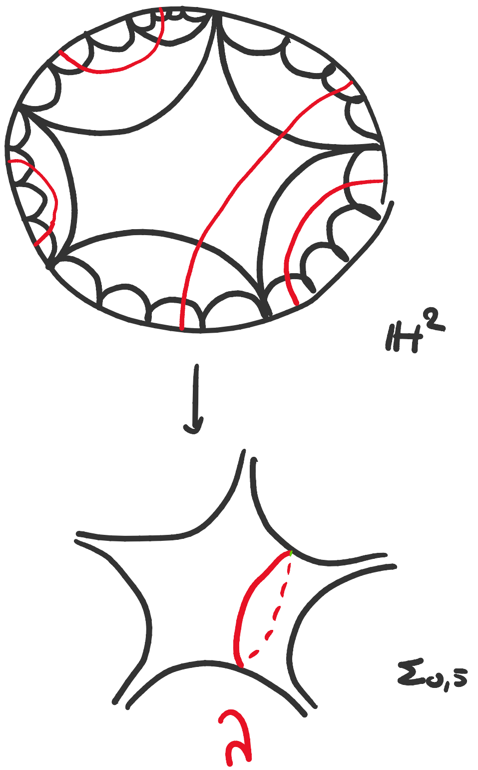
λ



locally, $K \times I$

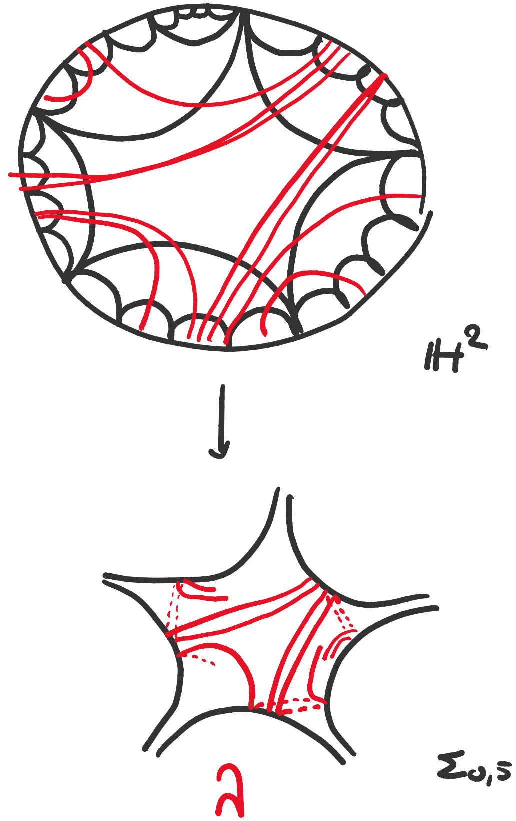
for some $K \subset I$ compact

Basic example:

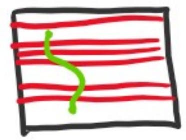


A simple closed geodesic is a lamination.

Generic picture:



A measure m on λ assigns weights to transverse arcs



$\mathcal{PML}(\Sigma)$ = the space of projective measured laminations on Σ

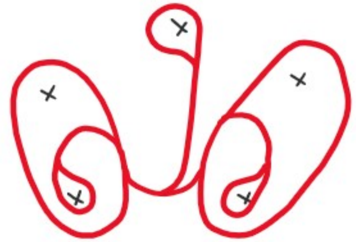
$(\lambda, m) \sim (\lambda, c \cdot m)$ for all $c \in \mathbb{R}_{>0}$

[Thurston] $\mathcal{PML}(\Sigma_{g,n}) \cong \mathbb{S}^{6g+2n-7}$ $\Rightarrow \mathcal{PML}(\Sigma_{0,5}) \cong \mathbb{S}^3$

Tool for studying $\mathcal{PML}(\Sigma)$: Train tracks.

"charts/local coordinates"

train track τ on Σ = "smoothly embedded" graph on Σ
with edges called **branches**



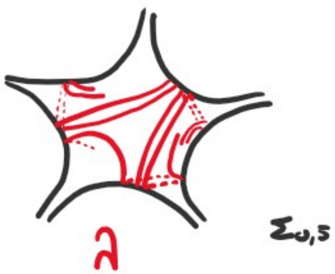
A **measure** on τ associates weights
to all branches such that

