Graphical small cancellation and groups of type *FP*

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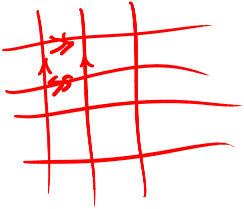
Groups of type F

G is type F if there is a finite Eilenberg–Mac Lane space for G.

Equivalently, G acts freely simplicially cocompactly on a contractible complex.



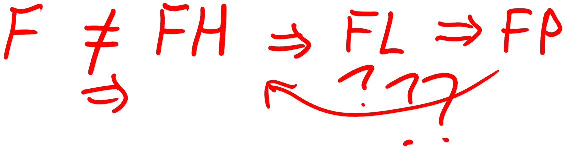
Groups of type *FP*

G is of type *FP* if there is a finite resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module by finitely generated projective modules.

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

Groups of type *FH*

G is of type *FH* if *G* acts freely, cocompactly, simplicially on an *acyclic* complex.



Bestvina-Brady groups

In the 1990's, Bestvina–Brady constructed groups of type *FH* that are not finitely presented.

In 2015 I constructed an uncountable family of such groups.

I used the same 'Morse theory' that Bestvina–Brady used.

Tom Brown and I now have an independent construction.

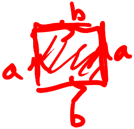
Acyclic kernels

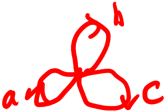
$G \curvearrowright E \qquad G/N \curvearrowright E/N$

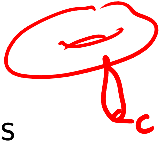
If G is of type F and $N \trianglelefteq G$ is *acyclic*, then G/N is type FH.

Small cancellation I

$[a,b] = aba^{-1}b^{-1}$







Think of relators as the boundaries of discs.

A *piece* is a [reduced] word that is either in two relators

or appears twice in the same relator.

Small cancellation II

A presentation is $C'(1/6)$ if any piece has length $< 1/6$ the length of any relator it appears in.

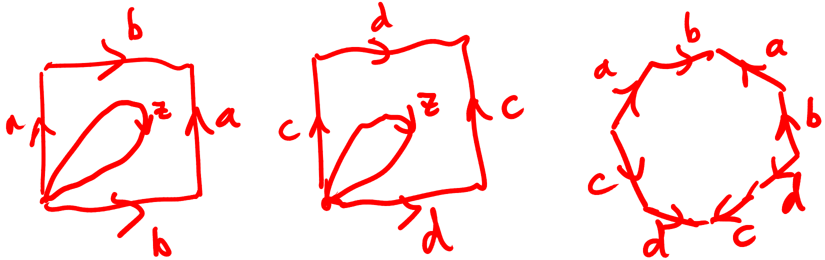
Theorem

- If a presentation is $C'(1/6)$, then its Cayley 2-complex is contractible and the relators embed into it.
- Any word equal to the identity contains more than half a relator.

Example

Make the closed orientable surface of genus 2 from two tori with discs removed.

$$\langle a, b, c, d \ : \ [a, b] = [c, d] \rangle$$



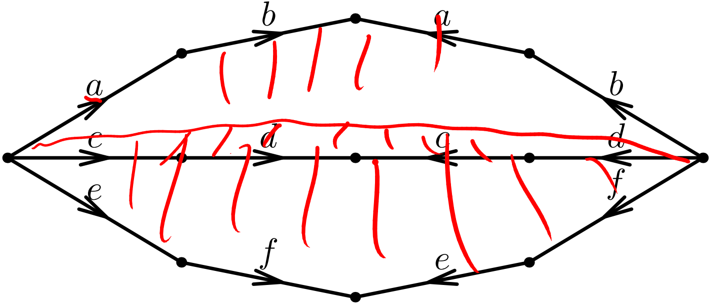
Non-example?

Identify the boundary circles of *three* punctured tori:

$$\langle a, b, c, d, e, f \;\; : \;\; [a, b] = [c, d] = [e, f] \rangle$$

The solution

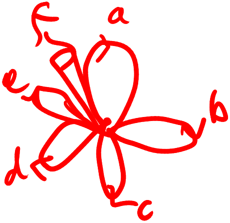
Take a single *graphical relator*



Theorem (Gromov, Ollivier, Gruber)

If a graphical presentation is $C'(1/6)$, then its graphical Cayley 2-complex is contractible and the graphical relators embed into it.

Any word equal to the identity contains more than half a simple cycle in one of the relators.



Spectacular complexes

A *spectacular complex* is a 2-complex with

Simplicial 1-skeleton;

Polygons Embed and are $C'(1/6)$;

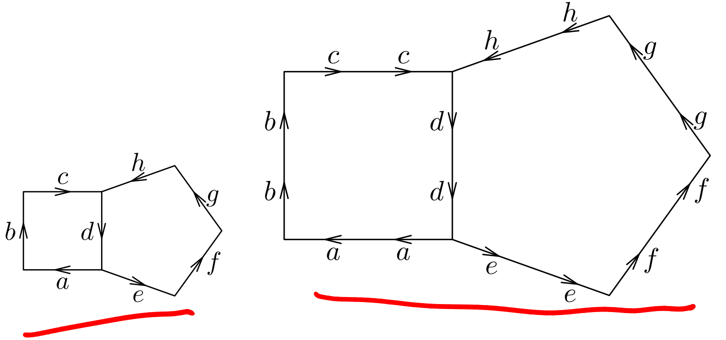
Two-dimensional ACyclic;

with lower bounds on Unbranched paths, Lengths of polygons

And Rotundity ($=$ girth).

