VEERING TRIANGULATIONS AND THEIR POLYNOMIALS

WARWICK Oct 2020

YAIR MINSKY

JOINT WORK W.

MICHAEL LANDRY & SAM TAYLOR

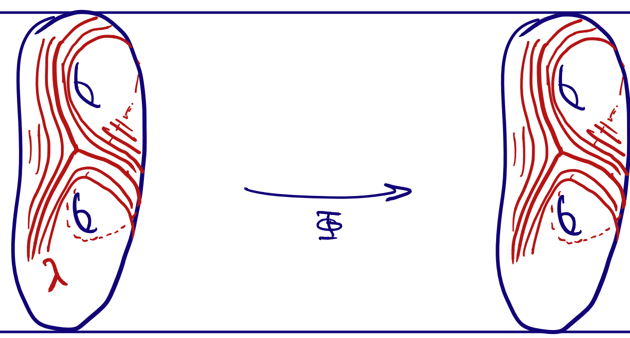
GOALS:

- Recall McMullen's polynomial Θ_F
  for fibred 3-mflds
- Recall Agol's veering triangulations
- Define the Veering polynomial V_τ
  & the Taut polynomial Θ_τ
- Θ_τ reduces to Θ_F in the fibred case
- $V_\tau = \Theta_\tau \cdot \prod (1 \pm g_i)$

McMullen's Teichmüller polynomial (1999)

M cpct, $h : M \to S^1$ fibration.
fibre $S = h^{-1}(p)$

monodromy $\Psi : S \to S$.
assume pseudo-Anosov.



$M = S \times \mathbb{R} / \Phi$

$\Phi : (x,t) \longmapsto (\Psi(x), t+1)$

λ = expanding foliation (or lamination) of Ψ.

$\mathcal{L}$ = suspension of λ. $= \lambda \times \mathbb{R} / \Phi$

Thurston & Fried: $\mathcal{L}$ is the same for all fibrations in the same face as h.

Faces of Thurston's norm:

$$x : H_2(M, \partial M) \longrightarrow \mathbb{R}$$

$$x(c) = \min \left\{ |\chi(T)| : [T] = c, \ T \text{ has no sphere components} \right\}$$

for $c \in H_2(M, \partial M; \mathbb{Z})$.

Thm (Thurston)

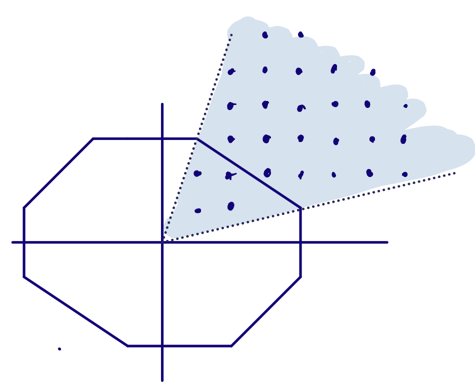
x on integer points extends to a norm.

The unit ball B is a polyhedron.

Fibres $[S]$ are in cones $\mathbb{R}_+ F$ over open top-dim faces of B.

and if one integer pt in $\mathbb{R}_+ F$ is a fibre, they all are.

(F is called a fibered face)



Thm (Fried) for all fibres in F, the suspension flow & foliation are the same up to isotopy.

The Teichmüller polynomial of a fibred face F:

$\Theta_F \in \mathbb{Z}[G]$ where $G = H_1(M,\mathbb{Z}) / \text{torsion}$

(the construction only uses $\mathcal{L}$).

Why is this a polynomial?

Write G multiplicatively: $x_1, \dots x_b$ generators.

$g \in G$ is a monomial $x_1^{n_1} \cdots x_b^{n_b}$

elt of $\mathbb{Z}[G]$ is $\sum_{i=1}^{m} a_i g_i$ $\quad a_i \in \mathbb{Z}$, g_i monomials.

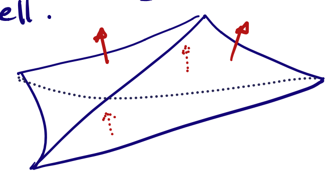
Veering triangulations.

M (oriented) has nonempty toroidal boundary.

