(A) Cleanup from last time: Recall $\mathcal{A} = a \cup \bar{a}$

We defined $\lambda : \mathcal{A}^* \circlearrowleft$, left to right reduction

Exercise: $\forall\, u, v \in \mathcal{A}^*$
1. $\lambda(v)$ is reduced.
2. $\lambda(\lambda(v)) = \lambda(v)$.
3. $\lambda(u \cdot v) = \lambda(\lambda(u) \cdot v)$

Theorem: Every free equivalence class contains a unique reduced word.

Exercise: $\{ab, baa\}$ do not generate $F(a)$

Fix $G = \langle a \mid R \rangle$. Write $u =_R v$ if $u \in [v]_R$.

Word Problem for G: Instance: $u, v \in \mathcal{A}^*$

Question: Is $u =_R v$?

Algorithms to compute normal forms is one way to answer this question. [Not necessarily the best way!]

Exercise: Consider $BS(1,2) = \langle a, b \mid ba\bar{b} = a^2 \rangle$

Show that
$$\left\{ \bar{b}^p a^q b^r \;\middle|\; \begin{array}{l} p, q, r \in \mathbb{Z} \quad p, r \geq 0 \\ p, r \geq 1 \Rightarrow q \text{ odd} \end{array} \right\}$$

is a system of equiv. class representatives. Give an algorithm to compute them. [Question: How do p, q, r depend on the initial word?]

Define Aut Out Inn

(B) Aut(G): Recall that we want to prove

Theorem: The word problem for $Aut(\mathbb{F}_n)$ is polynomial time.

Here $\mathbb{F}_n$ is shorthand for $F(\mathcal{a})$, with $n = |\mathcal{a}|$.

To make sense of this we need

Theorem [Nielsen] $Aut(\mathbb{F}_n)$ is finitely generated.

In fact the following automorphisms suffice

$$i_a(b) = \begin{cases} \bar{a}, & \text{if } b = a \\ b, & \text{if } b \neq a \end{cases}$$

$$\rho_{ab}(c) = \begin{cases} ab, & \text{if } c = a \\ c, & \text{if } c \neq a \end{cases}$$

$$\lambda_{ba}(c) = \begin{cases} ba, & \text{if } c = a \\ c, & \text{if } c \neq a \end{cases}$$

for $a, b, c \in \mathcal{a}$. and extend to $\mathbb{F}_n$.

Exercise: λ_{ba} can be obtained from i, ρ.

Rmks: In fact $Aut(\mathbb{F}_n)$ is finitely presented as well [Nielsen, McCool] So one could try to proceed directly from the finite presentation $\langle i, \rho \mid R \rangle$. However, it would be a mistake to refuse the wonderful gift of the action of $Aut(G)$ on G!

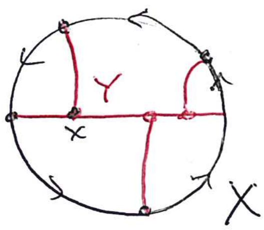
Obviously if $\varphi \in \mathrm{Aut}(F_n)$ then $\varphi = \mathrm{Id}$ iff $\forall a \in \mathcal{A}$, $\varphi(a) = a$.

I have found, in the literature, three proofs of Nielsen's theorem.

- Following Nielsen [Lyndon-Schupp, I.2]
  Too algorithmic!
- Following Whitehead [Stallings, 1999]
  Too topological!
  * Rmk: Can replace $\#_n S^1 \times S^2$ by a handle-body
- Following Stallings [Wade 2012 preprint]
  This is just right.

(c) Graphs: Let (X, x) be a finite connected graph with oriented edges and basepoint $x \in V(X)$.