$c = a + b$

$P(\tau) \xrightarrow{\text{embedding}} \mathcal{PML}(\Sigma)$

polytope of projective
measures on τ

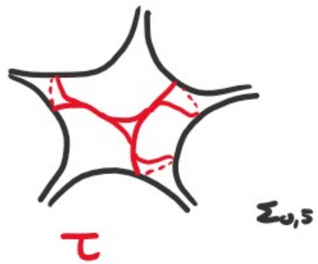
Note: Integer weights define multicurves



measured
laminations

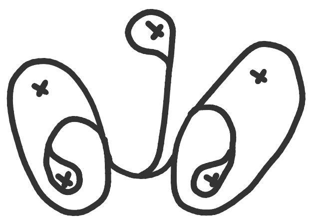
$\Longleftrightarrow$

measured
train tracks

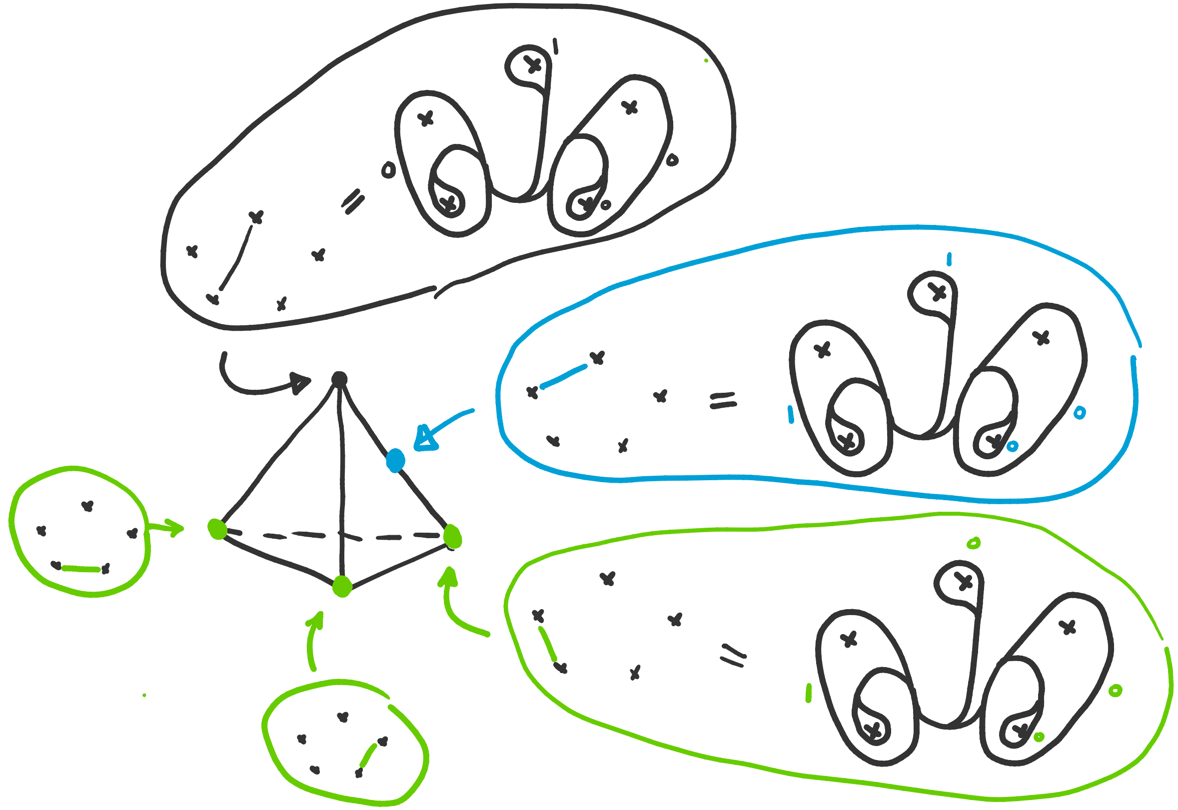


Example.

τ :



$P(\tau)$:



Define an injective map $\eta : \mathcal{C}(\Sigma) \hookrightarrow \mathcal{PML}(\Sigma)$ by

* If α is a curve on Σ, $\eta(\alpha)$ is the geodesic representative of α (with counting measure)
* Extend linearly to edges of $\mathcal{C}(\Sigma)$:

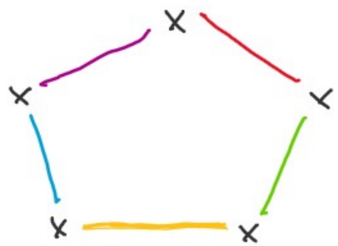
$$p \longmapsto t\,\eta(\alpha) + (1-t)\,\eta(\beta)$$

Motivation: Want to understand the topology of $\eta(\mathcal{C}(\Sigma)) \subset \mathcal{PML}(\Sigma)$ and its complement. [Gabai]

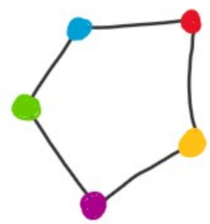
Explicit goal: For particular finite subgraphs of $\mathcal{C}(\Sigma_{0,5})$, find out whether their image in $\mathcal{PML}(\Sigma_{0,5}) \cong S^3$ is knotted.

