

Da vide Spriano

CURTAINS, WALLS AND STABLE CYLINDERS

$G \times T$ Oct 25

PLAN

CAT(0) CUBE COMPLEXES

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CAT(0) CUBE COMPLEXES



HYPERPLANES / SPACE WITH WALLS
[Sageev, Chatterji - Niblo, Nica]

▣ CUBICAL WORLD

▣ NON CUBICAL EXAMPLE

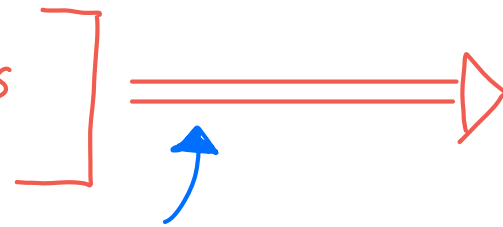
▣ FORMALISM

PLAN

CAT(0) CUBE COMPLEXES



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■ CUBICAL WORLD

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■ FORMALISM

DUALISABLE
SYSTEMS [Petit-Zolloum]

MOTIVATION / FIRST APPLICATION

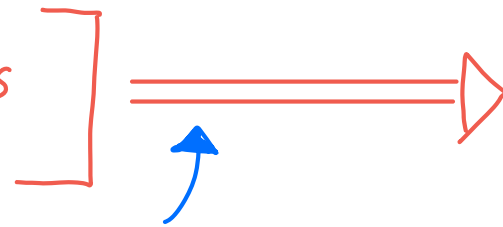
CAT(0) SPACES [Petit-S-Zolloum]

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MOTIVATION / FIRST APPLICATION

CAT(0) SPACES [Petit-S-Zolloum]

Thm [LAZAROVICH-SAGEEV]

Hyperbolic CUBICAL GROUPS ADMIT
GLOBALLY STABLE CYLINDERS

OVERVIEW / RECALL :

CAT(0) CUBE COMPLEXES

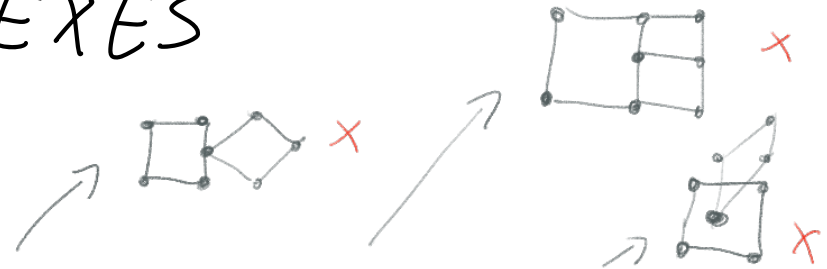
OVERVIEW / RECALL :

CAT(0)

CUBE COMPLEXES

↳ CUBES $([0, 1]^n)$

GLUED ALONG SURJECTIVE ISOMETRIES ONTO FACES

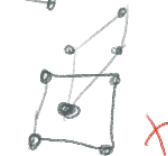


OVERVIEW / RECALL :

CAT(0) CUBE COMPLEXES

↳ CUBES $([0,1]^n)$

GLUED ALONG SURJECTIVE ISOMETRIES ONTO FACES



- SIMPLY CONNECTED
- LINK CONDITION

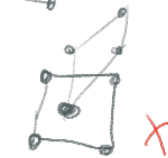
well behaved hyperplanes

OVERVIEW / RECALL :

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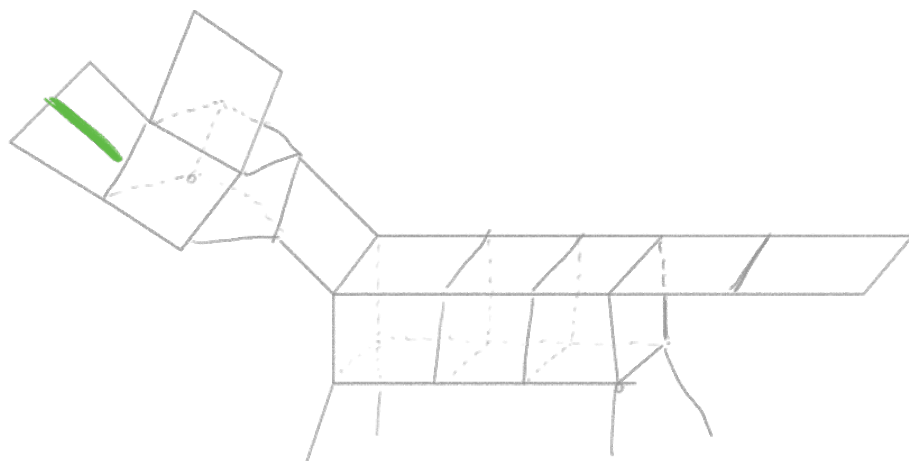
GLUED ALONG SURJECTIVE ISOMETRIES ONTO FACES



-
- SIMPLY CONNECTED
 - LINK CONDITION

well behaved hyperplanes

→ ① RESTRICT A COORDINATE TO $\frac{1}{2}$

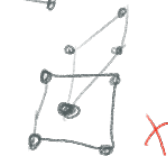


OVERVIEW / RECALL :

CAT(0) CUBE COMPLEXES

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GLUED ALONG SURJECTIVE ISOMETRIES ONTO FACES



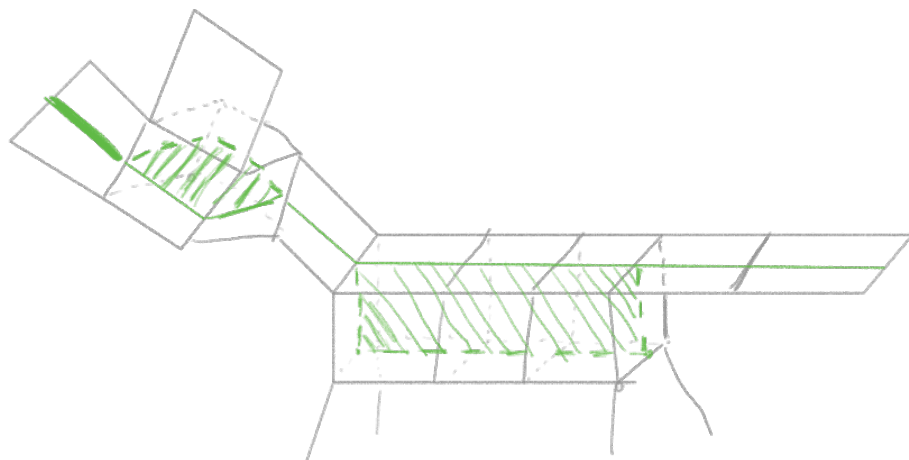
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well behaved hyperplanes

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② PROPAGATE



OVERVIEW / RECALL :

CAT(0) CUBE COMPLEXES

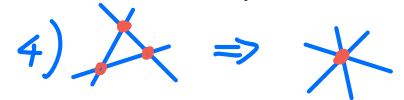
↳ CUBES $([0,1]^n)$

GLUED ALONG SURJECTIVE ISOMETRIES ONTO FACES

1) SEPARATE

2) 2-sided

3) CONVEX / NO SELF-INTERSECT

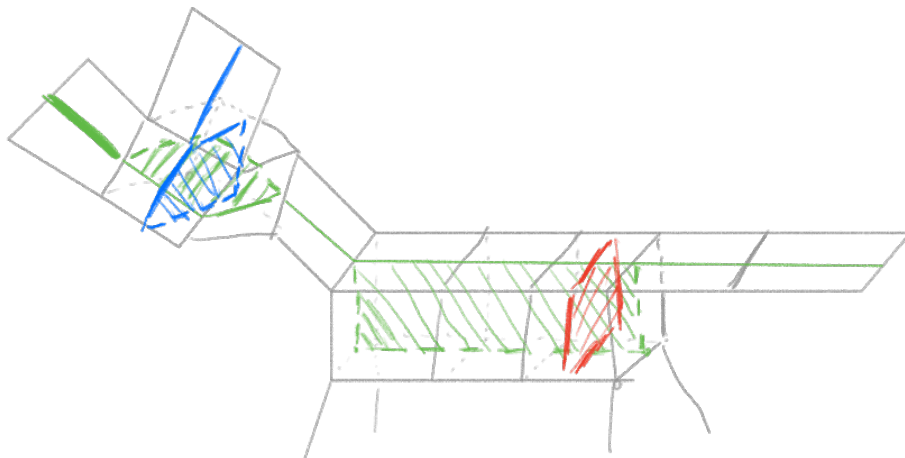
4)  $\Rightarrow$ 

well behaved hyperplanes

① RESTRICT A COORDINATE TO $\frac{1}{2}$



② PROPAGATE



- $\left(\begin{array}{l} \bullet \text{SIMPLY CONNECTED} \\ \bullet \text{LINK CONDITION} \end{array} \right)$

