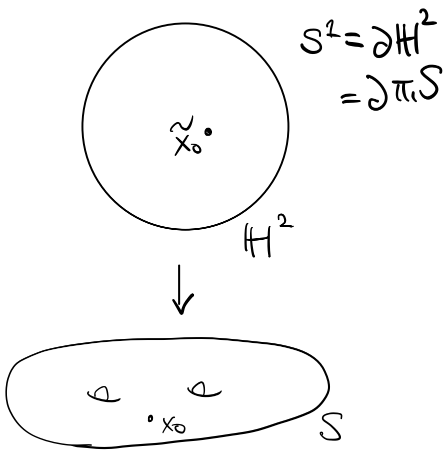
Optimal Regularity of Mapping Class Group Actions on the Circle

November 18, 2021
Geometry and Topology Online

Sam Sang-hyun Kim (KIAS)
Thomas Koberda (UVA)
Cristóbal Rivas (USACH)

⓪ Nieslen's Action of a MCG on S^1



$f : (S, x_0) \xrightarrow{\approx} (S, x_0)$

$\rightsquigarrow \tilde{f} : (\tilde{S}, \tilde{x}_0) \xrightarrow[\text{q.i.}]{\approx} (\tilde{S}, \tilde{x}_0)$

$\rightsquigarrow \partial \tilde{f} : S^1 \longrightarrow S^1$

indep. of isotopy

$\mathrm{Mod}(S, x_0) := \mathrm{Homeo}^+(S, x_0) / \text{isotopy}$

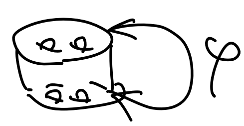
Nielsen $\quad \mathrm{Mod}(S, x_0) \hookrightarrow \mathrm{Homeo}^+ S^1 \qquad$ How smooth?

$M^1 :=$ S^1 or $I = [0,1]$

(Farb-Franks '01, Parwani '07, cf. Mann-Wolff '20)

$$\mathrm{Mod}(S_{g,p}) \not\hookrightarrow \mathrm{Diff}^1_+(M^1) \qquad g \gg 0$$

φ : pseudo-Anosov and $\quad M_\varphi := S_g \times I / (x, 0) \sim (\varphi(x), 1)$



$$\begin{tikzcd}
1 \arrow[r] & \pi_1 S_g \arrow[r] \arrow[dr] & \pi_1 M_\varphi \arrow[r] \arrow[d] & \pi_1(S^1) \cong \langle \varphi \rangle \arrow[r] \arrow[d] & 1 \\
& & \mathrm{Mod}(S_g, x_0) & \mathrm{Mod}(S_g) \arrow[ur] &
\end{tikzcd}$$

$$\widetilde{\mathrm{Mod}}(S_g, x_0) \to \mathrm{Mod}(S_g)$$
$$\downarrow$$
$$\mathrm{Homeo}(S^1)$$

(N. Tholozan) $\pi_1 M_\varphi \cong \langle \pi_1(S_g), \varphi \rangle \hookrightarrow \mathrm{Diff}^1_+ S^1$

ⓞ Virtual Embedding

<u>Conjectures</u> For $g \geq 3$,

(1) $\mathrm{Mod}(S_{g,p}) \overset{\text{virt}}{\not\twoheadrightarrow} \mathbb{Z}$ (Ivanov Conjecture)

(2) $\mathrm{Mod}(S_{g,p}) \underset{\text{virt}}{\not\hookrightarrow} \mathrm{Diff}^1_+ M^1$

(3) $\mathrm{Mod}(S_{g,p})$ has property (T)

i.e. every isometric action on $\mathcal{H}$ fixes a pt

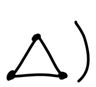
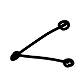
- (1) $\Rightarrow$ (2) cf. Thurston '74
- (3) $\Rightarrow$ (1) $G^{(T)} \Rightarrow \beta_1 G = 0$
- (3) $\Rightarrow$ (2) $G^{(T)} \Rightarrow G \not\hookrightarrow \mathrm{Diff}^{1.5}_+ M^1$ cf. Navas '03

<u>Baik-K-Koberda '19</u> $\mathrm{Mod}\, S_{g,p} \underset{\text{virt}}{\hookrightarrow} \mathrm{Diff}^2_+ M^1$

$\Leftrightarrow \mathrm{Mod}\, S_{g,p}$: virt. free $\Leftrightarrow 3g - 3 + p \leq 1$

ⓞ Right-angled Artin group

Γ : finite graph $A(\Gamma) := \langle v \in V\Gamma \mid [a,b] = 1 \;\; \forall \{a,b\} \in E\Gamma \rangle$

$A($  $) = \mathbb{Z}^3$ $A($  $) = \mathbb{Z} \times F_2$

RAAG: • Every nontrivial subgp $\twoheadrightarrow \mathbb{Z}$

- $A\Gamma \hookrightarrow G \Longrightarrow \forall H \leq_{f} G,\ A\Gamma \hookrightarrow H.$
- residually torsion-free nilpotent (RTFN)

