Outer space for RAAGs : Part II

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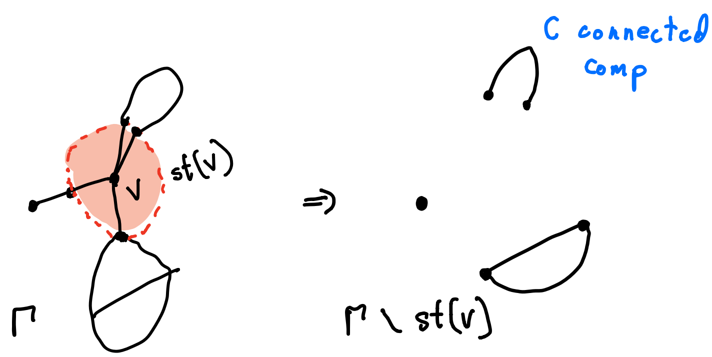
Generating $Out(A_\Gamma)$: (Laurence–Servatius)

1. Graph auts: $Aut(\Gamma)$
2. Inversions: $v \mapsto v^{-1}$
3. Partial conjugations:

   $C \mapsto vCv^{-1}$



4. Transvections: If $lk(v) \subseteq st(w)$, $v \mapsto vw$
   (a) If $lk(v) \subseteq lk(w)$: fold $v \leq_f w$
   (b) If $st(v) \subseteq st(w)$: twist $v \leq_t w$

Untwisted subgp $U(A_\Gamma) = \langle ①, ②, ③, ④a \rangle$

Idea: To get a space on which $Out(A_\Gamma)$ acts, parametrize marked metric spaces with $\pi_1 = A_\Gamma$.

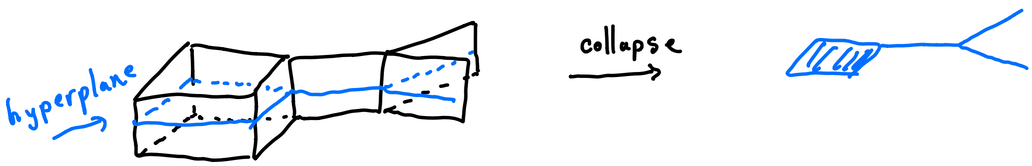
Step 1: (Charney-Stambaugh-Vogtmann)

Construct an outer space for $U(A_\Gamma)$.

A **Γ-complex** X is a locally CAT(0) cube complex satisfying

① $\pi_1(X) = A_\Gamma$

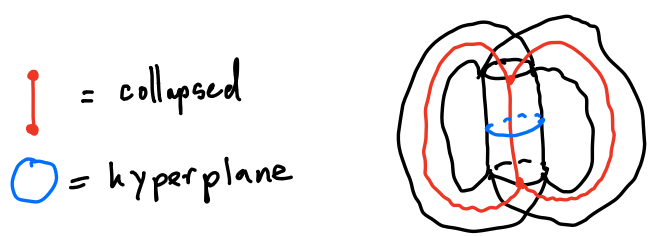
② $\exists$ collapse map $c : X \longrightarrow S_\Gamma$ which is a h.e.



Ex: $\Gamma = \overset{a}{\bullet}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}$

$A_\Gamma = F_2 \times \mathbb{Z}$

$X = \underset{a,c}{\Theta} \times \underset{b}{S^1}$



Thm: (CSV) $\exists$ contractible space K_Γ on which $U(A_\Gamma)$ acts properly

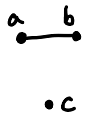
A point in K_Γ is (X, ρ)

- X Γ-complex
- $\rho : X \longrightarrow S_\Gamma$ s.t.

$$\begin{tikzcd} S_\Gamma \arrow[r, "c^{-1}"] \arrow[rr, bend left, "\in U(A_\Gamma)"] & X \arrow[r, "\rho"] & S_\Gamma \end{tikzcd}$$

say ρ is untwisted

Example of K_Γ:

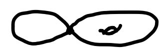
Γ: 

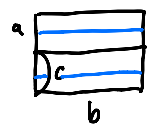
$A_\Gamma = \mathbb{Z}^2 * \mathbb{Z}$

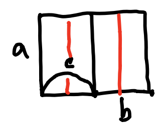
$U(A_\Gamma) \approx \mathbb{Z}^2$

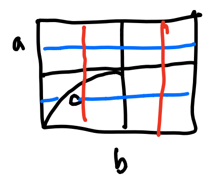
$\varphi_a : c \longmapsto ac$

$\varphi_b : c \longmapsto bc$

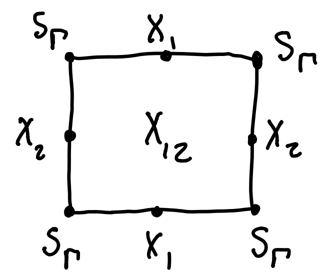
$S_\Gamma =$ 

$X_1 =$ 

$X_2 =$ 

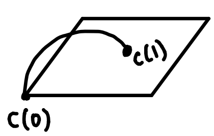
$X_{12} =$ 

$K_\Gamma = \mathbb{R}^2$



What should full outer space look like?

a, b can twist onto each other $\Rightarrow GL_2\mathbb{Z} \leq \mathrm{Out}(A_\Gamma)$



- $\mathrm{Out}(A_\Gamma) = \mathbb{Z}^2 \rtimes \underline{GL_2\mathbb{Z}}$
- Allow any flat metric on T^2

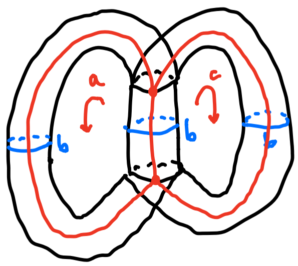
$\Rightarrow \mathbb{R}^2 \tilde{\times} \mathbb{H}^2$

Step 2: Allow general parallelotope metrics.