Key FACT:

X A CAT(0) CUBE COMPLEX, $X^{(1)}$ IS A GRAPH $\rightsquigarrow$ METRIC $d_{(1)}$

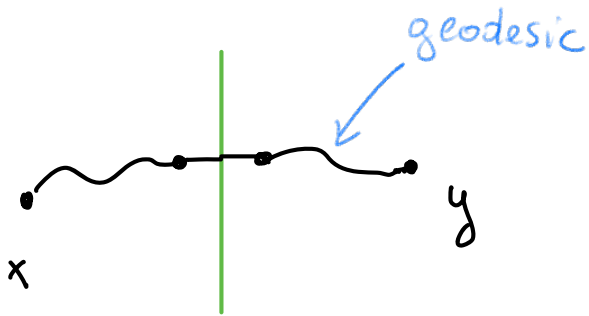
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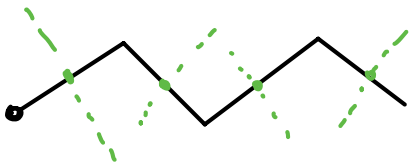
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Sketch



WALLS = length of geo

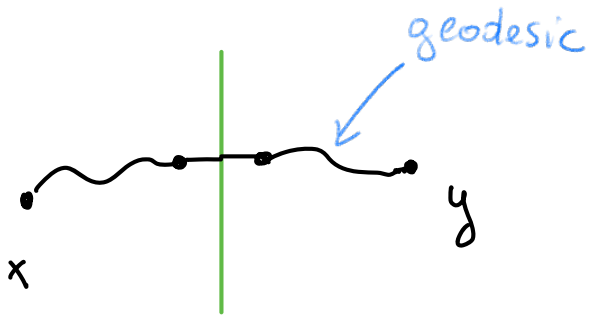


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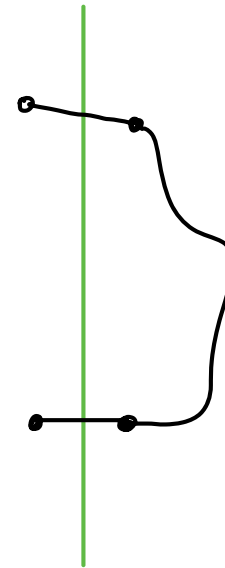
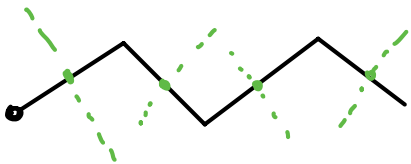
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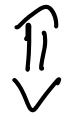
WALLS = length of geo



↗ Convexity

ASIDE / DIFFERENT PERSPECTIVE

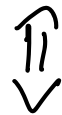
Thm: Γ_{GRAPH} IS 1-SKELETON OF A CAT(0) CUBE CX



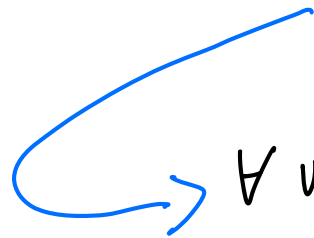
Γ IS A MEDIAN GRAPH

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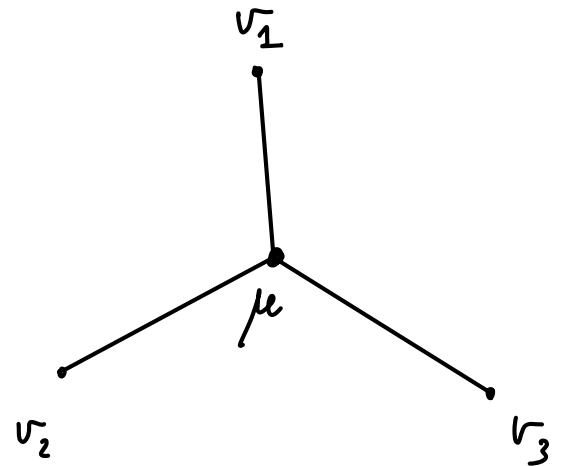
Γ is a MEDIAN GRAPH



$\forall v_1, v_2, v_3 \in V(\Gamma)$

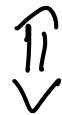
$\exists! \mu = \mu(v_1, v_2, v_3) :$

$$d(v_i, v_j) = d(v_i, \mu) + d(\mu, v_j)$$



ASIDE / DIFFERENT PERSPECTIVE

Thm: Γ_{GRAPH} is 1-SKELETON OF A CAT(0) CUBE CX

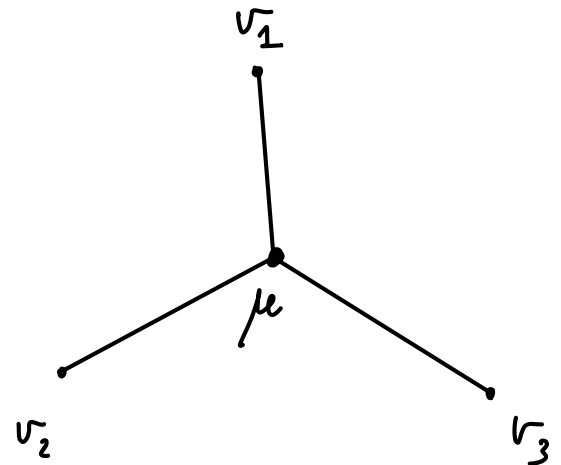
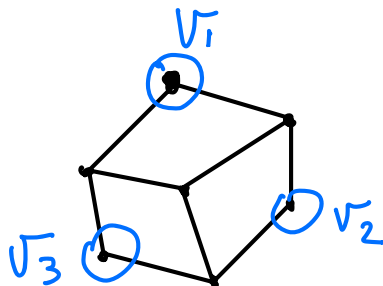


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NON EX



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GEOMETRY $\longleftrightarrow$ HYPERPLANES

GIVEN

- SET OF VERTICES
- SET OF HYPERPLANES
- WHICH VERTICES EACH Hyp.
IS SEPARATING

ONE CAN RECONSTRUCT $X^{(1)}$ (AND X)

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$\rightarrow X^{(0)} = h^- \sqcup h^+$

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WALLSPACES

DEF A **SPACE WITH WALLS** is a PAIR (X, W) where
[X is a set, $h \in W$ is a bipartition $X = h^+ \sqcup h^-$ *non empty*
SO THAT
• $\forall x, y \in X, |\{h \in W \text{ separating } x \text{ from } y\}| < \infty$

WALLSPACES

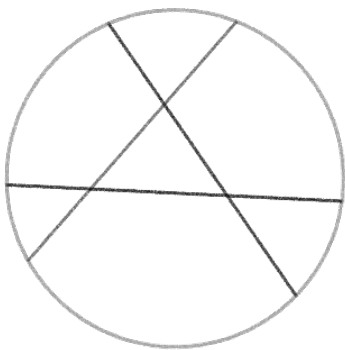
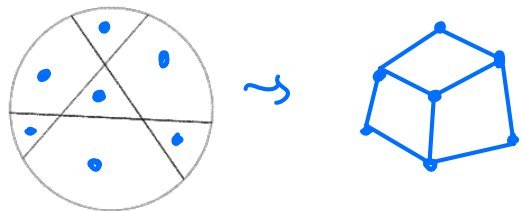
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NAÏVE
 DUAL OF (X, W)

HAS

- VERTICES = $\coprod_0 (X \setminus W)$
- EDGES = SEP. BY ONE WALL



WALLSPACES

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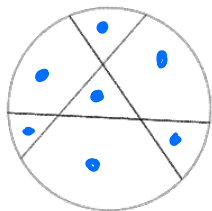
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NAÏVE