3. Finding knots in $\eta(\mathcal{C}(\Sigma_{0,5})) \subset \mathcal{PML}(\Sigma_{0,5})$

The pentagon P in $\mathcal{C}(\Sigma_{0,5})$



$\Sigma_{0,5}$



$\subset \mathcal{C}(\Sigma_{0,5})$

Remark: In the hyperbolic structure on $\Sigma_{0,5}$ given by gluing two regular ideal hyperbolic pentagons using the identity map, the curves in P are the 5 curves of minimal length.

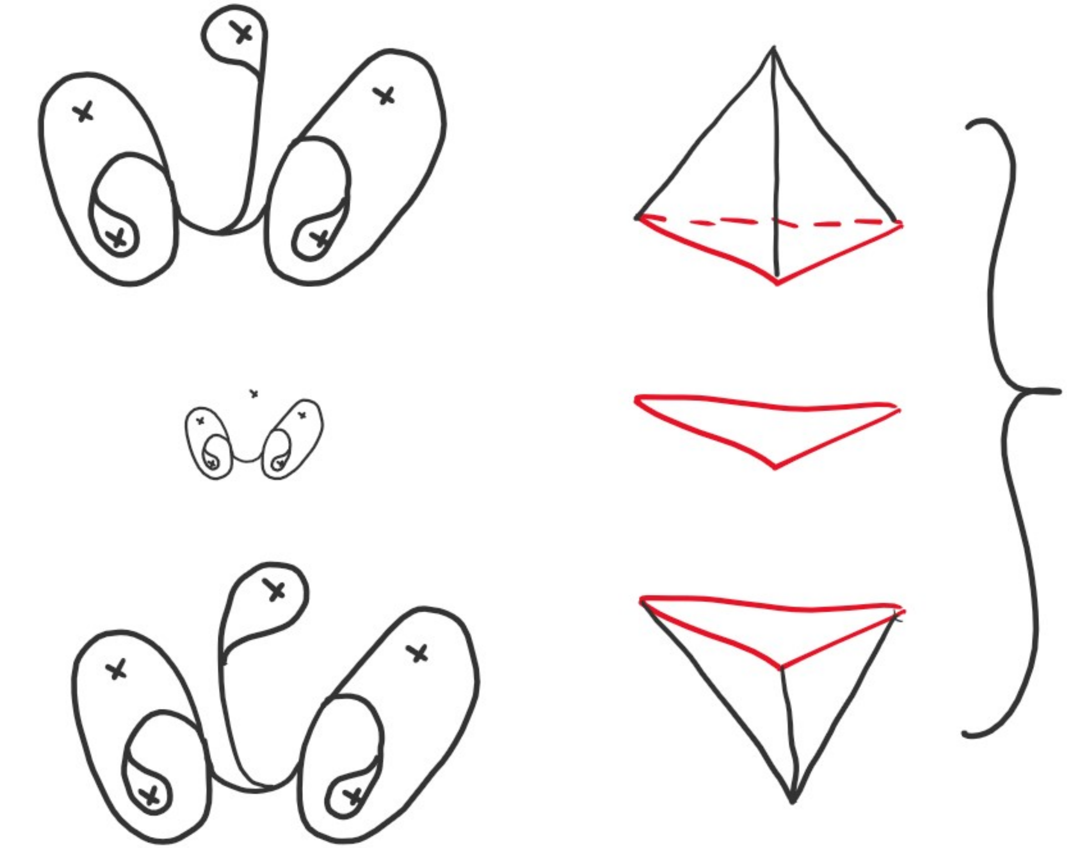
Lem. $\eta(P) \subset \mathcal{PML}(\Sigma_{0,5})$ is an unknot.