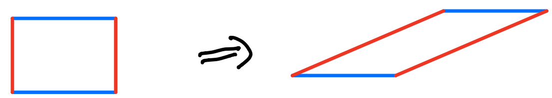
A twisted metric on a Γ-complex replaces cubes by parallelotopes, subject to compatibility relations $\Rightarrow$ metric locally CAT(0)

Ex: $\Gamma = \overset{a}{\bullet}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}$

$a, c \underset{t}{\leq} b$



Replace 3 cylinders with different parallelograms:



Def: $\mathcal{O}_\Gamma = \left\{ (X, d, h) \;\middle|\; \begin{array}{l} X \ \Gamma\text{-complex} \\ d \text{ twisted metric} \\ h: X \to S_\Gamma \text{ h.e.} \end{array} \right\}$

$Out(A_\Gamma)$ acts by post-composition.

Thm (B-Charney-Vogtmann) $\mathcal{O}_\Gamma$ is a contractible space with a proper action of $Out(A_\Gamma)$

Strategy: Define intermediate space

$$T_\Gamma = \left\{ (X, \mathcal{F}, \rho) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ collection of parallelotopes} \\ \rho \text{ untwisted} \end{array} \right\}$$

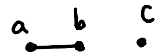
$$\begin{array}{ccc} & T_\Gamma & \\ {}^{\sigma}\swarrow & & \searrow^{\Theta} \\ \{(X,\rho)\} = K_\Gamma & & \mathcal{O}_\Gamma = \{(X,d,h)\} \end{array}$$

ρ untwisted

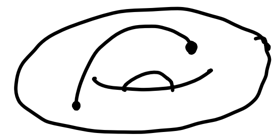
- σ "straightens" $(X, \mathcal{F})$
- $\Theta(X, \mathcal{F}, \rho) = (X, d_{\mathcal{F}}, \rho)$

① σ is a def. retraction

② Θ is a fibration with contractible fibers.

Example fiber: $\Gamma =$ 

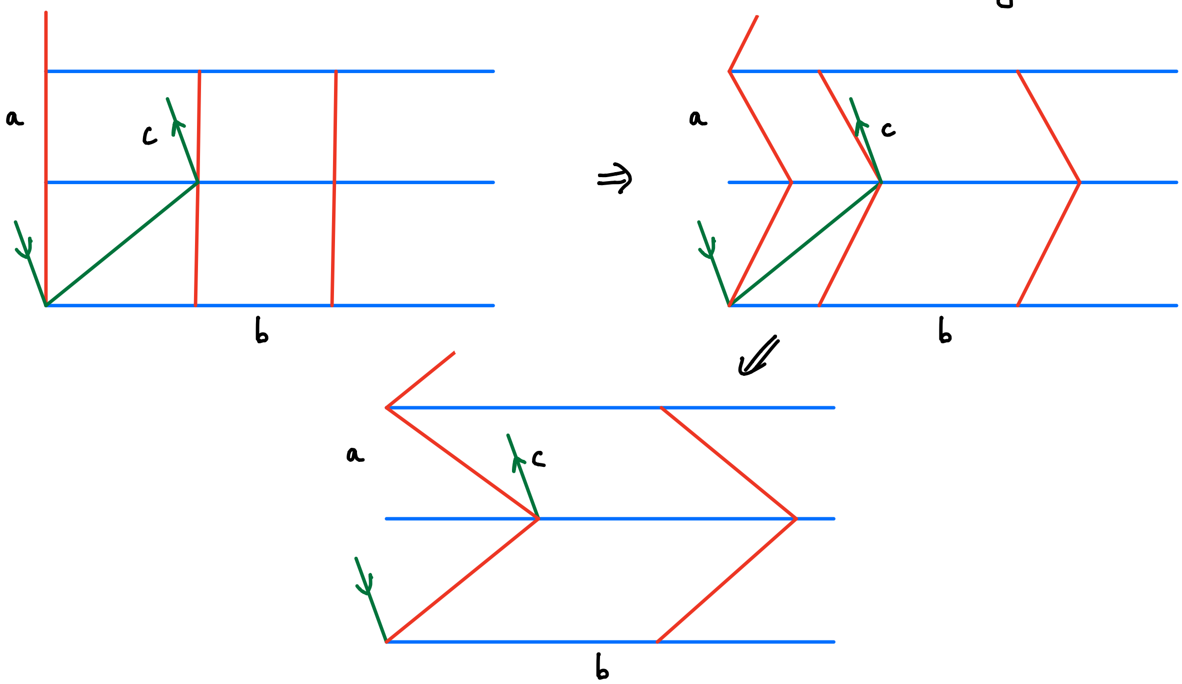
$c \longmapsto ac$, $c \longmapsto bc$



a,b torus with c-edge attached

$a \leq_t b$

$\langle a, b \rangle$-plane in $\tilde{X}$



Future Directions

1. Is there a natural compactification of $\mathcal{O}_\Gamma$?
2. Fixed points of finite subgroups?
3. Train tracks? Dynamics?