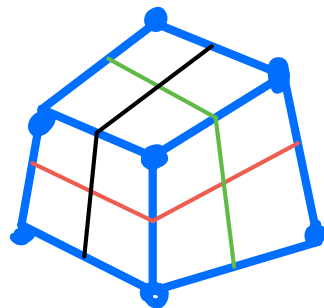
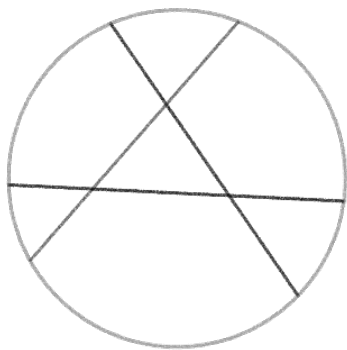
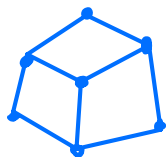
DUAL OF (X, W)

HAS

- VERTICES = $\prod_0 (X \setminus W)$
- EDGES = SEP. BY ONE WALL



$\leadsto$



NOT MEDIAN

WALLSPACES

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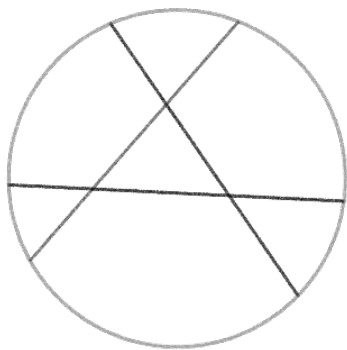
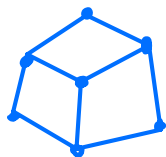
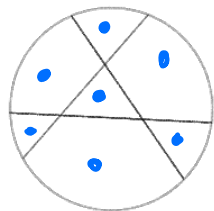
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NAÏVE

DUAL OF (X, W)

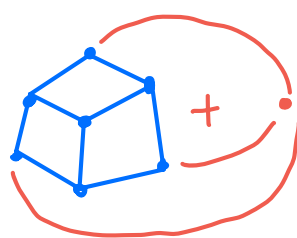
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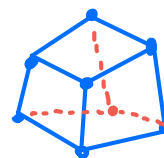


SAGEEV'S INTUITION

THE NAÏVE DUAL CAN BE
 COMPLETED TO A CAT(0) CUBE CPLX



=



WALLSPACES

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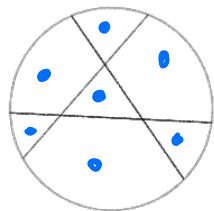
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NAÏVE

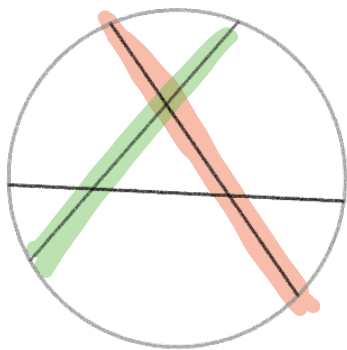
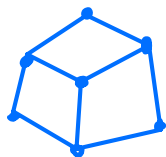
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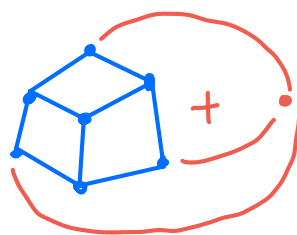


$\leadsto$



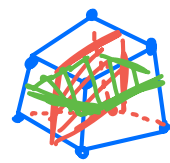
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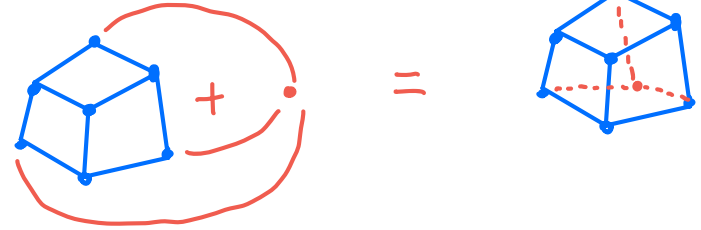
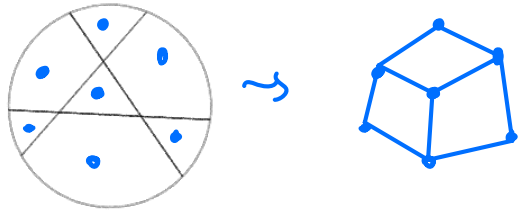


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ULTRAFILTERS



REGIONS BETWEEN N WALLS



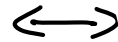
COMING FROM
THE ORIGINAL SPACE



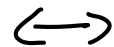
BEING SEPARATED BY A WALL



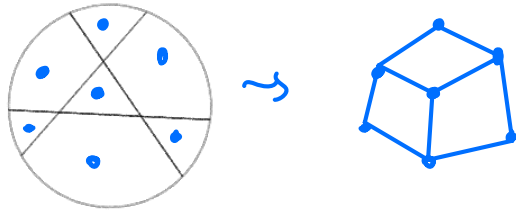
DISTANCE



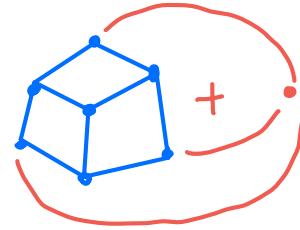
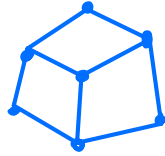
DUAL SPACE



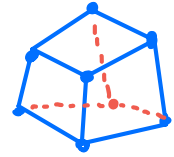
ULTRAFILTERS



$\leadsto$



$=$



REGIONS BETWEEN N WALLS

$\longleftrightarrow$

ULTRAFILTERS φ

$\hookrightarrow$ COMING FROM
THE ORIGINAL SPACE

$\longleftrightarrow$

BEING SEPARATED BY A WALL

$\longleftrightarrow$

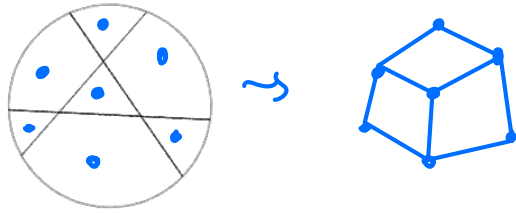
DISTANCE

$\longleftrightarrow$

DUAL SPACE

$\longleftrightarrow$

ULTRAFILTERS

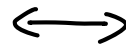


REGIONS BETWEEN N WALLS



ULTRAFILTERS φ

COMING FROM
THE ORIGINAL SPACE

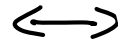


$i: X \rightarrow \text{ULTRA}(W)$
 $x \mapsto \varphi_x$

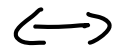
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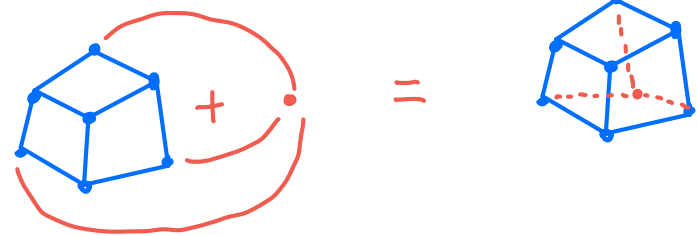
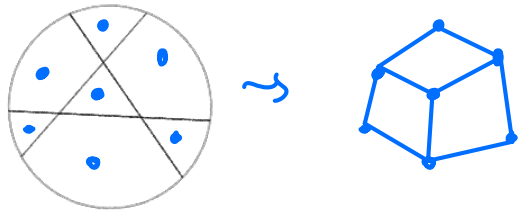
DISTANCE



DUAL SPACE



ULTRAFILTERS

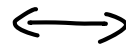


REGIONS BETWEEN N WALLS



ULTRAFILTERS φ

COMING FROM
THE ORIGINAL SPACE



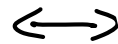
$i: X \longrightarrow \text{ULTRA}(W)$
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BEING SEPARATED BY A WALL

