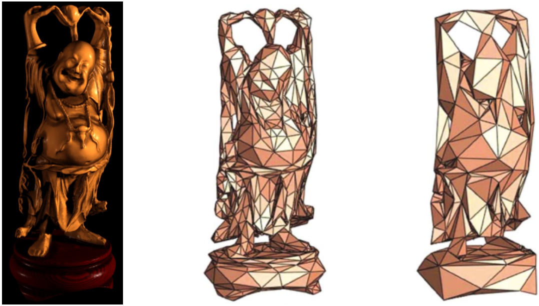
Short canonical decompositions of non-orientable surfaces

Arnaud de Mesmay (CNRS, LIGM, Université Gustave Eiffel, Paris)

Joint work with Niloufar Fuladi and Alfredo Hubard.

Three reasons to decompose a surface

1. Practical matters: surface parameterization.



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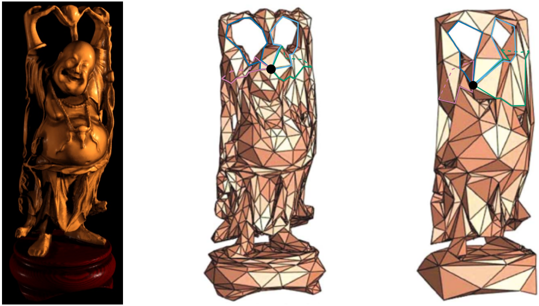

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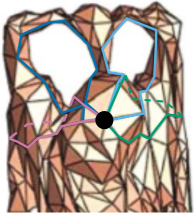
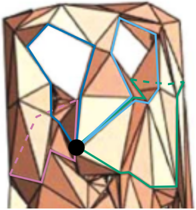


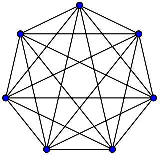


- In order to compute a parameterization (i.e., homeomorphism) between two surfaces of non-zero genus, the first step is generally to cut them open into a disk.
- One way to do that is to cut along a fixed system of loops.

Three reasons to decompose a surface

2. Visualization: How to represent an embedded graph?

- The complete graph on 7 vertices can be drawn without crossings on a torus.



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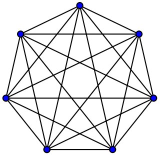

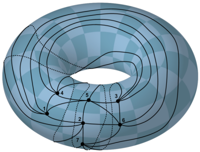

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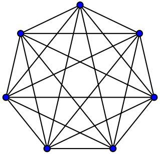

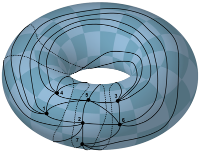

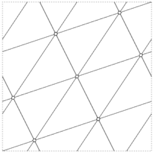

Three reasons to decompose a surface

3. Combinatorial group theory: How to change bases?

- Given an orientable surface S, and a family of simple curves inducing a presentation of the fundamental group:

$$\pi_1(S) = < a, b, c, d \mid abcd\overline{abcd} = 1 >$$

- How do I go from this presentation to my "favorite" presentation?

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- Here, one can take $a_1 = \overline{dc}a$, $b_1 = bcd$, $a_2 = \overline{c}$ and $b_2 = \overline{d}$. In general, how to bound the length of these words?

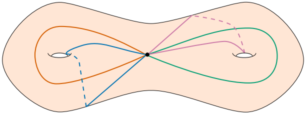
Joint crossings

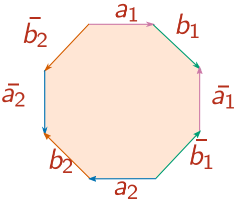
- In all three questions we aim to control the *complexity* of some decomposition.
- A graph G embedded on a surface S is an injective map $G \to S$.
- An embedding is *cellular* if the faces are topological disks.

Cutting one graph along another

Let G_1 and G_2 be two graphs cellularly embedded on a surface S of genus g. Is there a homeomorphism $h : S \to S$ such that $h(G_1)$ and G_2 cross transversely and not too much?

