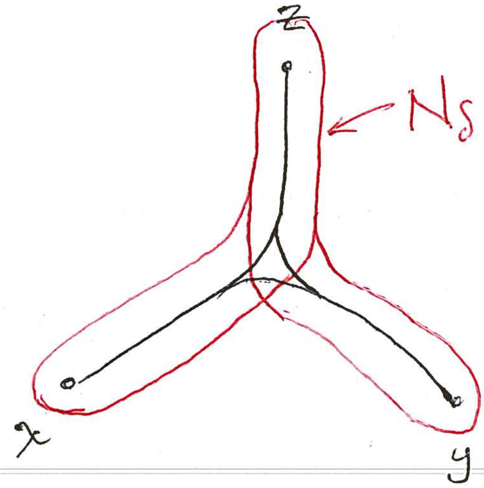
Last time we showed how the compressed word problem for $G$ (CWP($G$)) leads to a solution for the word problem for finitely generated subgroups of Aut($G$).
In particular: Lohrey's solution to CWP($\mathbb{F}_m$) lets us solve WP(Aut($\mathbb{F}_m$)) in poly-time.
This time:

<u>Thm</u>: Suppose $G$ is Gromov hyperbolic. Then the CWP($G$) is poly-time. [[free gps, surface gps, free products of finite gps ...]]

(A) <u>Hyperbolicity</u>: Suppose that $X$ is a graph that is connected, uniformly locally finite (ie. bounded valence). Equip $X$ with the <u>edge metric</u>: All edges have length one. If $x, y \in V(X)$ then we denote a generic geodesic in $X$, from $x$ to $y$, by $[x,y]$. (To ease the notational burden we surpress the parametrization)
A triangle $T(x,y,z)$ of geodesics $[x,y], [y,z], [z,x]$ is <u>$\delta$-slim</u> if $[x,y] \subseteq N_\delta([y,z] \cup [z,x])$ and similarly for the other two sides.

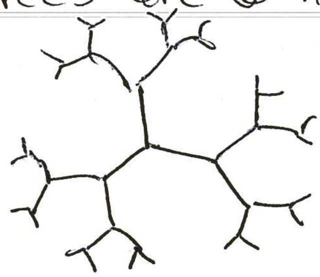
Picture:



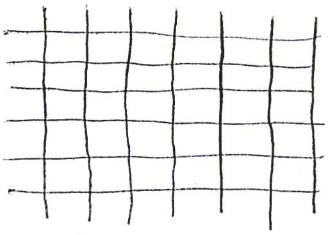

Def: [Gromov, Rips]

We say $X$ is $\delta$-hyperbolic if every geodesic triangle is $\delta$-slim.

Ex: Trees are $0$-hyperbolic

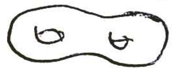


Exercise: The integer lattice



is not $\delta$-hyperbolic, for any $\delta$.

Ex: Cayley graph of $\pi_1(S_2)$



$\leftarrow \mathbb{H}^2$.

## (B) Quasi-geodesics [Cannon, Gromov]

Suppose $\rho : [0,N] \longrightarrow X$ is an edge path [i.e. $\rho$ sends ~~adjacent~~ adjacent integers to adjacent vertices, sends $[i, i+1]$ to the connecting edge.]

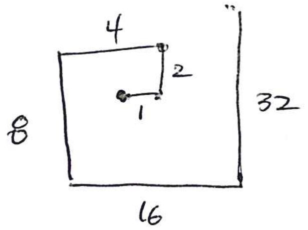
We say $\rho$ is an $(a,b)$-quasi-geodesic if

$\forall\ p, q \in [0,N]$ we have

$$\boxed{|p-q| \leq a \cdot d_X(\rho(p), \rho(q)) + b}$$

[Note that $d_X(\rho(p), \rho(q)) \leq |p-q|$ because $\rho$ is an edge path.]

<u>Exercise</u>: Consider the path




in $\mathbb{Z}^2$.

Show this is a $(3,0)$-quasi-geodesic. [So quasi geod. in $\mathbb{Z}^2$ can be strange--]

---

We will need a "strong" version of the Morse Lemma.