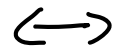


$\varphi(h) \neq \psi(h)$

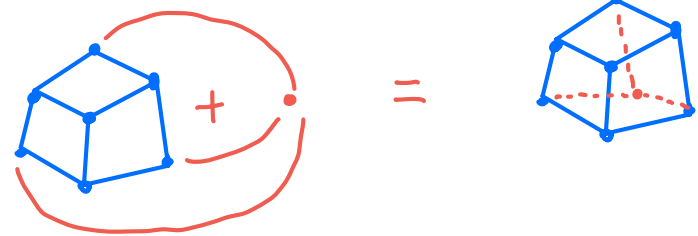
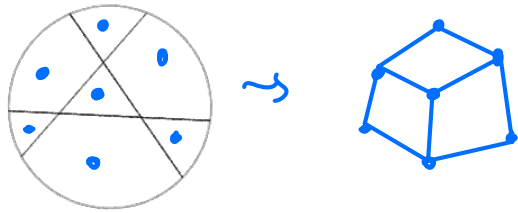
DISTANCE



DUAL SPACE



ULTRAFILTERS



REGIONS BETWEEN N WALLS



ULTRAFILTERS φ

COMING FROM
THE ORIGINAL SPACE



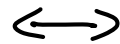
$i: X \longrightarrow \text{ULTRA}(W)$
 $x \longmapsto \varphi_x$

BEING SEPARATED BY A WALL



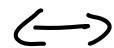
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DISTANCE

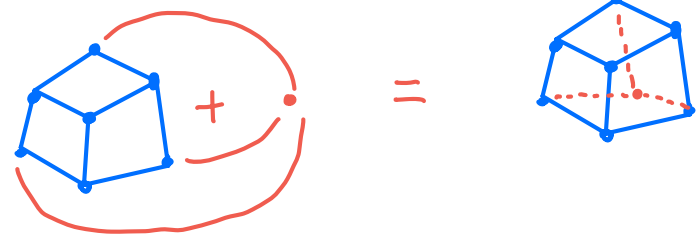
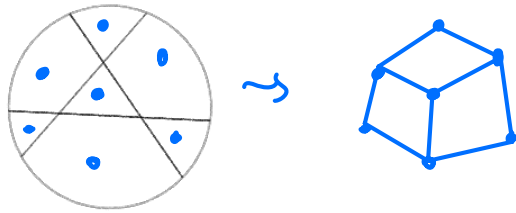


$d(\varphi, \psi) = |\{h \in W: h \text{ separates}\}|$

DUAL SPACE



ULTRAFILTERS

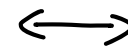


REGIONS BETWEEN N WALLS



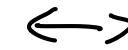
ULTRAFILTERS φ

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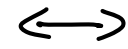
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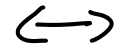
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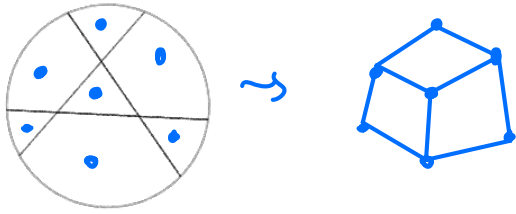
$d(\varphi, \psi) = |\{h \in W: h \text{ separates}\}|$

DUAL SPACE



$\hat{X}$

ULTRAFILTERS

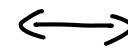


REGIONS BETWEEN N WALLS



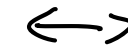
ULTRAFILTERS φ

COMING FROM
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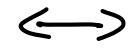
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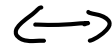
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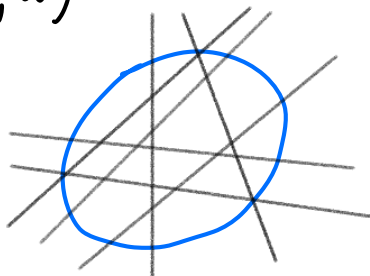


$\hat{X}$

Thm1 [Sageev, Chatterji-Niblo, Nice]
 $\hat{X}$ is a CAT(0) cube complex

A word of WARNING

(X, w)



$\hat{X}$

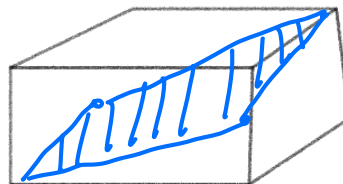
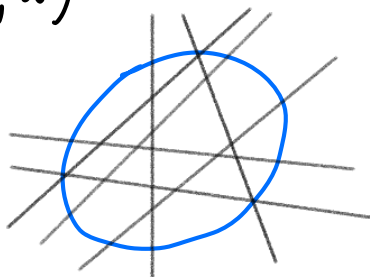


Image of X can be small in $\hat{X}$

A word of WARNING

(X, w)



$\hat{X}$

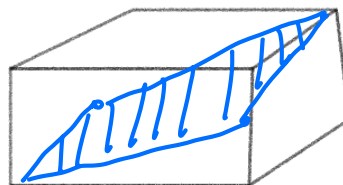
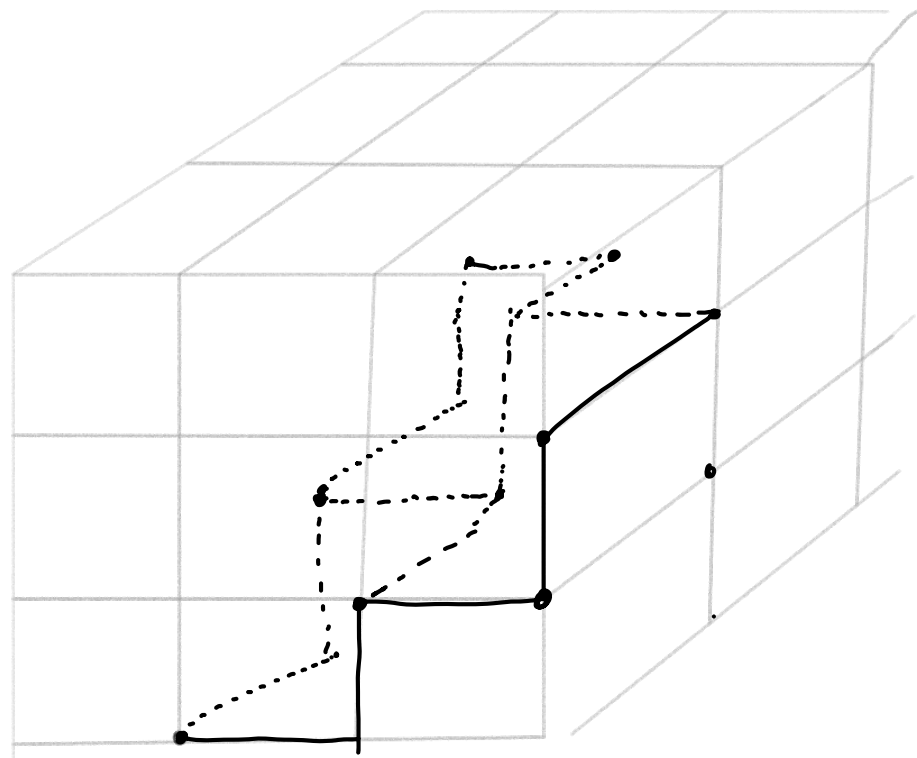
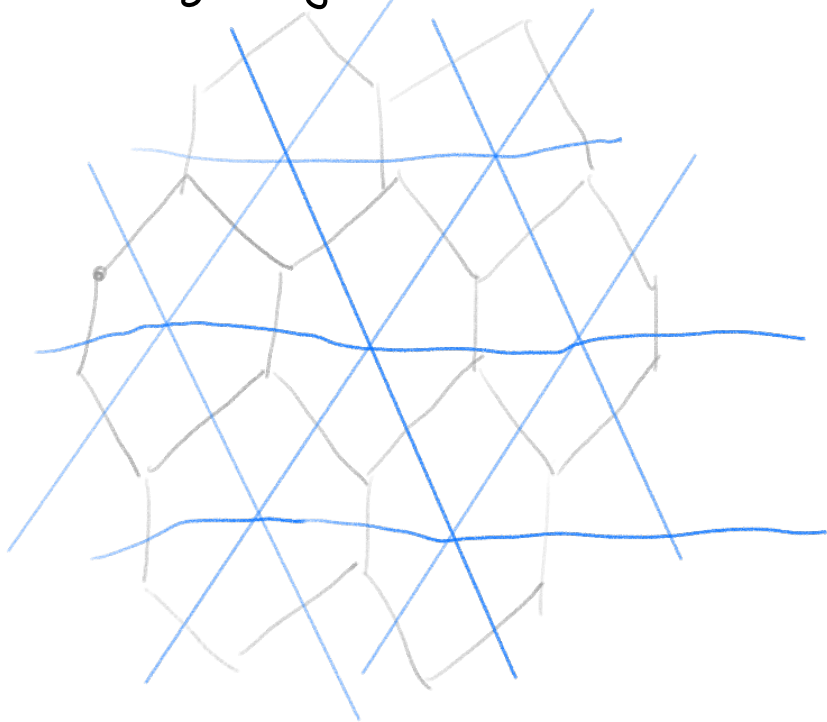
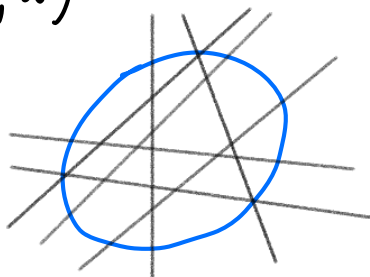


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$\hat{X}$

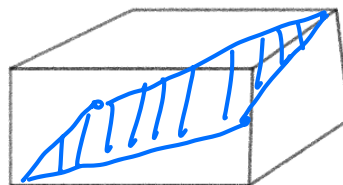
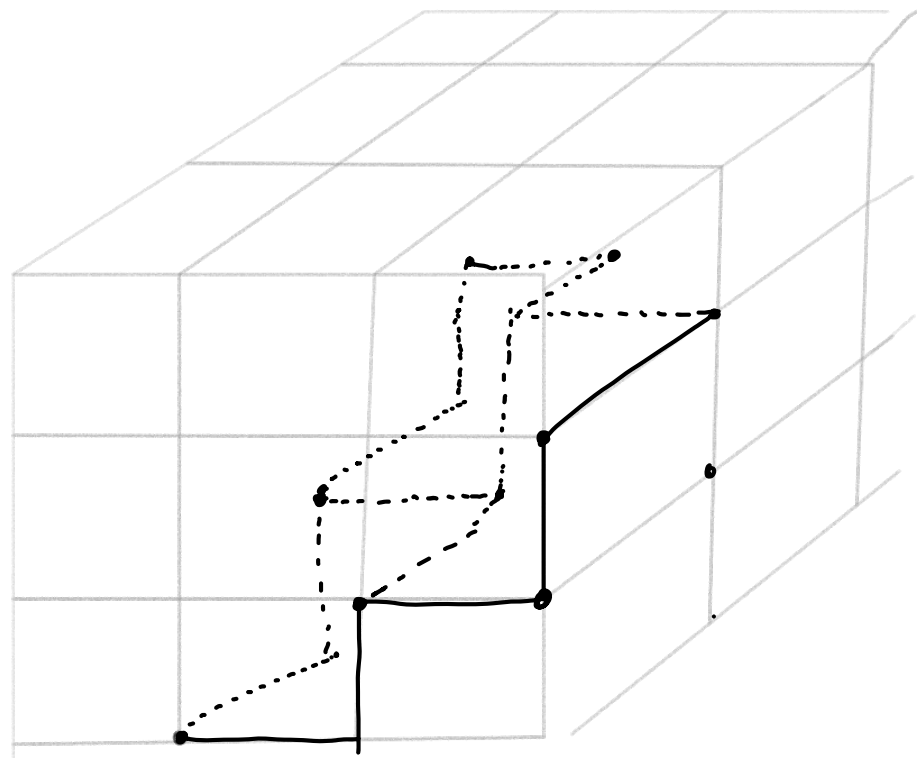
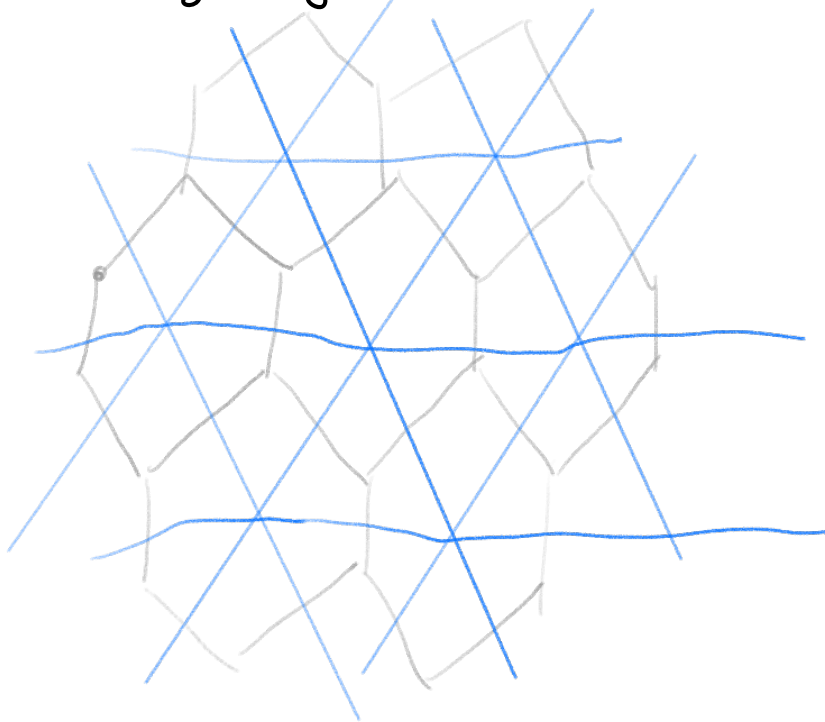


Image of X can be small in $\hat{X}$



→ "FINE" if X is hyperbolic

Beyond SAGEEV'S CONSTRUCTION

RECALL: (X, W) ^{BIPARTITIONS} S.T. $|\{h \in X \text{ sep } x, y\}| < \infty$

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WALLS IN CAT(0) SPACES

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WALLS IN CAT(0) SPACES

- UNIQUE GEODESICS
- γ geod, $\pi_\gamma : X \rightarrow \gamma$ closest point projection is well-def ($\exists!$)



Beyond SAGEEV'S CONSTRUCTION

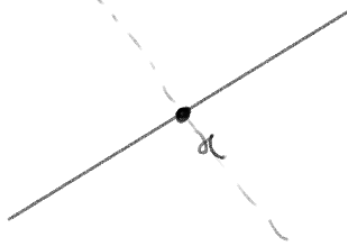
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Naïve idea: $\gamma = [a, b]$, $x \in (a, b) \leadsto$ WALLS $\pi^{-1}((a, x]) \nmid \pi^{-1}((x, b))$



Beyond SAGEEV'S CONSTRUCTION

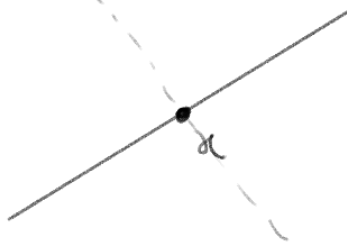
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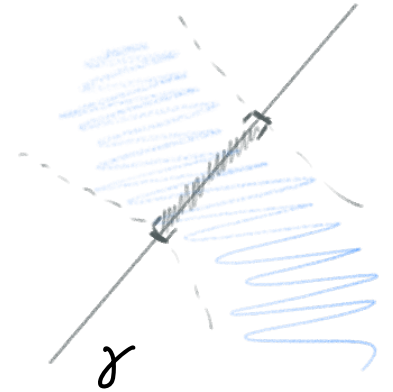


Naïve
WALLS

$d(x, y) = \infty$ for $x \neq y$

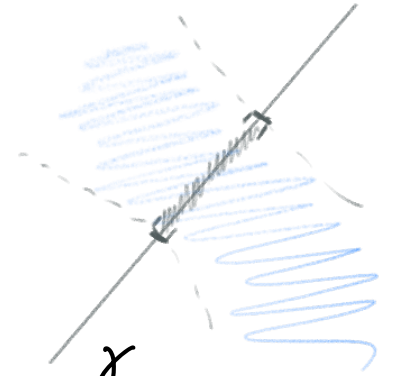
 CURTAINS 

Def A CURTAIN is $\pi_\gamma^{-1}((\text{PROPER}) \text{ UNIT SUBINTERVAL})$



CURTAINS

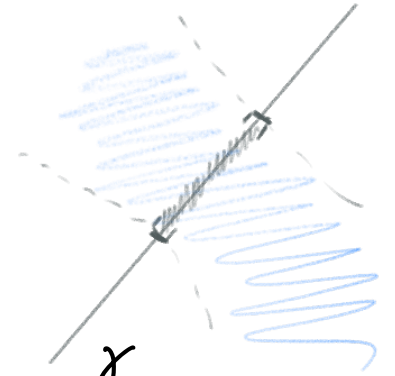
Def A CURTAIN is $\pi_Y^{-1}((\text{PROPER}) \text{ UNIT SUBINTERVAL})$



$$d_{\text{CURTAIN}}(x, y) = \left| \left\{ \text{MAX Set of } \underline{\text{DISJOINT}} \text{ CURTAINS sep. } x, y \right\} \right| (+1)$$