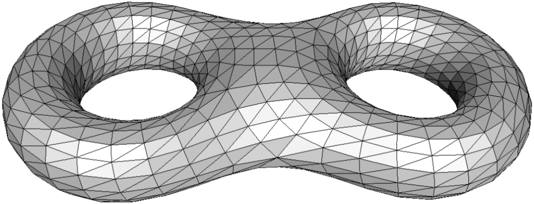
For the examples above, pick for G_2 my favorite embedded graph, the *orientable canonical system of loops*:





Graph duality

To any cellularly embedded graph one can associate a *dual graph* where vertices and faces are inverted.



Following an edge in the primal graph is the same as *crossing* the dual edge.

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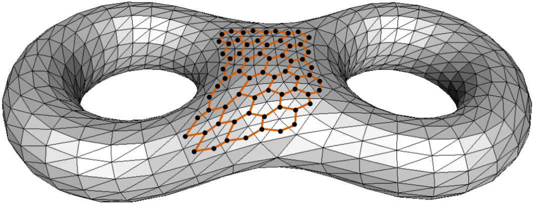

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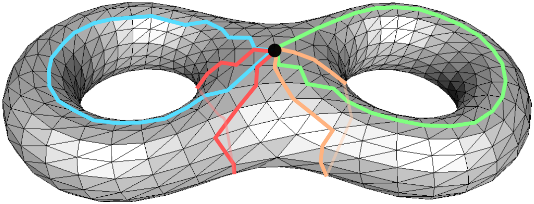

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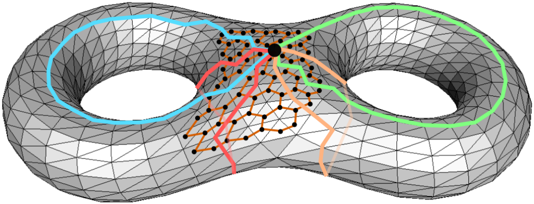

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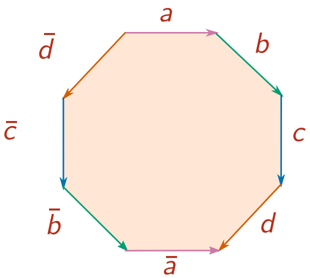

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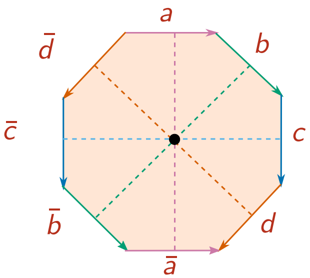

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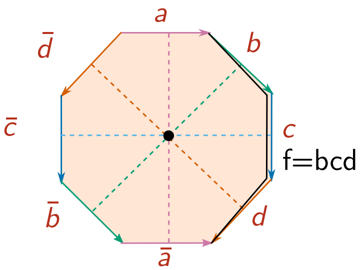
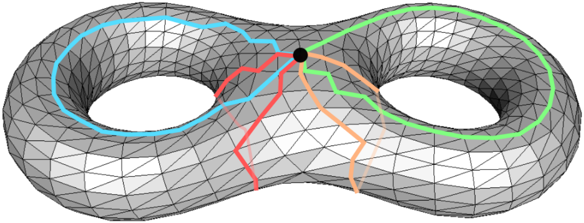


Following an edge in the primal graph is the same as *crossing* the dual edge.

Canonical decomposition of orientable surfaces

Theorem (Lazarus, Pocchiola, Vegter, Verroust '01)

Let G be a graph embedded on an orientable surface S of genus g. Then there exists a canonical system of loops, so that each loop crosses each edge of the graph at most 4 times. Dually, there exists a canonical system of loops so that each loop uses each edge of the graph at most 4 times.



In terms of length, the canonical system of loops has length $O(gn)$, this is tight.

Other cutting shapes?