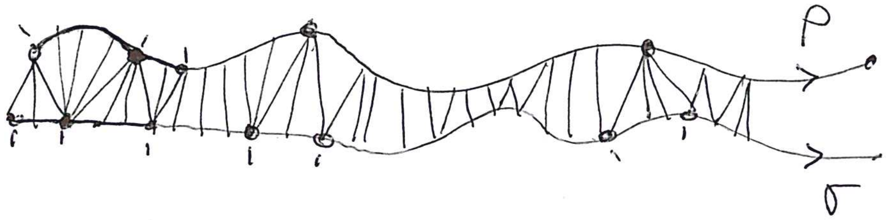
<u>Def</u>: Two edge paths $\rho : [0,N] \to X$ and $\sigma : [0,M] \to X$ <u>R-ramble together</u> if there is a relation $F \subseteq [0,N] \times [0,M]$ st.

- $F(n,m) \Rightarrow d_X(\rho(n), \sigma(m)) \leq R$
- $F(n)$ and $F^{-1}(m)$ are intervals $\forall n, m$. [and these meet in at most endpts]

Asyc. fellow travelling

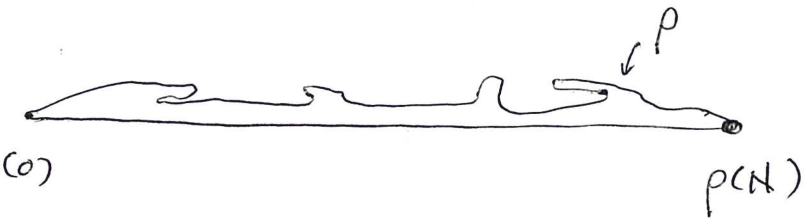
"leap frog"

<u>Picture</u>:




<u>Morse Lemma</u>: $\forall \delta, a, b$ $\exists R = R(\delta, a, b)$ so that if $X$ is $\delta$-hyperbolic and $\rho : [0,N] \to X$ is an $(a,b)$-quasi-geodesic then $\rho$ and the geodesic $[\rho(0), \rho(N)]$ R-ramble together.

<u>Picture</u>:




In particular $\rho$ cannot back track or side track too much.

---

We call $R$ the "stability constant."

We end this part of the lecture with an important local-to-global property.

<u>Lemma</u> [Cannon ~~[illegible]~~ Thm 4] $\forall \delta, a, b \ \exists K = K(\delta, a, b)$ and $c, d$ so that if $X$ is $\delta$-hyperbolic and $p: [0, N] \to X$ is a <u>$K$-local</u> $(a,b)$ quasigeodesic <u>then</u> $p$ is a $(c,d)$-quasi geodesic.

---

<u>Def</u>: $p : [0,N] \to X$ is a $K$-<u>local</u> $(a,b)$ quasi-geodesic if $p : [i, i+K] \to X$ is an $(a,b)$ quasi-geodesic for all $i = 0, 1, 2, \ldots N-K$.

---

[As usual, the local-to-global principle fails for $\mathbb{Z}^e$: Consider a square of side-length $K$.]

---

(c) <u>Short-lex</u>: Recall $\mathcal{A} = a \cup \bar{a}$. Let $X = X(G, \mathcal{A})$ be the Cayley graph, with edges labelled by $\mathcal{A}$.

If $u \in \mathcal{A}^*$ then we write $u_*$ for the short-lex representative of the group element $[u]$. This gives the lex-earliest geodesic from $1_G = [\varepsilon]$ to $[u]$ in $X$.

<u>Thm</u> [Cannon] The language $\{ u_* \mid u \in \mathcal{A}^* \}$ is regular

[the language of short-lex ~~[illegible]~~ representatives.]

---

We will pair this with an important exercise:

<u>Exercise</u>: The <u>compressed membership problem</u> for regular languages is poly-time.

Just to be clear:

CMP(L): Instance: compressed word $\mathbb{A}$ over $\mathcal{S}^*$

Question: Is $w_{\mathbb{A}}$ in $L$?

