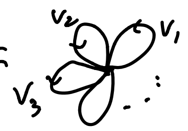
Outer Space for RAAGs

Joint work with Cory Bregman + Karen Vogtmann

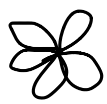
$\Gamma = (V, E)$ finite simplicial graph

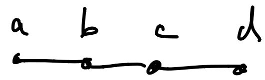
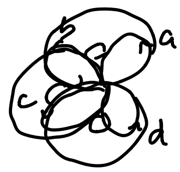
$A_\Gamma = \langle V \mid v_i v_j = v_j v_i$ if $v_i - v_j \in E \rangle$ right-angled Artin group

$S_\Gamma = K(A_\Gamma, 1)$-space, Salvetti complex

$=$  $\cup$ k-torus for each k-clique in Γ } locally CAT(0) cube complex

Eg:

Γ = discrete
$A_\Gamma = F_n$
$S_\Gamma =$ 

$\Gamma =$ 
$A_\Gamma =$
$S_\Gamma =$ 

Γ = complete
$A_\Gamma = \mathbb{Z}^n$
S_Γ = n-torus

Today: $\mathrm{Out}(A_\Gamma) = \mathrm{Aut}(A_\Gamma) / \mathrm{Inn}(A_\Gamma)$

$$\mathrm{Out}(F_n) \xleftrightarrow[\mathrm{Out}(A_\Gamma)]{?} \mathrm{Out}(\mathbb{Z}^n) = GL_n(\mathbb{Z})$$

Most of the work to date on $\mathrm{Out}(A_\Gamma)$ uses algebraic methods.

Our goal: develop geometric tools to study these groups.

At the extremes we have

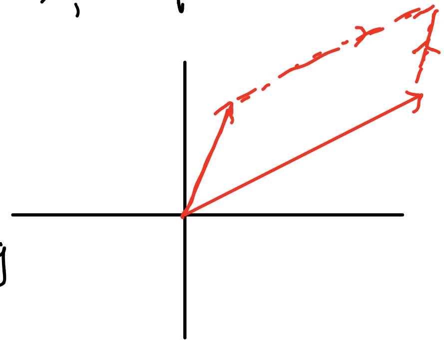
$$\mathrm{Out}(F_n) \curvearrowright CV_n \xleftrightarrow{?} GL_n(\mathbb{Z}) \curvearrowright \text{homogeneous space}$$

To find: a contractible space $\mathcal{O}_\Gamma$ with a proper action of $\mathrm{Out}(A_\Gamma)$.

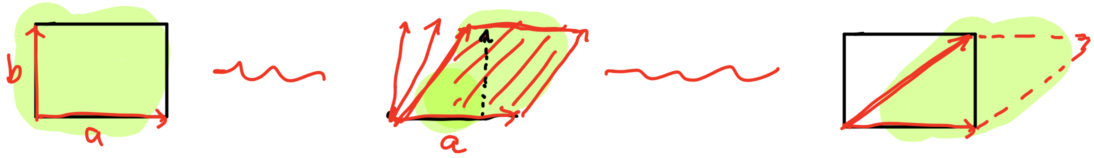
Case $A_\Gamma = \mathbb{Z}^n$: $\quad Out(A_\Gamma) = GL_n(\mathbb{Z})$, $\quad S_\Gamma = T^n$

$GL_n(\mathbb{Z}) \curvearrowright Q_n := GL_n(\mathbb{R})/O_n(\mathbb{R})$

= space of flat metrics on T^n with a marking $g: \pi_1(T^n) \xrightarrow{\cong} \mathbb{Z}^n$



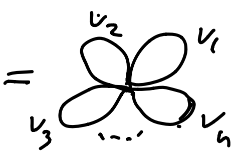
$\alpha \in GL_n(\mathbb{Z})$ acts on Q_n by changing the marking



$\alpha: b \mapsto ba$
$\quad a \mapsto a$

(T^n, g) $\qquad$ $\uparrow$ different metric torus $\qquad$ $(T^n, \alpha g)$

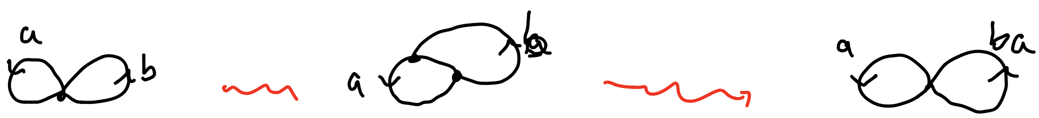
same metric
different marking

Case $A_\Gamma = F_n$: $\quad Out(F_n)$, $\quad S_\Gamma = R_n =$ 

$Out(F_n) \curvearrowright CV_n$ = Culler-Vogtmann's Outer Space

$$CV_n = \left\{ (\Theta, g) \;\middle|\; \begin{array}{l} \Theta = \text{metric graph} \\ g: \pi_1(\Theta) \xrightarrow{\cong} F_n \end{array} \right\} \Big/ \sim$$

$Out(F_n)$ acts by changing the marking.



$\alpha: b \mapsto ba$
$\quad a \mapsto a$

(R_n, g) $\qquad$ different graph $\qquad$ $(R_n, \alpha g)$

same graph
different marking

Idea: Meld these two constructions.

Start with a space with fundamental group A_Γ (how about S_Γ?)

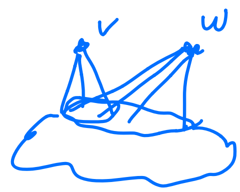
Then modify it as necessary to allow us to move from one marking to another. We will need to modify both the combinatorial structure and the metric!

Generating set for $Out(A_\Gamma)$ (Laurence, Servatius)

1. graph autos: $\Gamma \xrightarrow{\cong} \Gamma$
2. inversions: $v \longmapsto v^{-1}$
3. partial conjugations: conjugate a component of $\Gamma \setminus st(v)$ by v

Γ:

4. transvections (folds) $v \longmapsto vw$ where $lk(v) \subseteq lk(w)$



5. transvections (twists) $v \longmapsto vw$ where $st(v) \subseteq st(w)$

The untwisted automorphism group is the subgroup $U_\Gamma \subset Out(A_\Gamma)$ generated by (1)–(4).

The group U_Γ behaves much like $Out(F_n)$.