Pf. Use charts given by the following train tracks:

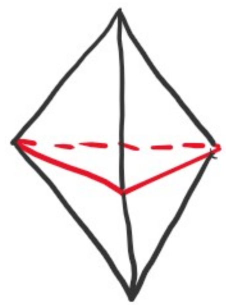
τ

ρ

σ



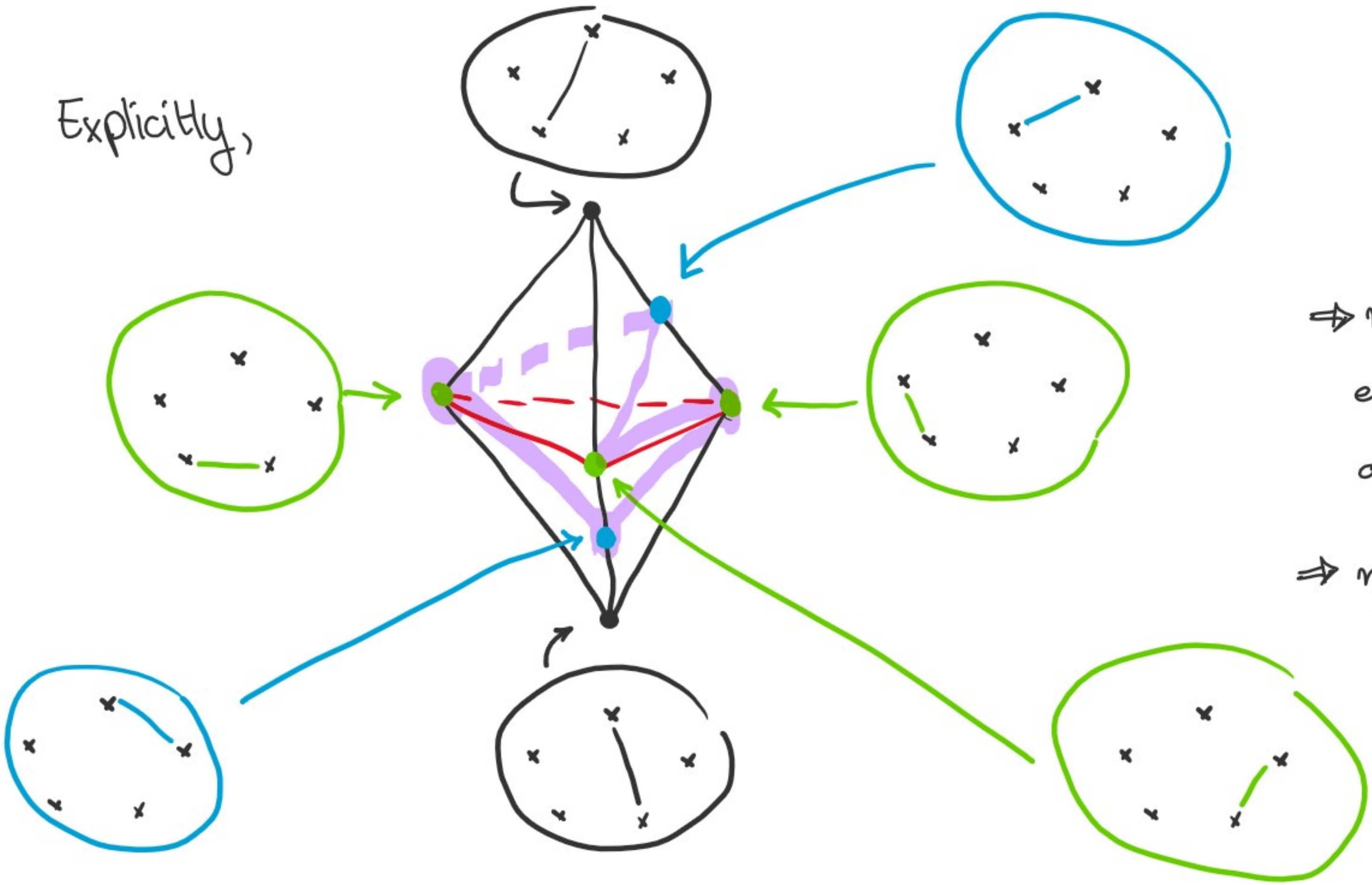
Claim. Any lamination carried by both τ and σ is carried by ρ.



$\downarrow$ embedding

$\mathcal{PML}(\Sigma_{0,5})$

Explicitly,

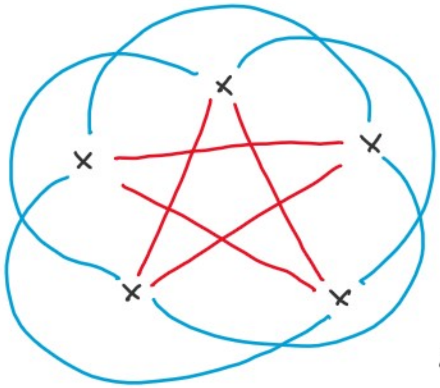


$\Rightarrow \eta(P)$ is a PL embedded circle on a PL 2-sphere

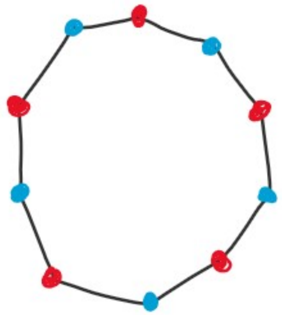
$\Rightarrow \eta(P)$ is an unknot.