---

Here is our first result:

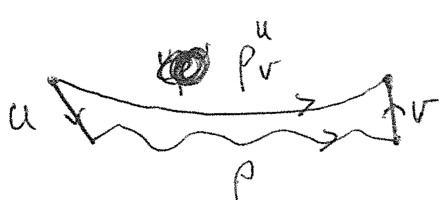
**Theorem I**: Fix $\delta, c, d$. Suppose $X = X(G, \mathcal{S})$ is $\delta$-hyperbolic. There is a poly-time algo. that, given a compressed $(c,d)$-quasi-geodesic $\mathbb{A}$, computes a compressed short-lex word $\mathbb{X}$ so that $w_{\mathbb{A}} =_G w_{\mathbb{X}}$.

---

To prove this we introduce short-lex bundles.

Suppose $\rho$ is a path. Let $\mathrm{Bund}(\rho, R)$ be the set of short-lex geodesics that $R$-ramble with $\rho$. We use the notation ${}^{u}\rho^{v}$ for the element of $\mathrm{Bund}(\rho, R)$ so that $[\rho^u_v] = [u \cdot \rho \cdot v]$

Picture:

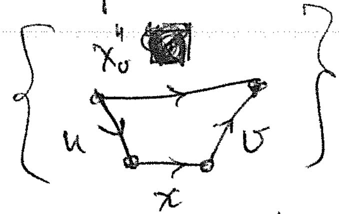



Here $u, v$ are short-lex and of length $\leq R$.

Note that the size of $\mathrm{Bund}(\rho, R)$ is at most $[\mathrm{vol}(B_1(R))]^2$, so is independent of $\rho$.

Pf of Theorem I: Take $R = R(\delta, c, d)$ the stability constant for $(c,d)$ quasi-geodesics. Suppose $A$ is a compressed $(c,d)$-quasi geodesic.

We build Bund$(A,R)$ from the bottom-up.

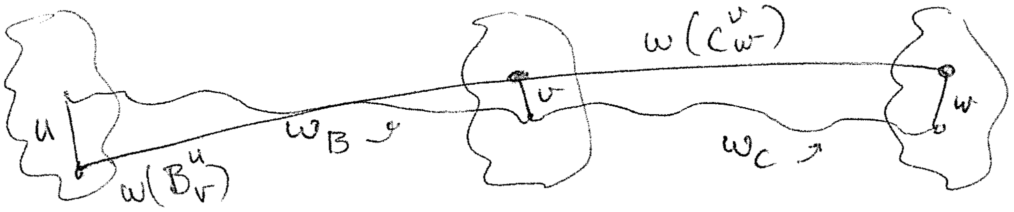
For every $x \in \mathcal{S}$ we have Bund$(x,R) = \Big\{$  $\Big\}$


For each short lex word $x_v^u$, if it R-rambles with $x$ then place it in Bund$(x,R)$.

Induction step: Suppose $A \to B \cdot C$. We are given Bund$(B,R)$ and Bund$(C,R)$ — these each contain at most $\mathrm{vol}(B_1(R))^2$ straight line programs [compressed short-lex geodesics that R-ramble with $w_B$ and $w_C$ respectively. So: Fix $v \in B_1(R)$ short lex. For all $B_v^u$ and $C_w^v$ we check if $w(B_v^u) \cdot w(C_w^v)$ is short lex. If so we add the first one we find to Bund$(A,R)$ with the root $A_w^u$.

In fact $A_\epsilon^\varepsilon$ is the desired program.

Picture:




Stability implies that Bund$(A,R) \subseteq$ Bund$(B,R) \cdot$ Bund$(C,R)$. //

## D Normal forms and trimming:

<u>Theorem II</u>: Suppose $X = X(G, \mathcal{A})$ is $\delta$-hyperbolic. Let $K = K(\delta, 1, 20\delta)$ be the local-to-global constant. There is a poly-time algo that, given a compressed word $\mathbb{A}$ over $\mathcal{A}$ computes a compressed $K$-local $(1, 20\delta)$ quasi-geodesic $\mathbb{X}$ s.t. $w_{\mathbb{A}} = w_{\mathbb{X}}$.