- What if my favorite embedded graph is not the canonical system of loops? Perhaps a polygonal scheme of the form $a_1 \ldots a_{2g} \overline{a_1 \ldots a_{2g}}$?

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- What if my favorite embedded graph is not the canonical system of loops? Perhaps a polygonal scheme of the form $a_1 \dots a_{2g} \overline{a_1 \dots a_{2g}}$?
- This is an open problem.

Negami's conjecture

Let G_1 and G_2 be two graphs cellularly embedded on a surface S of genus g. Is there a homeomorphism $h : S \to S$ such that each edge of $h(G_1)$ crosses each edge of G_2 at most a constant number of times?

Best known bound:

Theorem (Negami '01)

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Canonical decompositions of non-orientable surfaces

- What about non-orientable surfaces? Can I at least cut along my favorite system of loops $a_1 a_1 \dots a_g a_g$?

Canonical decompositions of non-orientable surfaces

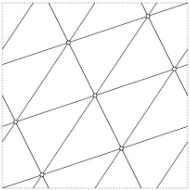
- What about non-orientable surfaces? Can I at least cut along my favorite system of loops $a_1 a_1 \dots a_g a_g$?
- Our main result is a positive answer:

Theorem (Fuladi,Hubard, dM '21+)

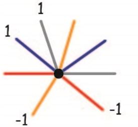
Let G be a graph embedded on a non-orientable surface S of genus g. Then there exists a canonical system of loops, so that each loop crosses each edge of the graph at most 30 times. Dually, there exists a canonical system of loops so that each loop uses each edge of the graph at most 30 times.

Reduction to the one-vertex case

- In both graphs one can contract a *spanning tree*, solve the problem on one-vertex graphs and uncontract the spanning tree locally.



- Such a one-vertex graph is entirely described by a *rotation system*: the cyclic ordering of the edges around the vertex, and, in the non-orientable case, the sidedness of the curves.



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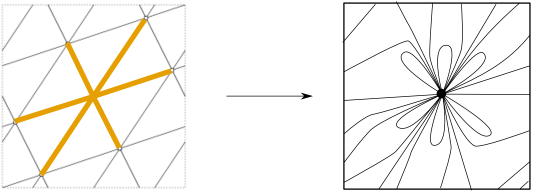

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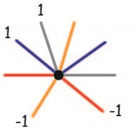
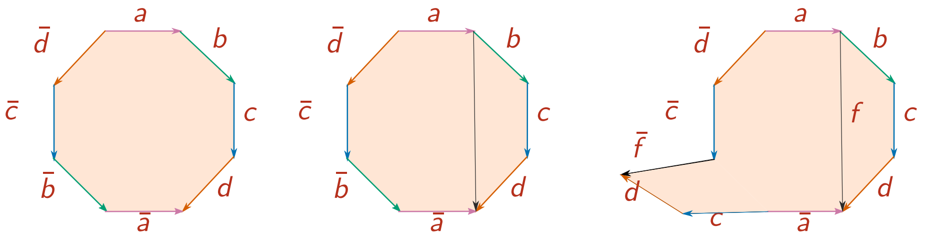


The orientable case: back to the classification of surfaces

Let's go back to our third problem.



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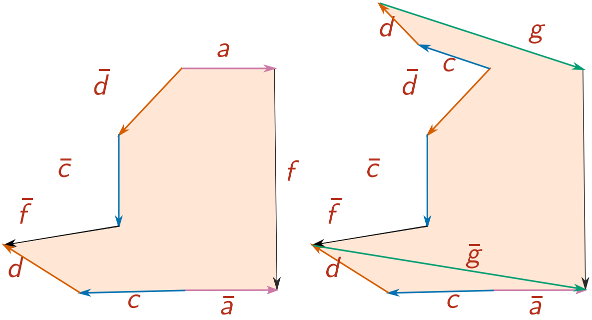
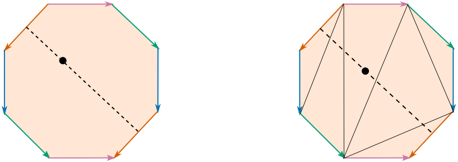


$f = bcd$, $g = \overline{dc}a$ and we are done.