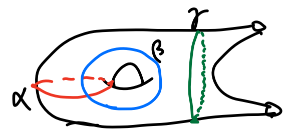
(Farb–Franks 2003) $\forall_{f.g.}$ RTFN $\hookrightarrow \mathrm{Diff}_+^1 I \ (\hookrightarrow \mathrm{Diff}_+^1 S^1)$

(K–Koberda '18) Γ : finite graph

$$A\Gamma \hookrightarrow \mathrm{Diff}_+^2 M^1 \iff (F_2 \times \mathbb{Z}) * \mathbb{Z} \not\hookrightarrow A\Gamma$$

(Koberda '12) $\Rightarrow (F_2 \times \mathbb{Z}) * \mathbb{Z} \hookrightarrow \mathrm{Mod}\, S_{g,p}$ for $3g-3+p \geq 2$

(Baik–K–Koberda '20) $\mathrm{Mod}\, S_{g,p} \xhookrightarrow{\text{vrt}} \mathrm{Diff}_+^2 S^1$
$\iff 3g-3+p \leq 1$



$\langle T_\alpha^N, T_\beta^N, T_\gamma^N, T_\delta^N \rangle \cong (F_2 \times \mathbb{Z}) * \mathbb{Z}$

(e) Critical Regularity

$r \geq 1 \qquad r = k + \tau \qquad k = \lfloor r \rfloor$

$$\mathrm{Diff}_+^r M^1 := \left\{ f : M^1 \xrightarrow{C^k} M^1 \ \middle|\ \sup_{x \neq y} \frac{|f^{(k)}x - f^{(k)}y|}{|x-y|^\tau} < \infty \right\}$$

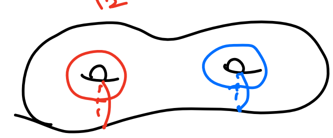
$$\mathrm{CritReg}_{M^1} G := \sup \{ r \geq 1 \mid G \hookrightarrow \mathrm{Diff}_+^r M^1 \}$$

<u>Thm (K–Koberda '20)</u> $\forall r \geq 1$ real, $\exists G_r^{f.g.}$ s.t. $\mathrm{CritReg}\, G_r = r$.

<u>MAIN THM</u> (K–Koberda–Rivas)

$$\forall \tau > 0, \quad (F_2 \times F_2) * \mathbb{Z} \not\hookrightarrow \mathrm{Diff}_+^{1,\tau} M^1$$

$F_2 \times F_2$



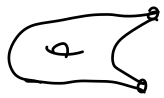
<u>COR</u> (KKR) $\quad 3g-3+p \geq 3$

$$\Longrightarrow \forall H \leq_{f.i.} \mathrm{Mod}(S_{g,p}),\ \mathrm{CritReg}_{M^1}(H) = 1$$

Question (open)

(1) $\mathrm{Mod}(S_{g,p}) \overset{\text{virt}}{\not\hookrightarrow} \mathrm{Diff}_+^1 S^1$?

(2) $(F_2 \times \mathbb{Z}) * \mathbb{Z} \not\hookrightarrow \mathrm{Diff}_+^{1,\tau} M^1$? $\tau > 0$

(3) $\mathrm{Mod}\Big($  $\Big) \underset{\text{virt}}{\not\hookrightarrow} \mathrm{Diff}_+^{1,\tau} M^1$? $\tau > 0$

e Ingredients

abt-Lemma (K–Koberda)

$a, b, t \in \mathrm{Diff}_+^1 M^1$, $\mathrm{supp}\, a \cap \mathrm{supp}\, b = \phi \implies \langle a, b, t \rangle \ncong \mathbb{Z}^2 * \mathbb{Z}$

In fact, for $u_0 := [\,[a^t, b^{tb}\,], a\,]$

$$\overline{\mathrm{supp}\, u_0} \subseteq \mathrm{supp}\, a \cup \mathrm{supp}\, b \cup \mathrm{supp}\, t$$

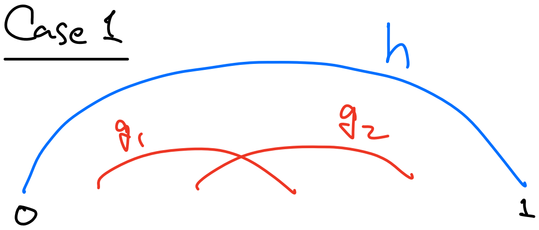
Lemma (KKR, cf. Navas '08)

$G \times H \hookrightarrow \mathrm{Diff}_+^{1,\tau} I$

$\implies \exists k = k(\tau) \gg 0,\ \forall g \in G^{(k)}, \forall h \in H^{(k)}$

$\mathrm{supp}\, g \cap \mathrm{supp}\, h = \phi$

Case 1



Bonatti–Wilkinson–Crovisier

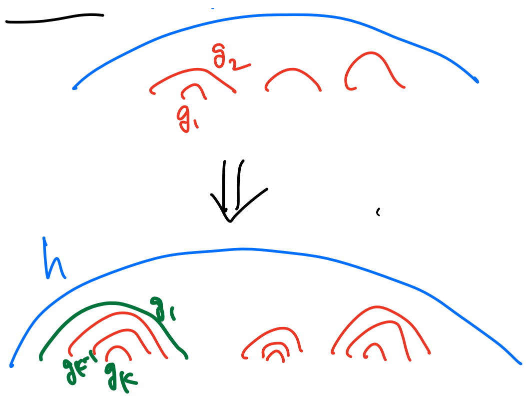
(or Deroin–Kleptsyn–Navas '07)

$\exists g_0, x_0 \in \mathrm{Fix}(g_0)$ s.t. $g_0'(x_0) > 1$

$\implies g_0'(h^n x_0) = g_0'(x_0) > 1$

$h^n(x_0) \longrightarrow \partial I$ $(\Rightarrow\Leftarrow)$

Case 2 h



$k \gg 0$

$\nexists$ chain $\iff \exists$ inv. meas

$\iff \exists$ int. fixed pt. of $g \in [G,G]$

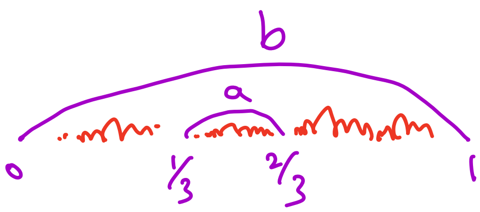
$g_k \in G^{(k)}$, $x_n \longrightarrow \partial I$ s.t.

$$g_k'(x_n) - 1 \geq e^{-V}\left(g_k'(x_0) - 1\right) \geq \varepsilon_k$$

bounded variation

$(\Rightarrow\Leftarrow)$

e lamplighter



$(\langle a\rangle \wr \langle b\rangle) \times \langle t\rangle$

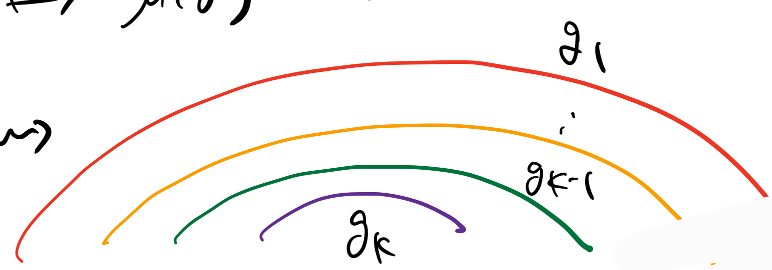
How smooth?

<u>Prop (KKR)</u> The above action admits $C^{\phi-\varepsilon} = C^{1.618\ldots-\varepsilon}$ action

<u>Question</u> Is this optimal?

Key Idea

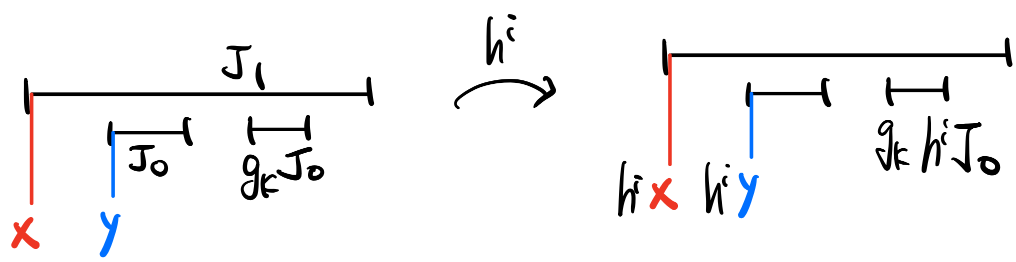
- $\exists$ G-invariant Radon measure on $\mathrm{int}\, I$
- $\tau_\mu : G \to \mathbb{R} \qquad \tau_\mu(g) = \mu[x, gx)$
- $g \in [G, G] \iff \tau_\mu(g) = 0 \implies \mathrm{Fix}\, g \cap \mathrm{int}\, I \neq \emptyset$
- $g_k \in G^{(k)} \rightsquigarrow$



Centralizer-Conradian Lemma (KKR)

$G^{\partial P} \leq \mathrm{Diff}^{1+\varepsilon} I \qquad c \in \mathrm{Diff}^1_+ I$ centralizes G

$\implies G \restriction_{\mathrm{supp}\, c}$ is Conradian



$$g_k'(u_i) = \frac{g_k h^i y - g_k h^i x}{h^i y - h^i x}$$

$$= 1 + \frac{g_k h^i y - h^i y}{h^i y - h^i x} = 1 + \frac{|h^i J_0|}{|h^i J_1|} \overset{i \to \infty}{\geq} 1 + \delta$$