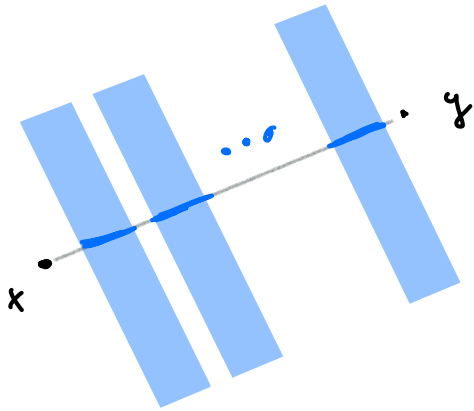
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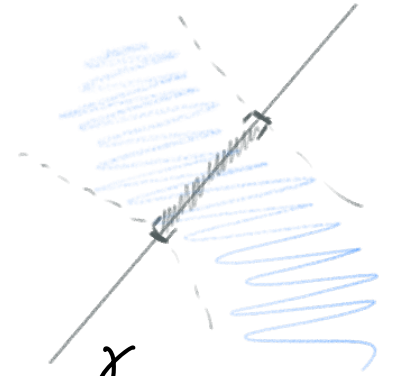
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Ex | $\mathbb{R}^2$



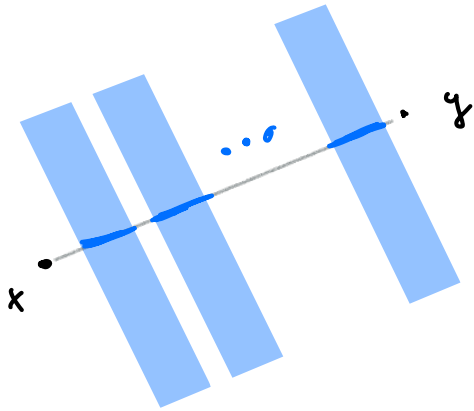
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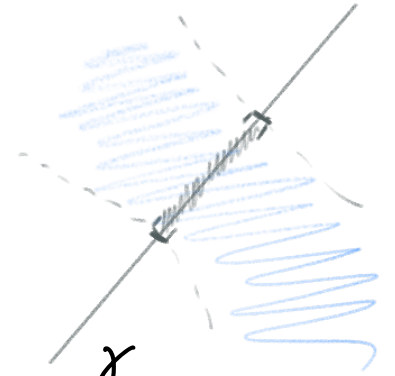
Ex | $\mathbb{R}^2$



$$\leadsto \left| d_{\text{CURTAIN}}(x, y) - d(x, y) \right| \leq 1$$

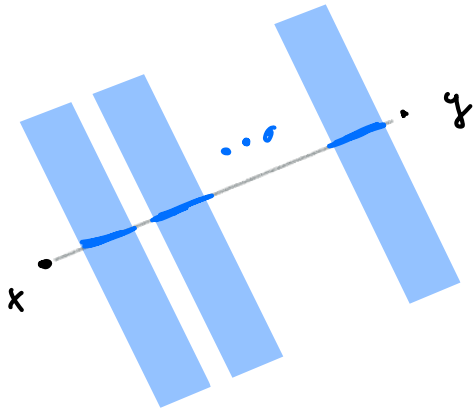
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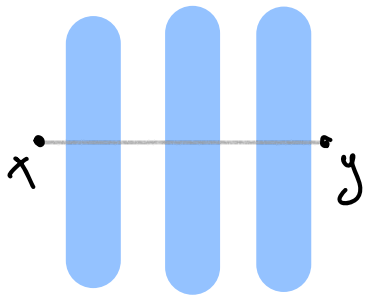
What did we gain?

-) "Cubical arguments" for CAT(0) spaces
-) Setup for more general dual construction

$X_{CAT(o)}$, $W = \{ \text{All walls coming from } \begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array} \} \}$

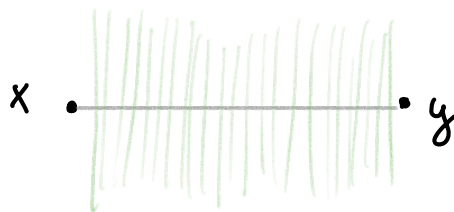
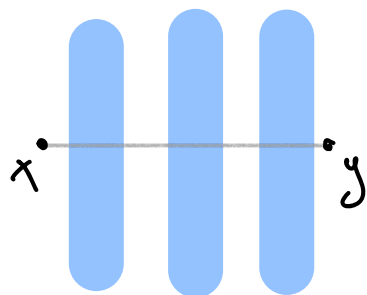
$X_{CAT(0)}$, $W = \{ \text{All walls coming from } \begin{array}{c} \text{---} \backslash \text{---} \\ \text{---} / \text{---} \end{array} \}$

$d(x, y) \sim \left\{ \begin{array}{l} \text{max set disj. CURTAINS} \\ \text{SEPARATING } x, y \end{array} \right\}$



$X_{CAT(0)}$, $W = \{ \text{All walls coming from } \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \}$

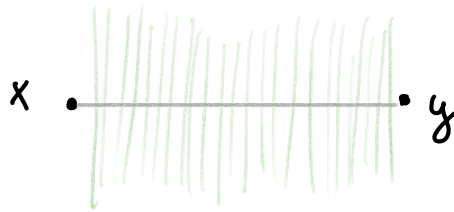
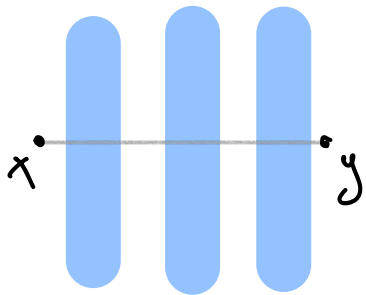
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$\triangle \infty$ many disj walls

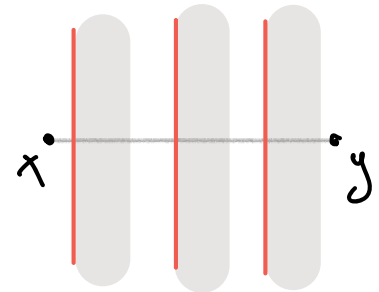
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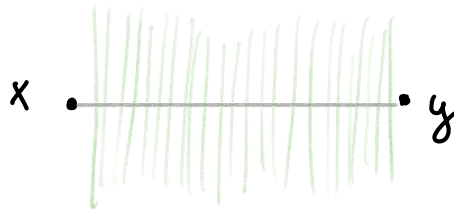
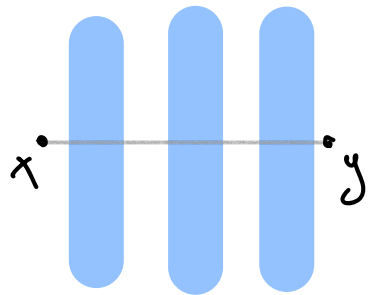
$\triangle \infty$ many disj walls

look only at
"far away" walls



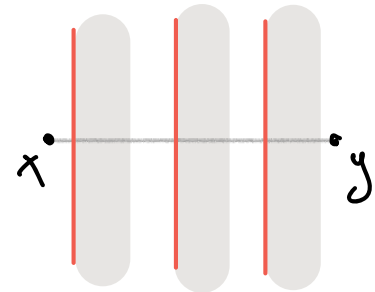
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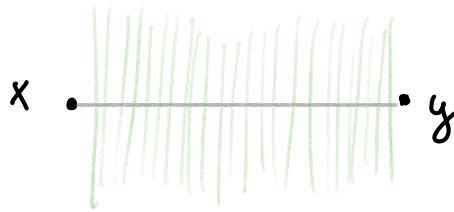
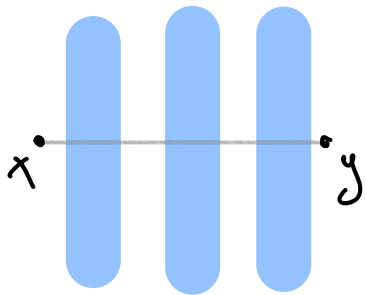