Remarks about this approach

- There are $O(g)$ cut-and-pasting steps.
- One must be very careful about not reusing edges, otherwise the size of the solutions blows up.
  $\rightarrow$ This is why this proof only works for the canonical system of loops.
- Any graph can be reduced to a one-vertex graph, but if there are more edges in the graph, it gets trickier.



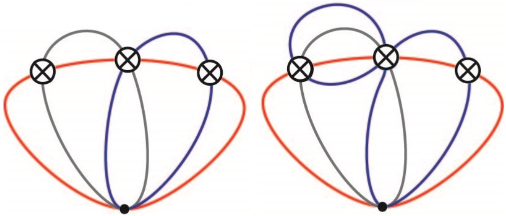
- In the non-orientable case, there are additional cut-and-pasting steps causing an $O(g)$-overhead.

A different approach

Theorem (Schaefer-Štefankovič '15)

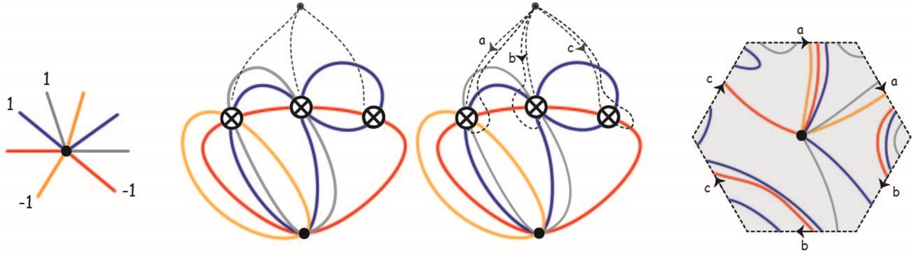
Any graph embeddable on a non-orientable surface can be embedded in a way that each edge crosses each cross-cap at most twice.

- Here we are talking about embeddings where cross-caps are *localized*.



- It is a conjecture of Mohar ('2009) that the theorem holds with *twice* replaced by *once* (when allowing to change the homeomorphism class of the embedding).

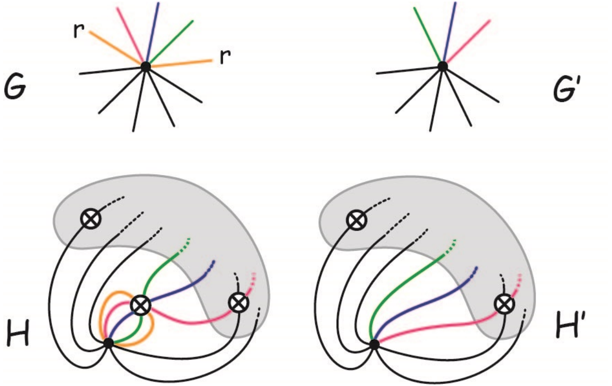
From cross-cap drawings to canonical systems of loops



- If one can control the (dual) diameter of the resulting drawing, one can control the length of the resulting system of loops.

Sketch of proof for the cross-cap drawings

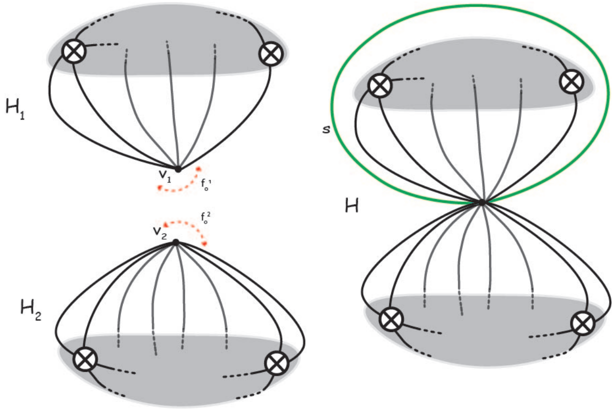
1. Induct on each loop depending on its topological type. Use the Euler characteristic as an accounting device to know that the correct number of crosscaps is used.



