The Kaplansky conjectures

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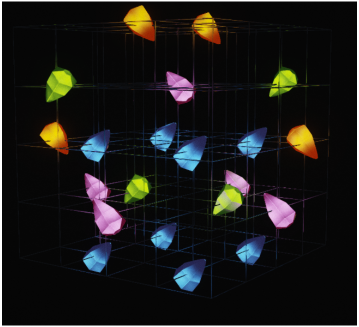


Group rings

Definition

Let G be a group and R be a ring. The group ring $R[G]$ is the ring

$$\left\{ \sum r_g g \mid r_g \in R, g \in G \right\}$$

of finite formal sums with multiplication

$$\left(\sum r_g g \right) \cdot \left(\sum s_h h \right) := \sum (r_g s_h)(gh).$$

If $G = \mathbb{Z} = \langle t \rangle$ then $R[G]$ is just the Laurent polynomials $R[t, t^{-1}]$.

The Kaplansky conjectures

The Kaplansky conjectures (Higman 1940, Kaplansky 1950s+)

Let G be a torsion-free group and let K be a field. Then the group ring $K[G]$ has

- no non-trivial units, i.e. $ab = ba = 1 \implies a = kg, k \in K \setminus \{0\}, g \in G$
- no non-zero zero divisors, i.e. $ab = 0 \implies a = 0$ or $b = 0$
- no non-trivial idempotents, i.e. $a^2 = a \implies a = 0$ or 1

Torsion-freeness is essential: for example, if $g \in G$ has order n then $(1 - g)(1 + g + \cdots + g^{n-1}) = 1 - g^n = 0$ so $1 - g$ is a zero divisor.

Relationship between the conjectures

For each $K[G]$:

unit conjecture $\implies$ zero divisor conjecture $\implies$ idempotent conjecture

A non-trivial idempotent x is a zero divisor since $x(x-1) = x^2 - x = 0$.

Suppose that $ab = 0$ for some non-zero $a, b \in K[G]$. There exists $c \in K[G]$ such that $bca \neq 0$ (Connell 1963). Now $(bca)^2 = bc(ab)ca = 0$ so that $(1 + bca)(1 - bca) = 1$ and we have non-trivial units.

We now know the unit conjecture is *strictly* stronger than the zero divisor conjecture.

A counterexample to the unit conjecture

Theorem (G., 2021)

Let P be the torsion-free group $\langle\, a, b \mid b^{-1}a^2b = a^{-2},\ a^{-1}b^2a = b^{-2} \,\rangle$ and set $x = a^2, y = b^2, z = (ab)^2$. Set

$$
\begin{aligned}
p &= (1+x)(1+y)(1+z^{-1}) \\
q &= x^{-1}y^{-1} + x + y^{-1}z + z \\
r &= 1 + x + y^{-1}z + xyz \\
s &= 1 + (x + x^{-1} + y + y^{-1})z^{-1}.
\end{aligned}
$$

Then $p + qa + rb + sab$ is a non-trivial unit in the group ring $\mathbb{F}_2[P]$.

P is the fundamental group of the Hantzsche–Wendt flat 3-manifold.

$$1 \to \mathbb{Z}^3 \to P \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1$$

A counterexample to the unit conjecture

Picking a suitable isometric action on $\mathbb{R}^3$ realizes the polynomials $p = (1+x)(1+y)(1+z^{-1})$, $q = x^{-1}y^{-1} + x + y^{-1}z + z$, $r = 1 + x + y^{-1}z + xyz$ and $s = 1 + (x + x^{-1} + y + y^{-1})z^{-1}$ as follows

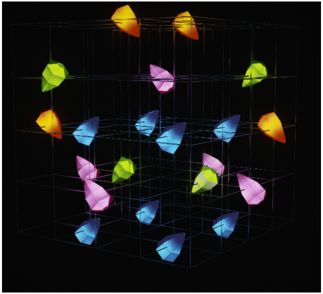


Figure 1: Source: quantamagazine.org

Murray gave non-trivial units for every $\mathbb{F}_q[P]$ (but their supports grow without bound).

The group of units

Corollary (G., 2021)

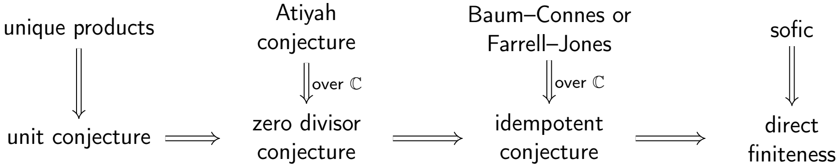
The group of units of $\mathbb{F}_2[P]$ is a torsion-free linear group that is not finitely generated and contains non-abelian free subgroups.

For infinite generation we have to first turn the unit into a family of units, which we can do via affine rescaling. We then map to $\mathbb{F}_2[D_\infty]$ and use Mirowicz's calculation of

$$(\mathbb{F}_2[D_\infty])^\times \cong (*_{j \in \mathbb{Z}} \oplus_{i \in \mathbb{N}^+} \mathbb{Z}/2) \rtimes D_\infty.$$

Conjectures everywhere

The other Kaplansky conjectures are especially interesting because of their relationship to other outstanding questions.



A ring is *directly finite* if $ab = 1 \implies ba = 1$. The direct finiteness conjecture makes no torsion-free assumption.

The Farrell–Jones conjecture implies the unit conjecture holds stably (corresponding Whitehead group is trivial).

Unique products

A naive combinatorial group theoretic condition implies the unit conjecture.

Definition

A group G *has unique products* if for finite subsets $A, B \subset G$ there is always some element uniquely expressible as gh for $g \in A, h \in B$.

E.g. free groups, surface groups, torsion-free nilpotent groups and one-relator groups, free-by-cyclic groups, special groups, many hyperbolic groups, Thompson's group F...

Finding torsion-free groups without unique products is a challenge.

Non-unique product groups

The known groups without unique products come in two flavours:

- small cancellation: Rips–Segev (1987), Steenbock (2015), Gruber–Martin–Steenbock (2015), Arzhantseva–Steenbock (2014+)
- small presentation: Promislow (1988), Carter (2014), Soelberg (2018)

Promislow showed that the Hantzsche–Wendt group P contains a 14-element set A such that $A \cdot A$ has no unique product.

Soelberg's group S is virtually the integral Heisenberg group.

Theorem (G. 2021+)

$\mathbb{F}_2[S]$ has non-trivial units.

The zero divisor conjecture is known for all 3 "small presentation" examples in the literature.

SAT solving

A non-trivial unit is hard to find but easy to verify. The problem

- *given* the multiplication table $A \times B \to G$ for finite sets,
- *decide* if there is a non-trivial solution in $\mathbb{F}_2[G]$ to $ab = 1$ with $\operatorname{supp} a \subseteq A, \operatorname{supp} b \subseteq B$

is in the complexity class NP.

We can naturally formulate it as an instance of the NP-complete problem:

Boolean satisfiability (SAT)

Given a Boolean formula in propositional logic, is there an assignment of the variables to `true` and `false` that makes the formula evaluate to `true` (i.e., that *satisfies* the formula)?

The standard input form for SAT solvers is conjunctive normal form, e.g.

$$(x \vee \overline{y}) \wedge (\overline{x} \vee y \vee z) \wedge (y \vee \overline{z})$$

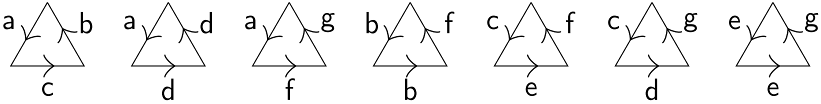
A new candidate group

One can use SAT to approach all the Kaplansky conjectures, as well as related questions: left-orderability and unique product property.

Theorem (G. 2021+)

The torsion-free group $\langle\, a, b \mid aba^2b^{-1}a^2b^{-2}, ab^3ab^4a^{-1}b \,\rangle$ *does not have the unique product property.*

This group is an arithmetic $\widetilde{A_2}$-lattice. A classifying space is:



It has Kazhdan's Property (T) and is not linear over $\mathbb{C}$ $\rightsquigarrow$ known techniques to prove zero divisor conjecture fail. (It does however satisfy both the Baum–Connes and Farrell–Jones conjectures.)

Questions?