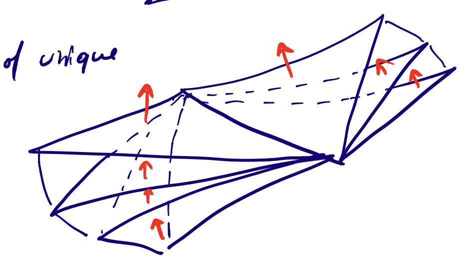
τ ideal triangulation, with:

- co-orientation of faces such that:
  2 incoming, 2 outgoing on each 3-cell.

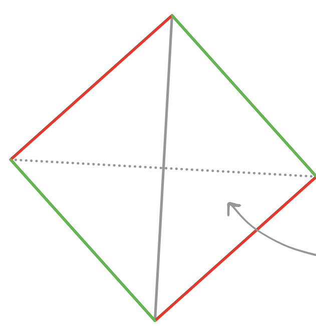
"transverse taut" structure (Hodgson-Rubinstein-Segerman-Tillman)



each edge is at bottom of unique 3-cell & at top of unique 3-cell.



- 2-coloring of edges ("left-veering", "right veering")
  s.t. each 3-cell when placed in std pos'n, looks like:



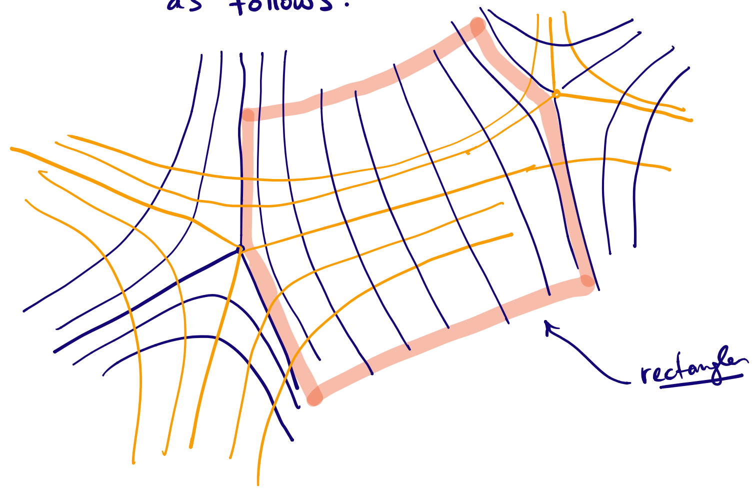
(coorientation toward us)

diagonal arcs allowed to be either color.

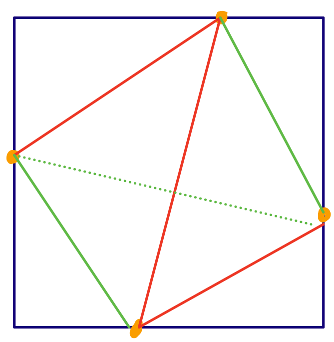
Fibred setting: Agol showed $\mathring{M} = M -$ (singular orbits of flow)
has a veering triangulation τ
which only depends on the fibred face F.

Gueritaud: τ can be obtained from structure of $\lambda^\pm$ as follows:

$\tilde{S}$:



maximal rectangle has singularities in all 4 edges



defines tetrahedron, color edges using **slope**.

Agol-Gueritaud: same construction yields veering triangulation for any pseudo-Anosov flow w/out perfect fits.

Schleimer-Segerman: (in progress): **vice versa**.

Defining polynomials. ("following McMullen's philosophy")

fix M, τ.

$G = H_1(M,\mathbb{Z})/\text{torsion}$.

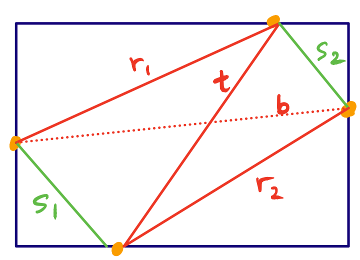
$G \circlearrowright \tilde{M}$ = cover associated to $\pi_1 M \longrightarrow G$

E = edges of τ.

$\tilde{E}$ = lifts to $\tilde{M}$.

Relations among edges: (motivated by the rectangular picture)

for each tetrahedron:



$b = s_1 + t + s_2$

would hold for widths if a transverse measure were defined.

consider formal combinations of edges / these relations

this should be the object on which a transverse measure would have to be defined.

work in $\tilde{M}$:

$\mathbb{Z}$-module: $\mathbb{Z}^{\tilde{E}} = \left\{ \sum a_i e_i \quad \begin{array}{l} a_i \in \mathbb{Z} \\ e_i \text{ edges in } \tilde{E} \end{array} \right\}$.