Subdividing a graphical relator

$a^{k^n} b^{k^n} c^{k^n} d^{k^n}, \; d^{k^n} e^{k^n} f^{k^n} g^{k^n} h^{k^n}$



A presentation

Fix $k > 0$, and let $Z := \{k^n : n \in \mathbb{N}\}$.

$H(\emptyset)$ has generators the directed edges of a spectacular complex K, with relators

the 'degree n subdivisions of the boundaries of polygons of K', for $n \in Z$.

More presentations

$Z = \{1, k, k^2, k^3, \ldots\}$

Let $S \subseteq Z$. The group $H(S)$ has same generators as $H(\emptyset)$

with graphical relators

* for $n \in Z - S$: the degree n subdivisions of boundaries of polygons P of K;
* for $n \in S$: the degree n subdivision of K^1.

Theorem

$g \geq 13$

These graphical presentations are $C'(1/6)$.

For $S \subset T$,

$H(S) \to H(T)$ is surjective with acyclic kernel.

$a^n b^n c^n \ldots m^n$ $\quad n < p$

$a^p b^p c^p \ldots m^p$ $\quad a^n b^n$

A slogan

Suppose $N \trianglelefteq G$ with $Q = G/N$, and that X is the Cayley 2-complex for G.

X/N has 1-skeleton the Cayley graph for Q, with 2-cells corresponding to the relators for G.

Polygon subgroups

$$H_P = \langle a_1, \ldots, a_l \ : a_1^n a_2^n \cdots a_l^n = 1 \ \ n \in Z \rangle.$$

$a_i \mapsto a_i^k$

$$H_P = \langle a_1, \ldots, a_l \ : a_1^{k^n} a_2^{k^n} \cdots a_l^{k^n} = 1 \ \ n \geq 0 \rangle.$$

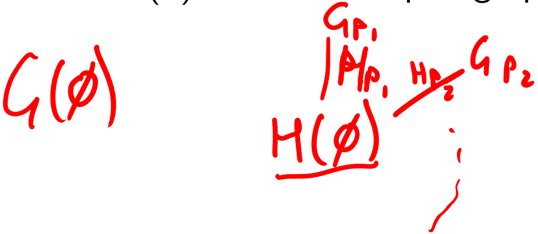
Here l is the number of sides of the polygon P.

H_P has an HNN-extension that is type F.

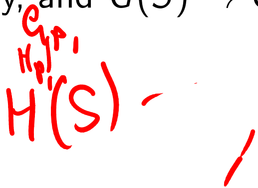
$G_P = \langle a_1, \ldots, a_l \mid a_1 \cdots a_l = 1, \ t a_i t^{-1} = a_i^k \rangle$

A group of type F

Build $G(\emptyset)$ as a star-shaped graph of groups:



$G(S)$ is defined similarly, and $G(S) \to G(T)$ has ~~acyclic~~ kernel.



Eilenberg-Ganea conjecture?

Each $G(S)$ has cohomological dimension 2.

For most $G(S)$, we do not have a 2-dimensional Eilenberg–Mac Lane space.

Making spectacular complexes

Fix a prime power q.

$G = PGL(2, q)$ acts triply-transitively on $K^0 = \mathbb{P}^1(q)$.

$|K^0| = q+1$

Start with the complete graph on K^0.

Making spectacular complexes II

Pick an element $g \in G$ of order $d > 2$ dividing $q \pm 1$.

The conjugacy class of g is closed under inverses.

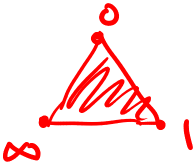
As a permutation of K^0 g is lots of d-cycles and either 0 or 2 fixed points.

- Attach d-gons to the complete graph using these d-cycles for g and all its conjugates.

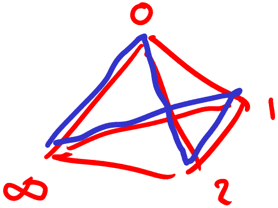
Triple transitivity implies that the intersection of two d-gons cannot contain a path of two or more edges.

Examples

$q = 2 \quad d = 3 \quad PGL(2,2) \cong \Sigma_3$

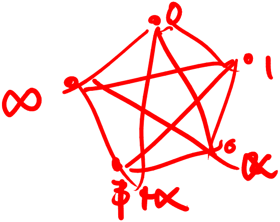


$q = 3 \quad d = 4 \quad PGL(2,3) \cong \Sigma_4$



a copy of $\mathbb{R}P^2$ made from 3 squares

$q=4 \quad d=\underline{3}$ $\qquad PGL(2,4) \cong A_5$



$\rightsquigarrow$ 2-skeleton of Δ^4

$q=4 \quad d=5$

2-skeleton of Poincaré homology sphere

A spectacular complex

The case $d = 7$, $q = 8$ gives 36 7-gons attached to a K^9.

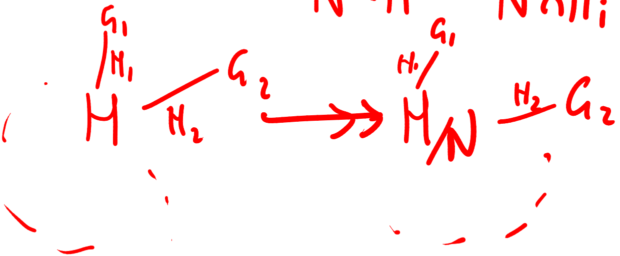
This complex has perfect fundamental group. $H_1 = 0$

Throwing away 8 of the polygons can give an acyclic complex.

Subdivide each edge into 5.

This complex made by attaching 28 35-gons to a subdivided K^9 is spectacular.

$N \triangleleft H \qquad N \cap H_i = \{1\}$



$* N$