2. The hardest curves to deal with are the *separating curves*.
3. *Our main contribution:* Fine control on the complexity of the resulting drawing to be able to connect the crosscaps.

The case of separating curves

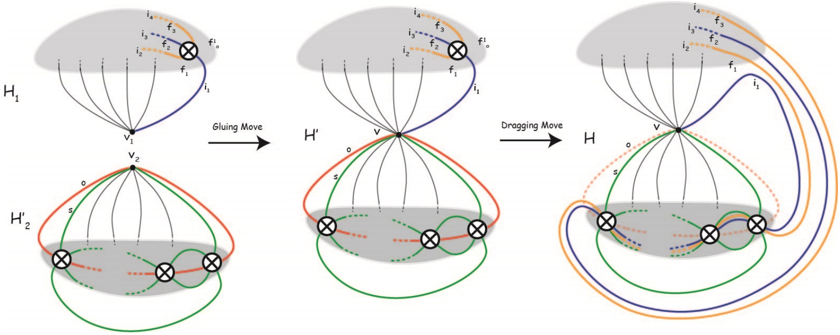
- Adding a separating curve between two non-orientable drawings is easy.



The case of separating curves

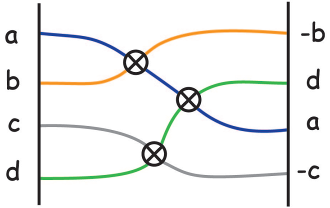
- Adding a separating curve between two non-orientable drawings is easy.

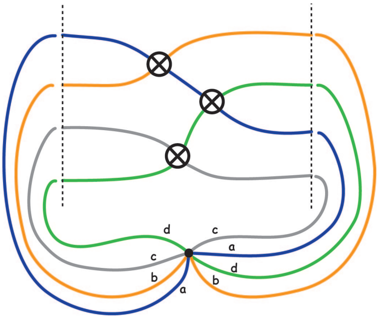
- But a graph of orientable genus g may require $2g+1$ crosscaps to be drawn
  $\rightarrow$ one needs to save a crosscap when one of the sides is orientable.



A completely different problem

- The *signed reversal distance* between two signed words is the minimum number of reversals to go from one to the other one.
- Very important in *computational biology*, computable in *polynomial time* [Hannenhalli-Pevzner '99].
- Strong similarities with crosscap drawings, which we leverage in our proof.





Another conjecture to finish

Negami's conjecture

Let G_1 and G_2 be two graphs with at most n edges embedded on a surface S of genus g. is there a homeomorphism $h : S \to S$ such that each edge of $h(G_1)$ crosses each edge of G_2 at most a constant number of times?

- If G_1 and G_2 are simple graphs (no loops and multiple edges), it is even open if one can achieve a *single* crossing.
- Two shortest paths cross at most once, hence:

Universal shortest path metric

Given a surface S of negative Euler characteristic, is there a [hyperbolic] metric on S so that any simple graph embeddable on S can be embedded so that edges are realized as shortest paths?

- In the plane this is Fàry's theorem.
- We [HKdMT '15] studied this problem in the orientable case and showed that *most* hyperbolic metrics do not work as $g \to \infty$.

And many open questions

1. What is the computational complexity of computing the *shortest* canonical system of loops? The *shortest* pants decomposition?
   - Not know to be polynomial-time nor NP-hard.
2. Canonical systems of loops allow cutting a graph with length $O(gn)$ and this is tight. Is there a better canonical cutting shape?
   - Known lower bound: $\Omega(n^{7/6})$ [Colin de Verdière Hubard dM'14].
3. Any system of loops can be turned into a canonical one which has total word length $O(g^2)$. Is it tight? (asked by [Lazarus '16]).

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Thank you! Questions?

One more move.

Concatenation move: