Orbit equivalence rigidity of irreducible actions of right-angled Artin groups

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joint work with Jingyin Huang (Ohio State University)

GaTO Seminar, Oct. 21, 2021

Group actions on probability spaces

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(X, μ) standard proba space (e.g. $X \approx ([0,1], \mathrm{Leb})$)
$G \curvearrowright X$ free, ergodic, measure-preserving (p.m.p.) action

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- *Bernoulli actions*: $G \curvearrowright [0,1]^G$ by shift
- *Profinite actions*: G residually finite, acting on its profinite completion, preserving the Haar measure

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3. *conjugate* if $\exists \alpha : G \to H$, $f : X \to Y$ iso: $f(gx) = \alpha(g) f(x)$

Flexibility and rigidity: some known results

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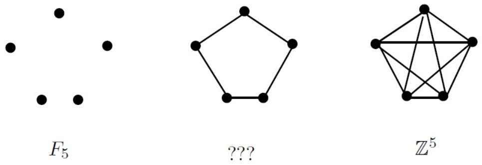
   - for irreducible actions of $F_p \times F_q$ (Monod-Shalom 06)
   - for Bernoulli actions of e.g. Property (T) groups (Popa 06)

Right-angled Artin groups (RAAGs)

Γ finite simple graph $\rightsquigarrow$ G_Γ right-angled Artin group:

- generators: vertices of Γ
- relators: $[v, w] = 1$ whenever v and w are adjacent



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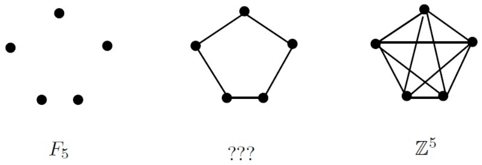

More generally: Graph product of $\{G_v\}_{v \in V\Gamma}$ over Γ: group obtained from $*_{v \in V\Gamma} G_v$ by further imposing that G_v and G_w commute whenever v, w are adjacent.

Failure of rigidity

Based on Ornstein-Weiss theorem + an argument of Gaboriau:

Proposition (H-Huang)

Let G be a RAAG with defining graph Γ. Let H be a graph product of countably infinite amenable groups over Γ. Then G and H have OE free, ergodic, p.m.p. actions.

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Better: Every action $G \curvearrowright X$ has a blow-up $G \curvearrowright \hat{X}$ (i.e. coming with a G-equivariant map $\hat{X} \to X$) which is OE to some H-action.

Idea of the construction

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For $h \in H_v$, let $h \cdot (x, z) := g \cdot (x, z)$, where $g \in G_v$ is s.t. $gz = hz$.

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Definition

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Theorem 1 (Strong rigidity, H-Huang)

Let G, H be two one-ended, centerless RAAGs.
If two irreducible actions $G \curvearrowright X$ and $H \curvearrowright Y$ are SOE, then they are conjugate.

Superrigidity of irreducible actions of RAAGs

Works of Furman, Monod-Shalom, Kida let us derive superrigidity.

Theorem 2 (Superrigidity, H-Huang)

Let G be a one-ended, centerless RAAG. Let H be a countable gp. If an irreducible action $G \curvearrowright X$ and a mildly mixing free, p.m.p. action $H \curvearrowright Y$ are SOE, then they are virtually conjugate.

Mildly mixing: $\liminf_{g \to +\infty} \mu(A \Delta gA) > 0$ whenever $0 < \mu(A) < 1$.

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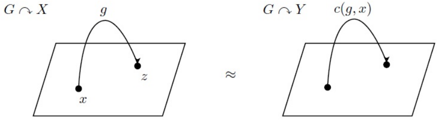
Rk: In the specific case of Bernoulli actions of G, work of Popa enables to remove the mild mixing assumption on the H-action.

Proof ingredients for Theorem 1 (Strong rigidity)

Say we have two OE irreducible actions $G \curvearrowright X$ and $G \curvearrowright Y$ of the pentagon RAAG G.

$\rightsquigarrow$ the same OE relation $\mathcal{R}$
$\rightsquigarrow$ $c : G \times X \to G$ the orbit equivalence cocycle



Goal: c is cohomologous to a group homomorphism $\alpha : G \to G$ (i.e. $\exists \theta : X \to G$ such that $\theta(gx)c(g,x)\theta(x)^{-1} = \alpha(g)$)

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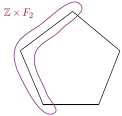

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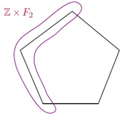

Step 1 (exploiting $\mathrm{CAT}(0)$ geometry):
Subrelations of $\mathcal{R}$ coming from restricting the action to a vertex group / a star subgroup can be "recognized".

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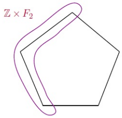

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$\rightsquigarrow$ c is cohomologous to c_v such that

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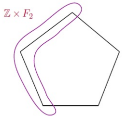

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Step 2 (based on Monod–Shalom, using irreducibility):
$\rightsquigarrow$ c_v is cohomologous to a cocycle which descends to a group isomorphism $G_{\mathrm{st}(v)}/G_v \to G_{\mathrm{st}(w)}/G_w$.

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Step 3 (cancelling ambiguities):
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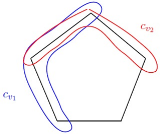

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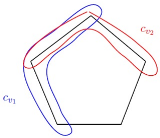

Step 4 (propagation, using commutation and irreducibility):
As G_v is part of a generating set of G where consecutive generators commute, c is cohomologous to a group homomorphism $G \to G$.

On the SOE classification of RAAGs

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