Formally:

$\mathcal{G} \subseteq 2^W$ defined $c \in \mathcal{G} \iff c = \{ \text{a set of walls coming from disjoint CURTAINS} \}$

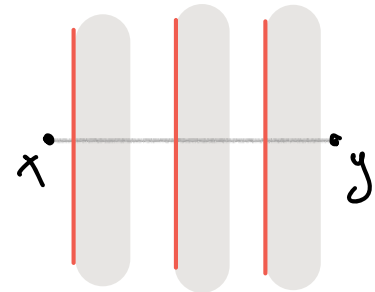
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$\triangle \infty$ many disj. walls

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Formally:

$\mathcal{G} \subseteq 2^W$ defined $c \in \mathcal{G} \Leftrightarrow c = \{ \text{a set of walls coming from disjoint CURTAINS} \}$

$d_{CURTAIN}(x, y) = \max_{c \in \mathcal{G}} \{ |c| : c \text{ separates } x \text{ from } y \}$

BOARD

Overview of Thm A

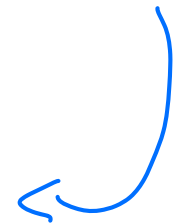
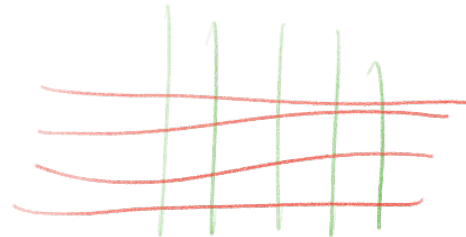
OVERVIEW OF THM A

INPUT: CAT(0) SPACE CURTAIN
MODEL  COLLAPSE ALL FLAT REGIONS

OVERVIEW OF THM A

INPUT: CAT(0) SPACE CURTAIN MODEL → COLLAPSE ALL FLAT REGIONS

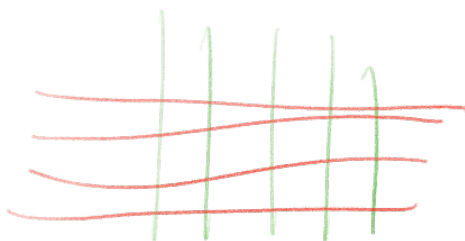
PENCILS OF FLATS



OVERVIEW OF THM A

INPUT: CAT(0) SPACE $\xrightarrow{\text{CURTAIN MODEL}}$ COLLAPSE ALL FLAT REGIONS

PENCILS OF FLATS



→ Define $\mathcal{G} = \left\{ \text{sets of walls that} \begin{array}{l} \cdot \text{ come from disjoint curtains} \\ \cdot \text{ ARE NOT PART of BIG PENCILS} \end{array} \right\}$

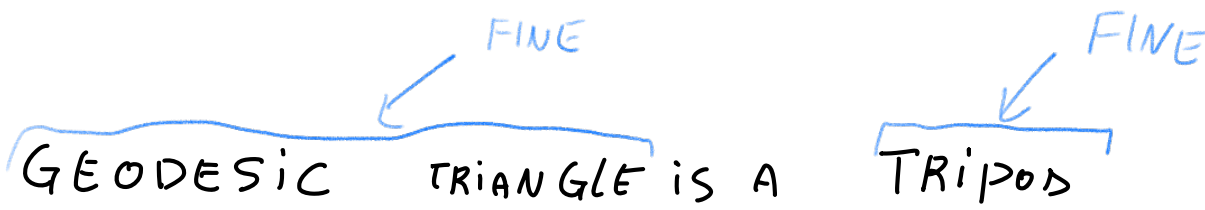
$$C \in \mathcal{G} = \{ h_1, h_2, \dots, h_n, \dots \}$$

STable Cylinders:

TREE: ANY GEODESIC TRIANGLE IS A TRIPOD

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STable Cylinders:

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HYPERBOLIC SPACE

THIN TRIANGLE

STable Cylinders:

TREE: ANY GEODESIC TRIANGLE IS A TRIPOD
↑
MAKE COARSE

STable Cylinders:

TREE: ANY GEODESIC TRIANGLE IS A TRIPOD
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MAKE COARSE

Def A (θ) -Cylinder for x, y is $C(x, y) \subseteq X$:
[x, y] $\subseteq C(x, y) \subseteq N_\theta \subset [x, y]$

Stable Cylinders:

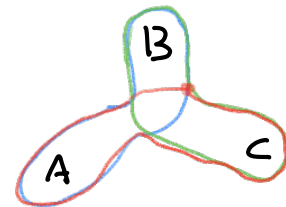
TREE: ANY GEODESIC TRIANGLE IS A TRIPOD
 ↑
 MAKE COARSE

Def A (θ) -Cylinder for x, y is $C(x, y) \subseteq X$:
 $[x, y] \subseteq C(x, y) \subseteq N_\theta \subset [x, y]$

Def THREE Cylinders (sets) $C(A, B), C(B, C), C(C, A)$ FORM A FAT TRIPOD

IF $C(A, B) \setminus C(C, A) = C(B, C) \setminus C(C, A)$

FOR ALL PERMUTATIONS OF $C(A, B), C(B, C), C(C, A)$



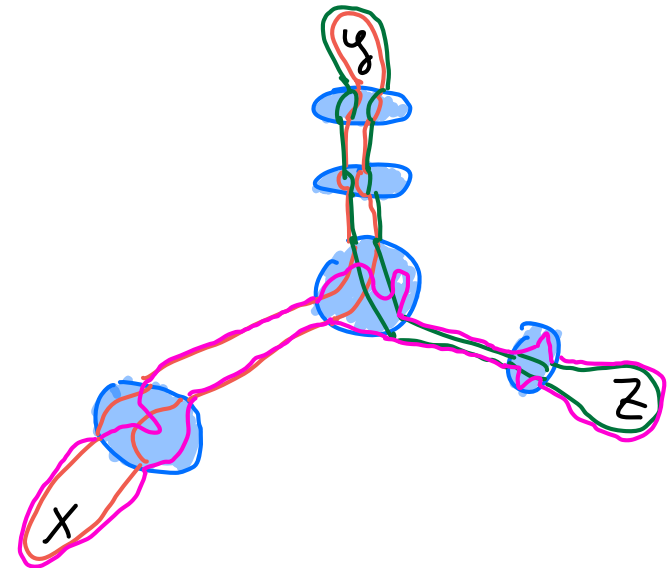
DEF A group G admits GLOBALLY STABLE CYLINDERS if $\forall x, y \exists$
 α (β -) cylinder $C(x, y)$ s.th.

-) $C(x, y) = C(y, x)$
-) $g C(x, y) = C(gx, gy)$
-) $\exists k, R$ s.th. $\forall x, y, z \exists k$ balls of radius R B_i

$C(x, y)$, $C(y, z)$, $C(x, z)$ FORM A FAT TRIPOD

in $X \setminus \cup B_i$

NOTE: GLOBALLY STABLE
 cylinders $\Rightarrow$ HYPERBOLICITY

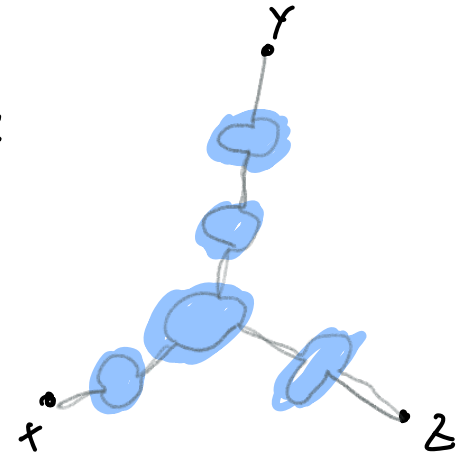


History / Motivation

History / Motivation

→ Introduced by Rips - SELA in '95

↳ CANONICAL REPRESENTATIVES : paths $p(x, y)$:

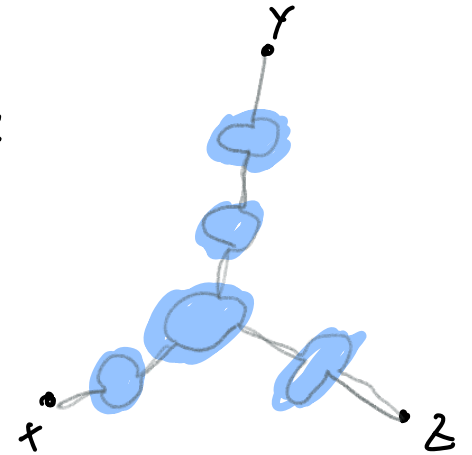


History / Motivation

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↳ CANONICAL REPRESENTATIVES : paths $p(x, y) :$

Q [RIPS - SELA]: Do torsion-free hyperbolic groups admit globally stable cylinders?