$\mathbb{Z}[G]$ module: for each $e \in E$ pick one lift $\tilde{e}$ to $\tilde{E}$.

another lift must be $g\tilde{e}$ for $g \in G$

($G \curvearrowright \tilde{E}$ freely)

so this gives a $\mathbb{Z}[G]$ action,

and $\cong \mathbb{Z}[G]^E$

one relation per tetrahedron gives map

$$L: \mathbb{Z}[G]^T \longrightarrow \mathbb{Z}[G]^E$$

can identify $T \simeq E$: tetrahedron $\longleftrightarrow$ bottom edge.

so

$$L: \mathbb{Z}[G]^E \longrightarrow \mathbb{Z}[G]^E$$

square matrix with polynomial entries.

$$\mathbb{Z}[G]^E \xrightarrow{L} \mathbb{Z}[G]^E \longrightarrow \mathcal{E} = \operatorname{coker}(L)$$

Define: $V_\tau = \det L \in \mathbb{Z}[G]$.

Remark: V_τ well-defined up to multiplication by a unit $\pm g$.

monomial

Note: V_τ is the generator of the Fitting Ideal of $\mathcal{E}$:

if $\mathcal{E}$ $\mathbb{Z}[G]$ module. Let $\mathbb{Z}[G]^r \xrightarrow{M} \mathbb{Z}[G]^s \to \mathcal{E} \to 0$ be any free resolution. then

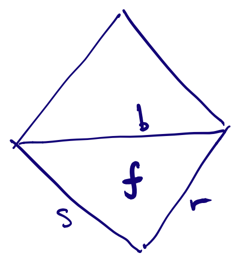
Fitt = ideal generated by $s \times s$ minors of M.

Independent of the resolution.

Another construction:

Face relations:

A face f lies at the bottom of a unique tetrahedron: - b = bottom edge of that tetrahedron.



$$L^{\Delta}(f) = b - s - r.$$

$$\mathbb{Z}[G]^F \xrightarrow{L^{\Delta}} \mathbb{Z}[G]^E \longrightarrow \mathcal{E}^{\Delta} \to 0$$

Let Θ_τ = generator of Fitting ideal of $\mathcal{E}^{\Delta}$

= gcd of $|E| \times |E|$ minors of L^{Δ}.

Anna Parlak:

- independent development of some of this
- explicit computations
- relation to Alexander polynomial.

Θ_τ and the Teichmüller polynomial.

M fibers. F face

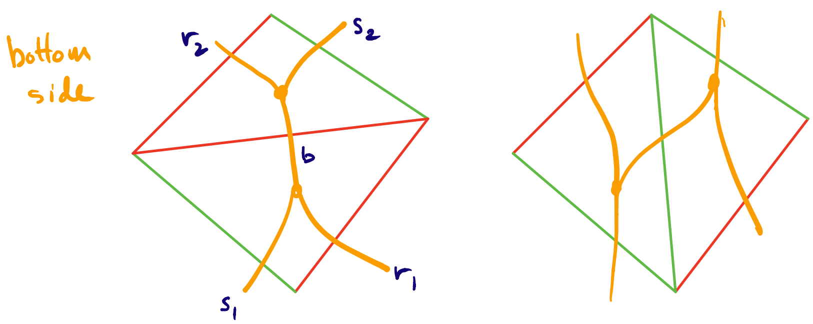
$\mathring{M} = M \setminus$ singular orbits $\qquad \mathring{F}$ = face for $\mathring{M}$ fibratn

τ veering triang for $\mathring{M}$ (unique given F)

<u>Thm</u> $\qquad \Theta_\tau = \Theta_{\mathring{F}}$

Essentially, construction of Θ_τ mimics McMullen construction using an invariant train track on the fibre.

local 'train-track' picture for τ:



face relations are the switch relations

What about on M?

inclusion $i : \mathring{M} \longrightarrow M$ induces map

$i_* : \mathbb{Z}[H_1(\mathring{M})/tor] \longrightarrow \mathbb{Z}(H_1(M)/tor)$

<u>Thm</u> $\qquad i_*(\Theta_\tau) = \Theta_F$

Θ_τ and V_τ

Thm $\quad V_\tau = \Theta_\tau \cdot V^{AB}$

where V^{AB} has the form $\prod (1 \pm g_i)$

$g_i \in G$ cycles arising from the dual graph of τ.