Asymptotic translation lengths on free factor and free splitting complex

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Free factor complex and free splitting complex

Let F_n be the free group with n generators.

- Free factor complex FF_n:
  - Vertices: conjugacy class of free factors of F_n.
  - Faces: sequences of free factors arranged by containment.

  FF_n is the simplicial completion of the Culler-Vogtmann outer space.
- Free splitting complex FS_n:
  - Face of dimension k: minimal simplicial action of F_n on tree with no edge stablizer, no valence 2 vertices and has k edge orbits. (i.e. write F_n as a graph of groups with trivial edge groups, with k edges).
  - Face gluing are via collapsing edges to a point.
  - Faces of dimension 1 has length 1.

Motivation: Curve graph and Curve complex on surfaces

Let S be a closed surface with genus > 1.

- Curve graph:
  - Vertices are isotopy classes of simple closed curves
  - Two vertices are connected by an edge of length 1 if the corresponding curves are disjoint.
- Curve complex:
  - Vertices are isotopy classes of simple closed curves
  - Vertices form a simplex iff the corresponding curves are disjoint.

Properties

- They are all Gromov hyperbolic. (Masur-Minsky, Bestvina-Feighn, Handel-Mosher)
- The mapping class group (or $Out(F_n)$) acts on them by isometry.
- The action is loxodromic iff
  - : Curve complex case: the mapping class is pseudo-Anosov. (M-M)
  - : FF_n: the $Out(F_n)$ element is fully irreducible. (B-F)
  - : FS_n: has filling attracting lamination. (H-M)

Asymptotic translation length

Asymptotic translation length, or stable length:
$l(g) := \lim_{n\to\infty} \frac{d(v, g^n v)}{n}$, where d is a vertex.
Curve graph case:

- $l(g)$ are rational numbers, with denominator bounded by some number depending only on the genus (Bowditch). Hence $l(g)$ can be calculated by finding geodesics on the curve graphs.
- $l(g) \gtrsim g(S)^{-2}$, where $g(S)$ is the genus, and this lower bound is optimal. (Gadre-Tsai)
- $l(g) \gtrsim g(S)^{-1}$, when g is in the Torelli group, and the lower bound is optimal. (Baik-Shin)

Thurston's norm and fibered cone

- Kin-Shin: the example in Gadre-Tsai for pseudo-Anosov maps with small asymptotic translation lengths can be made to be within a single fibered cone.
- Baik-Shin-Wu: one can further find an upper bound for all maps within the same fibered cone.
- Thurston's fibered cone:
  - $\psi : S \to S$ pseudo-Anosov, $M = S \times [0,1]/\sim$, $(\psi(x), 0) \sim (x, 1)$: mapping torus of ψ. $\alpha \in H^1(M;\mathbb{Z})$ pullback from the projection on S^1.
  - Thurston norm: $\beta \in H^1(M;\mathbb{Z})$, $\|\beta\| = \min \max\{0, -\sum_i \chi(S_i)\}$ where S is a dual of β.
  - Thurston norm can be extended to $H^1(M;\mathbb{R})$ as PL norm, the unit ball is a rational polytope. The cone over the fact which contains α is the fibered face containing α, in which any primitive integer class β represents a fiber of M over the circle, hence a pseudo-Anosov map ψ_β on the fiber S_β.

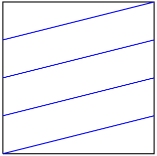
- Theorem (Hyungryul Baik-Hyunshik Shin-W) Let L be a rational slice of a proper subcone of the fibered cone P, passing through origin. Then, for any primitive integer element $\beta \in L$, $l(\psi_\beta) \lesssim \|\beta\|^{-1-1/(d-1)}$, where $d = \dim(L)$.

Analogy for $Out(F_n)$

- ψ: a graph map representing a fully irreducible element in $Out(F_n)$, M: its mapping torus.
- Dowdall-Kapovich-Leininger: there is a "cone of sections" or "McMullen cone" containing the pullback of generator of $H^1(S^1)$, where every primitive integer class β represent a fully irreducible outer automorphism ψ_β. Let the negative Euler characteristic of the fiber be $\|\beta\|$.
- Theorem (Hyungryul Baik-Dongryul Kim-W) Let L be a rational slice of a proper subcone of the McMullen cone P, passing through origin. Then, for any primitive integer element $\beta \in L$, $l(\psi_\beta) \lesssim \|\beta\|^{-1-1/(d-1)}$, where $d = \dim(L)$, l is the asymptotic translation length on the 1-skeleton of FF_n or FS_n.

Proof sketch

- Relate to the sphere complex.
- Find an description of the fiber corresponding to β:



- See that in the fiber there are large regions that locally look like an abelian cover of the original space.
- It takes many iterations of the lifted map to allow one fundamental domain to cover the whole region.

General case and Remaining questions

Further questions:

- Can one find an lower bound? Is the upper bound optimal?
- Can we use FS_n and FF_n to study asymptotic translation length on curve complexes?

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