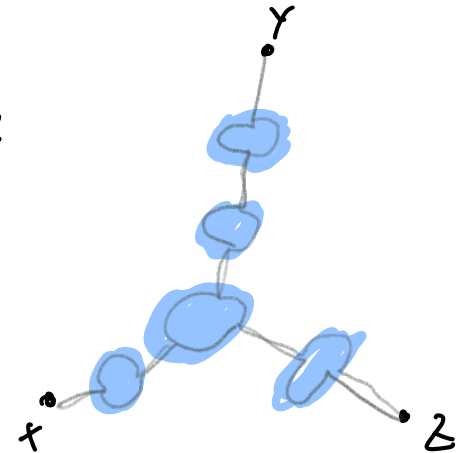


History / Motivation

→ Introduced by RIPS - SELA in '85

↳ CANONICAL REPRESENTATIVES : paths $p(x, y) :$

(Q [RIPS - SELA]: Do torsion-free hyperbolic groups admit globally stable cylinders?



"USED" To prove

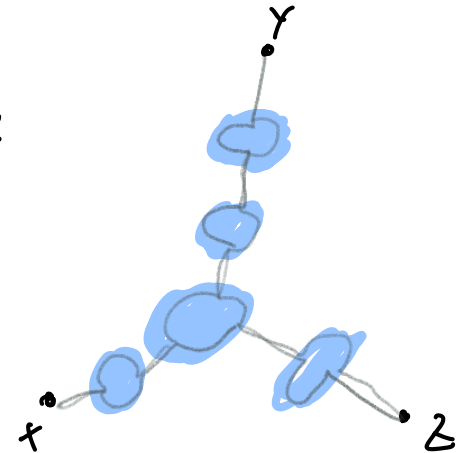
→ [RIPS - SELA]: Solving equations in T-free hyp. groups

History / Motivation

→ Introduced by RIPS - SELA in '85

↳ CANONICAL REPRESENTATIVES: paths $p(x, y)$:

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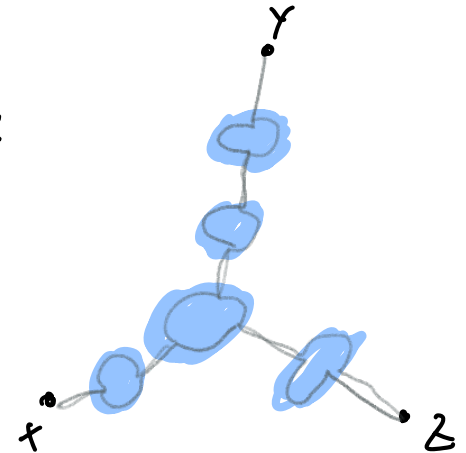
$$[G:H] = [G:K] < \infty \Rightarrow H \cong K.$$

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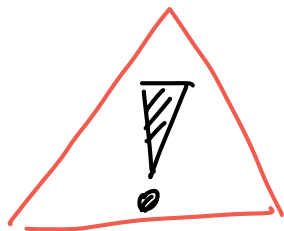
→ [LAZAROVICH]: G non-elt. hyperbolic, H, K s.th.

$$[G:H] = [G:K] < \infty \Rightarrow H \cong K.$$

→ [LAZAROVICH - SAGEEV]: Hyperbolic cubical groups have globally stable cyl.

Naive Hope: Use dualisable systems to do cubical arguments on hyperbolic groups.

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LAZAROVICH-SAGEEV: NEED FINITE DIMENSIONALITY OF THE CUBE CPLX.

— ~~△~~ VERY NOT TRUE IN GENERAL

OVERVIEW THM 13

OVERVIEW THM B

G RESIDUALLY FINITE

BESTVINA - BROMBERG -
FUJWARA, PETYT



G

QUASIMEDIAN
QUASI-ISOMETRIC

$\prod_{n < \infty}$ QUASI-TREES T_i

OVERVIEW THM 13

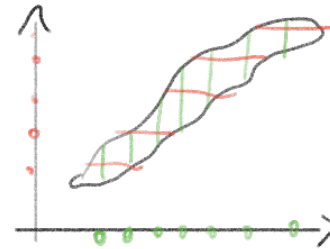
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$G \xrightarrow[\text{QUASI-ISOMETRIC}]{\text{QUASIMEDIAN}} \prod_{n < \infty} \text{QUASI-TREES } T_i$

$\forall$ QUASI-TREE T_i , $W_i = \left\{ \begin{array}{l} \text{PARTITIONS OBTAINED} \\ \text{REMOVING A LARGE} \\ \text{BALL} \end{array} \right\}$



OVERVIEW THM 13

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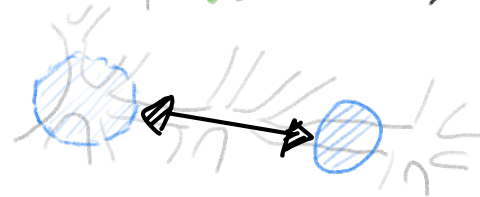
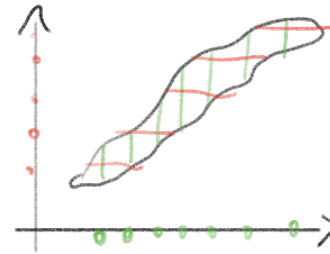
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OVERVIEW THM 13

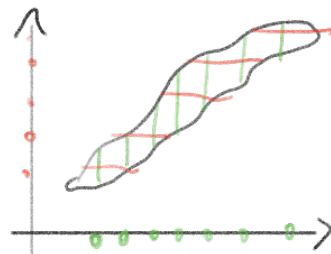
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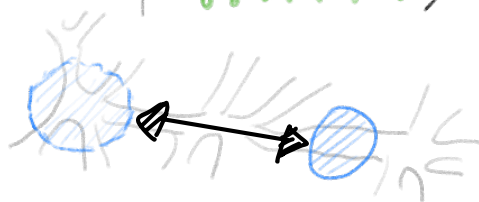


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if $c \in G_i$, $|c| \geq 2$, $h \in W_i$, h can cross at most one wall in c

THINK OF HAVING WALLS IN FINITELY MANY COLOURS (ONE FOR EACH COORDINATE)

AND G_i are sets of the i -TH COLOUR

• Define h far from x, y if

$$\max_{c \in \mathcal{C} = \bigcup G_i} \left\{ |c| : \begin{array}{l} c \text{ intersects } h \text{ and} \\ c \text{ separates } x, y \end{array} \right\} \leq M$$

OVERVIEW THM B

$$G \text{ RESIDUALLY FINITE} \xrightarrow[\text{QUASI-ISOMETRIC}]{\text{QUASIMEDIAN}} \prod_{n < \infty} \text{QUASI-TREES } T_i$$

BESTVINA - BROMBERG - FUJWARA, PETYT

$\Rightarrow$ Thm B'

Let G be hyperbolic and such that G admits an equivariant, QUASI-MEDIAN, quasi-isometric embedding in a finite product of QUASI-TREES AND CAT(0) CUBE COMPLEXES.
 THEN G ADMITS GLOBALLY STABLE CYLINDERS

$$G \xrightarrow{\text{Q.I., Q.M.}} \left(\prod \text{Q.TREES} \right) \times \text{CAT}(0) \text{ C.C.} \Rightarrow \text{GLOBALLY STABLE CYLINDERS}$$

OVERVIEW THM B

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Q | Is THIS TRUE FOR ALL HYPERBOLIC GROUPS?

Thank You For

Your Attention