The Mapping Class Group of $\#_n (S^2 \times S^1)$

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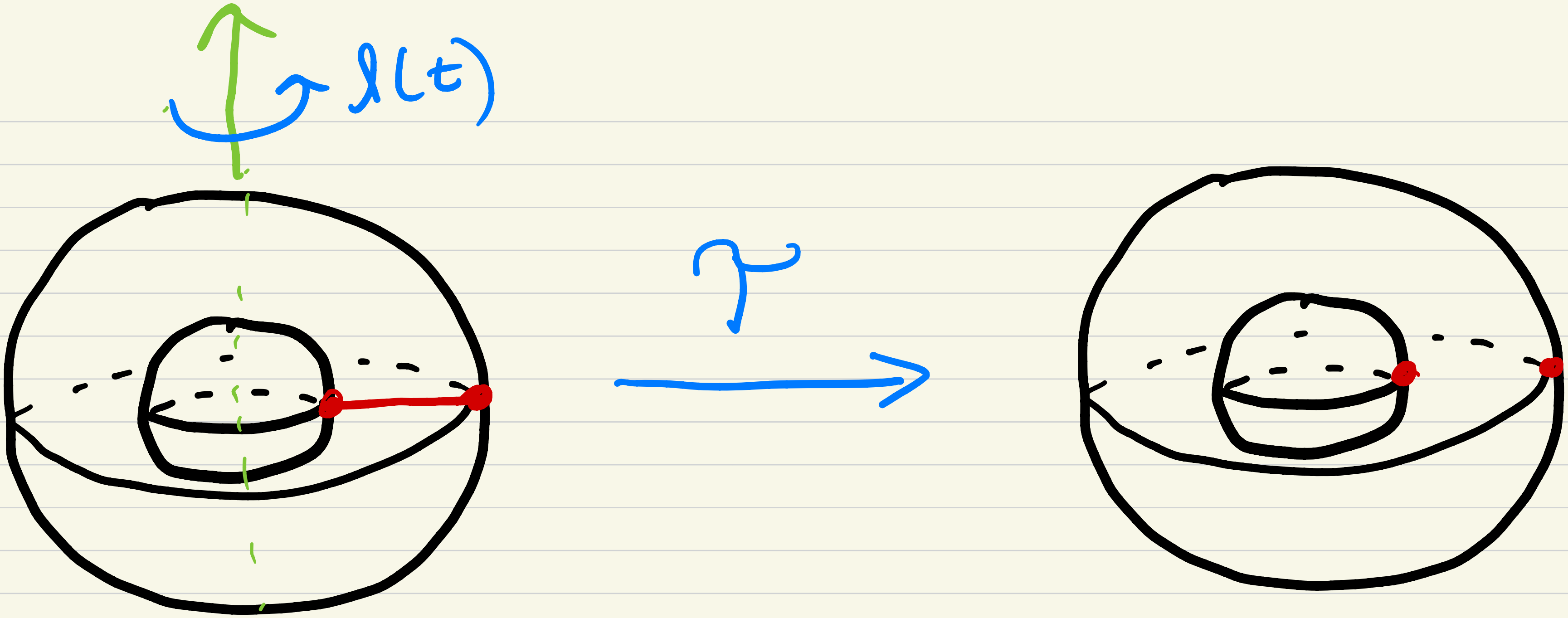
Sphere Twists: $S = S^2 \hookrightarrow M^3$

Recall: $\pi_1(SO(3), id) = \langle \ell \mid \ell^2 = 1 \rangle$

Define $T : S \times [0,1] \longrightarrow S \times [0,1]$

$$(s, t) \longmapsto (\ell(t) \cdot s, t)$$



$S^2 \times [0,1]$ $\qquad\qquad$ $S^2 \times [0,1]$

Twist Subgroup

$$\mathrm{Twist}(M^3) = \langle T_S \mid S \subset M^3 \rangle \leq \mathrm{Mod}(M^3)$$

$$\mathrm{Mod}(M_n) \xrightarrow{\rho} \mathrm{Out}(\pi_1(M_n))$$

Trivializations: M^3 oriented

An *(oriented) trivialization* of TM^3 is a section σ of the frame bundle:

$$\sigma : M^3 \longrightarrow Fr(TM^3)$$
$$p \longmapsto \sigma(p)$$

<u>Main Theorem</u> (BBP):

Let $[\sigma_0]$ be the homotopy class of a trivialization σ_0 of TM_n, and let $(\mathrm{Mod}\ M_n)_{[\sigma_0]}$ be its stabilizer.

Then we have:

(1) $\mathrm{Mod}\,(M_n) = \mathrm{Twist}\,(M_n) \rtimes (\mathrm{Mod}\, M_n)_{[\sigma_0]}$

(2) $\mathrm{Twist}\,(M_n) \cong H^1(M_n; \mathbb{Z}/2)$ $\quad *$

(3) $(\mathrm{Mod}\ M_n)_{[\sigma_0]} \cong \mathrm{Out}\, F_n$

<u>Have:</u> $1 \to \mathrm{Twist}\, M_n \to \mathrm{Mod}\, M_n \xrightarrow{\varphi} \mathrm{Out}\, F_n \to 1$

Standard Lemma The sequence

$$1 \to A \to G \longrightarrow Q \longrightarrow 1$$

splits iff $\exists$ a crossed homomorphism

$$\lambda : G \longrightarrow A \quad \text{s.t. } \lambda|_A = \mathrm{id}_A .$$

$$\lambda(g_1 g_2) = \lambda(g_1)^{g_2} \lambda(g_2)$$

Moreover, if such a λ exists, then we can choose a splitting $Q \longrightarrow G$ w/ image $\ker \lambda$ (so that

$$G = A \rtimes \ker \lambda).$$

Catalogue of Actions

- $Fr(TM^3) \circlearrowleft GL_3^+(\mathbb{R})$

$$(\Upsilon : \mathbb{R}^3 \longrightarrow T_p M^3) \cdot A$$

$$= \Upsilon \circ A : \mathbb{R}^3 \longrightarrow T_p M^3$$

- $Triv(M^3) \circlearrowleft C(M^3, GL_3^+\mathbb{R})$

$$(\sigma : M^3 \longrightarrow Fr(TM^3)) \cdot (\phi : M^3 \longrightarrow GL_3^+\mathbb{R})$$

$$(\sigma \cdot \phi)(p) = \underbrace{\sigma(p) \cdot \phi(p)}_{\text{prev. action}}$$

- $\mathrm{Triv}(M^3) \circlearrowleft \mathrm{Diff}^+(M^3)$

$$\begin{tikzcd} M^3 \arrow[d, "\sigma"'] & & M^3 \arrow[ll] \\ \mathrm{Fr}(TM^3) \arrow[rr, "(Df^{-1})_*"'] & & \mathrm{Fr}(TM^3) \end{tikzcd}$$

- $C(M^3, GL_3^+\mathbb{R}) \circlearrowleft \mathrm{Diff}^+ M^3$

$$\phi^f = \phi \circ f$$

All actions compatible:

$$(\sigma \cdot \phi)^f = \sigma^f \cdot \phi^f$$

Derivative crossed homomorphism

$$\mathcal{D} : \mathrm{Diff}^+(M^3) \longrightarrow C(M^3, GL_3^+\mathbb{R})$$

Twisting crossed homomorphism

$$\mathrm{Mod}\, M^3 \longrightarrow [M^3, GL_3^+\mathbb{R}]$$

<u>Key Prop</u> $\mathcal{J}^{\circ}(T_S) \in H^1(M^3; \mathbb{Z}/2)$ is the Poincaré dual of $[S] \in H_2(M^3; \mathbb{Z}/2)$.

<u>Sketch of proof</u>:

Let $[\Upsilon] = T_S$; let γ be a loop in M^3.
(WLOG Υ fixes γ pointwise.)

$$[0,1] \xrightarrow{\gamma} M^3 \xrightarrow{\mathcal{D}(\Upsilon)} GL_3^+(\mathbb{R})$$

image is $\mathcal{J}^{\circ}(T_S)([\gamma])$

<u>Cor</u> $\mathcal{J}: \mathrm{Mod}(M_n) \longrightarrow H^1(M_n; \mathbb{Z}/2)$

restricts to an isomorphism

$$\mathrm{Twist}(M_n) \cong H^1(M_n; \mathbb{Z}/2)$$

Main Theorem (Almost!) (BBP):

Let $[\sigma_0]$ be the homotopy class of a trivialization σ_0 of TM_n, and let $(\mathrm{Mod}\ M_n)_{[\sigma_0]}$ be its stabilizer.

Then we have:

(1) $\mathrm{Mod}(M_n) = \mathrm{Twist}(M_n) \rtimes \cancel{(\mathrm{Mod}\ M_n)_{[\sigma_0]}}$ $\mathrm{Ker}\ \mathfrak{J}$

(2) $\mathrm{Twist}(M_n) \cong H^1(M_n; \mathbb{Z}/2)$ as a $\mathrm{Mod}(M_n)$-module

(3) $\cancel{(\mathrm{Mod}\ M_n)_{[\sigma_0]}} \cong \mathrm{Out}\ F_n$

$\mathrm{Ker}\ \mathfrak{J}$

$\operatorname{Ker} \circlearrowright \supseteq (\operatorname{Mod} M_n)_{[\sigma_0]}$ ✓

Remains to show: $\operatorname{Ker} \circlearrowright$ acts trivially on HTriv $(= \{ [\sigma] \mid \sigma \in \operatorname{Triv} \})$