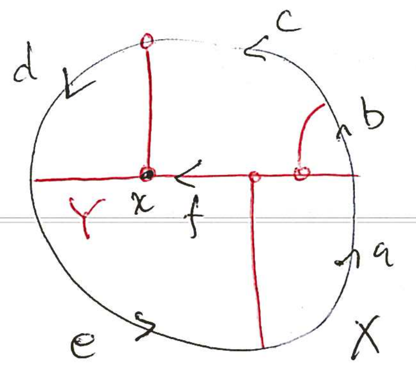
X is a tree if X has no simple loops (ignoring orientations on edges). $Y \subseteq X$ is a spanning tree of $V(X) \subseteq V(Y)$. Example:



Note that $\pi_1(X, x)$ is a free group, generated by the loops determined by the edges of $X - Y$. Now:

(i) reversing an edge $e \subseteq X - Y$ gives the automorphism $i_e : \pi_1(X, x) \circlearrowleft$.

(iv) Suppose $Z \leq X$ is another spanning tree with $Y - \{f\} = Z - \{e\}$. Picture



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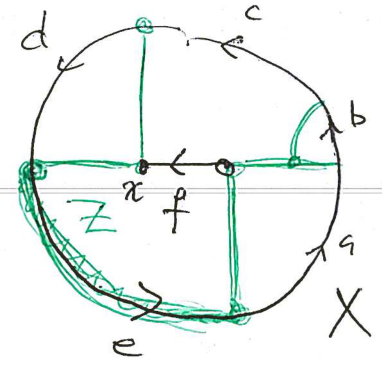

Old generators		New generators written in terms of the old ones
a	$\longrightarrow$	$ea\bar{e}$
b	$\longrightarrow$	$eb\bar{e}$
c	$\longrightarrow$	ec
d	$\longrightarrow$	d
e	$\longrightarrow$	e

In general, swapping a single edge gives a Whitehead transformation

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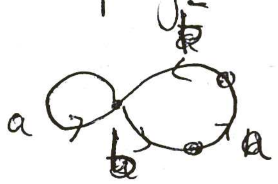
Exercise: Whitehead transformations are products of Nielsen trans.

However, X has only finitely many spanning trees, so we only get finitely many automorphisms this way. We need to change X itself...

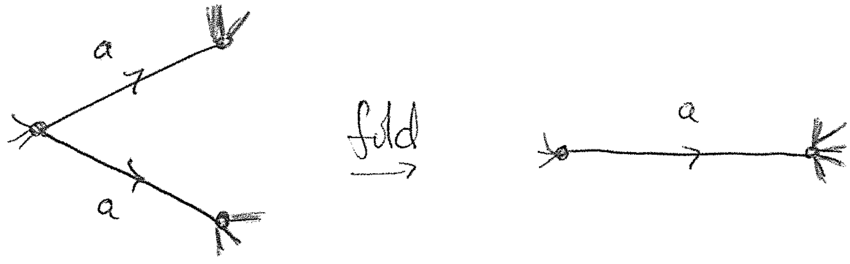
(D) Labelled graphs: Label every edge of X by an element of $\mathcal{A} = \mathcal{a} \cup \bar{\mathcal{a}}$, say $a(e)$ is the label of $e \in E(X)$. If we reverse e then change the label $[\, a(\bar{e}) = \overline{a(e)} \,]$. Reading the labels along a loop gives a homomorphism

$\pi_1(X, x) \longrightarrow F(\mathcal{a})$.