$\square$

The decagon D in $\mathcal{C}(\Sigma_{0,5})$



$\Sigma_{0,5}$



$\subset \mathcal{C}(\Sigma_{0,5})$

Thm [TW]

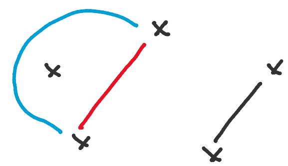
* $\eta(D)$ is an unknot.
* $\eta(D)$ and $\eta(P)$ form a Hopf link.

Idea of proof. Similar to proof of lemma, but need more train track "charts".

Moreover:

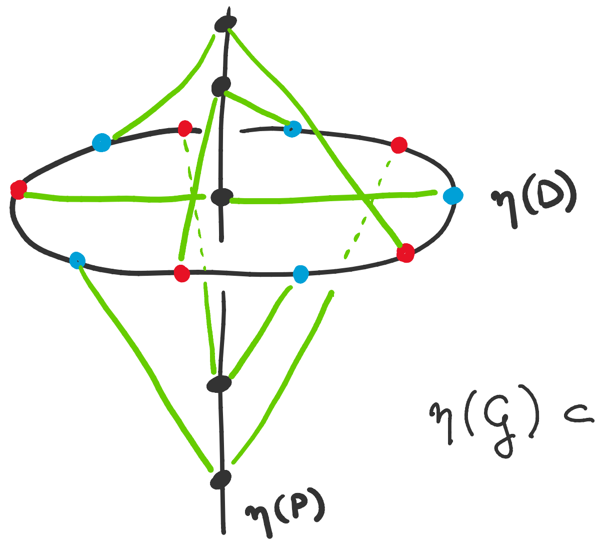
$\mathcal{G}$ = the induced subgraph of $\mathcal{C}(\Sigma_{0,5})$ on P and D.

$= P \cup D \cup \{10 \text{ extra edges}\}$



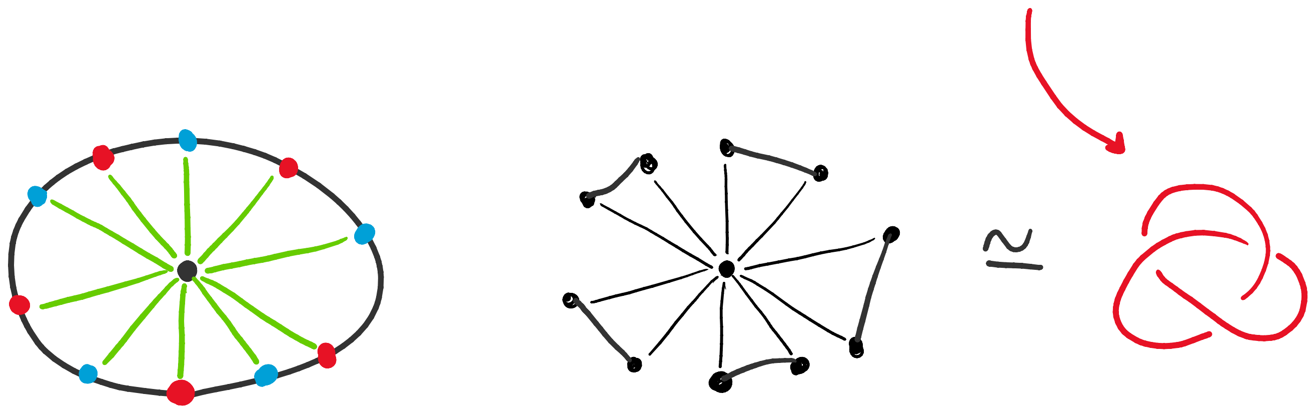
(every pentagon-curve is disjoint from 2 "opposite" decagon-curves)

Then



$\eta(\mathcal{G}) \subset \mathcal{PML}(\Sigma_{0,5})$

Consequence: Find an explicit trefoil knot in $\eta(\mathcal{G})$.



"petal projection"
of the trefoil knot