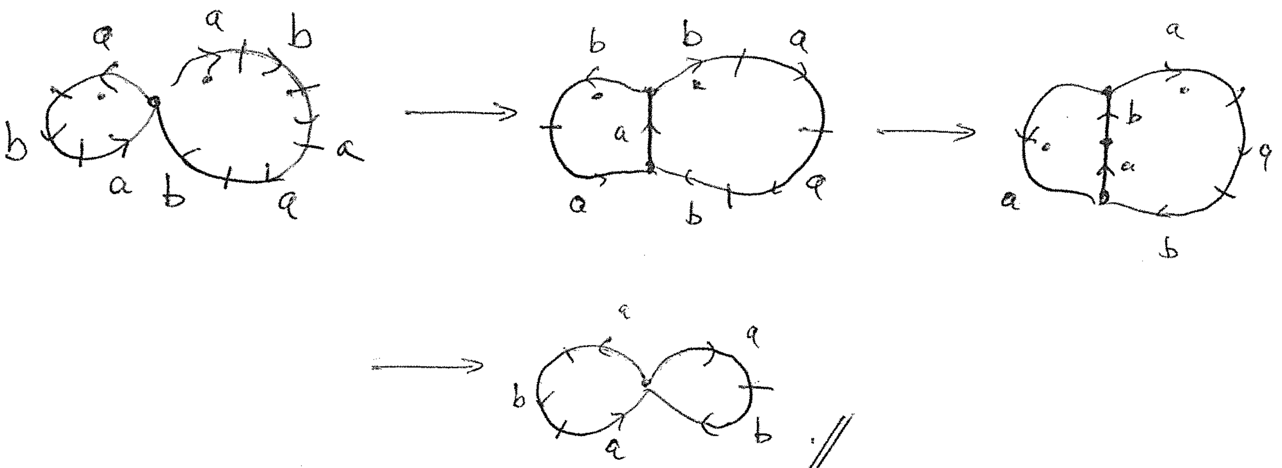
$\langle a, ba\bar{b} \rangle$

Folding: If $e, f \in E(X)$ have the same initial vertex and the same label then we may fold X at $\{e,f\}$ to get $X/_{e\sim f}$ a new labelled graph.

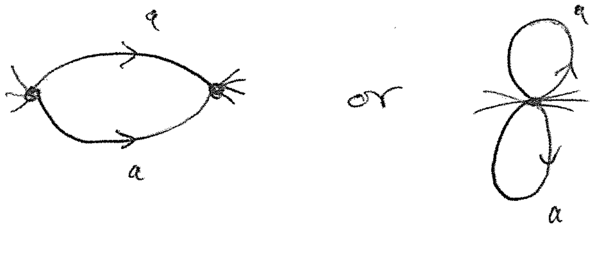


X $\qquad\qquad\qquad$ $X/_{e\sim f}$.

Example:



Note that if e,f also share their terminal vertex

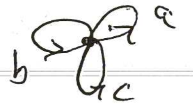


then the rank of $\pi_1(X,x)$ drops. However the image of π_1 in $F(a)$ remains the same.

Thus: If $\pi_1(X,x) \leq F(a)$ then such bad folds cannot happen.

To arrange this: we call $W \subseteq \mathcal{F}^*$ a **basis** if (all $w \in W$ are reduced and) there is $\varphi \in \mathrm{Aut}(F_n)$ with $\varphi(a) = W$. write $W_\varphi = W$.

Let X_φ be homeomorphic to the rose R_n  $n=3$
with edges labelled by W_φ.

Lemma: Any maximal seq of folds starting with X_φ terminates with the standard labelled rose. 

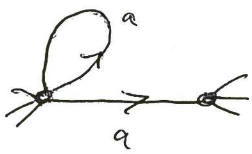
Pf: 'a' is the only reduced word in its equivalence class. //

Ⓔ Trees and folding: Fix any spanning tree $Y_0 \subseteq X_\varphi$. This gives an isomorphism

$$\pi_1 \xrightarrow{\Phi_0} F(a)$$

$$[e \subseteq X - Y] \longrightarrow w_e \in W_\varphi.$$

Any fold away from $X-Y$ leaves Φ_0 alone. If a fold involves a edge of $X-Y$ then swap edges to get a tree Z that avoids the fold. This changes Φ_0 by a Whitehead transformation.

[There is a case to check ]

Folding X_φ to the standard rose realizes Φ_0 (and thus φ) as a product of Whitehead transforms. //

[Gromner of graphs, confluence and termination, local confluence strong local confluence.]