---

<u>Here is the idea of the proof:</u>

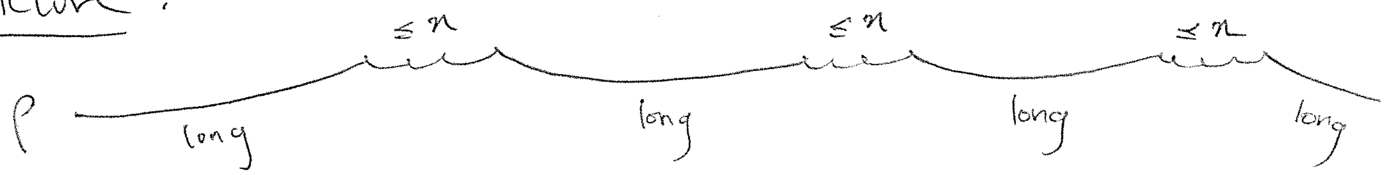
Let $(c, d)$ be the constants given by the loc to glob. Lemma.

Let $R = R(\delta, c, d)$ be the stability constant. Fix $H \gg \max\{R, K, \delta\}$

we define normal forms as follows.

$p \in NF(n)$ if
- $n = 0$, $p$ is short lex, $|p| \leq H$.
- $n \geq 1$, $p$ is piecewise short-lex where
  - (i) each piece is <u>short</u> (length in $[H, 2H-1]$) or <u>long</u> ( $\cdots$ $[2H, 5 \cdot n H]$ )
  - (ii) any collection of consecutive short pieces has at most $n$ pieces
  - (iii) $p$ is a $K$-local $(1, 20\delta)$ quasi-geodesic.

<u>Picture</u>:

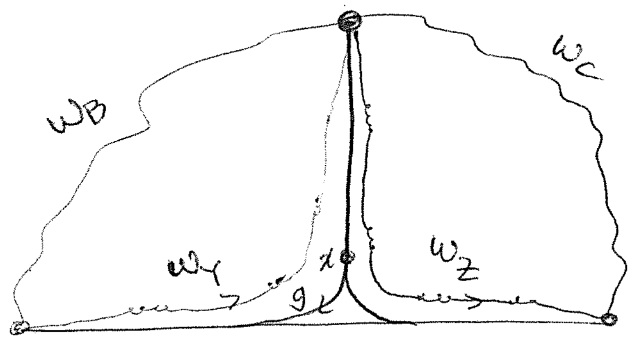


Note that condition (iii) of the normal form only needs to be checked within distance $K$ of the break points.

---

Suppose $A \to B \cdot C$. By induction we have $Y, Z \in V_X$ so that $w_Y$ and $w_Z$ are in $NF(n-1)$.

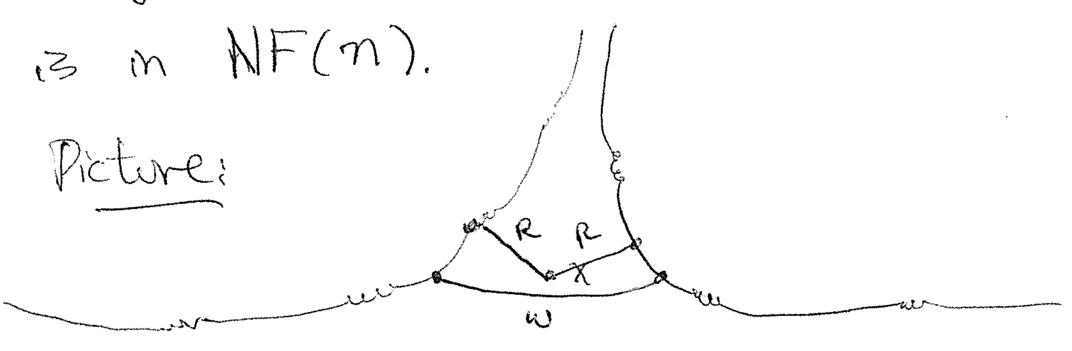
Picture:




~~Use Thm I to find all shortlex geod. that ramble with $\overline{w_Y}$ and ramble with $w_Z$.~~

Apply Thm I to find a $\overline{Y_*}$ compressed shortlex rep of $\overline{w_Y}$. (*) And Plandowski! Apply Thm I and binary seach to find the maximal prefix of $\overline{Y_*}$ that $R$-rambles with $w_Z$. This locates the body of the geodesic triangle. Trim off the spike and insert a long segment (at most linear in $n$!) to get $w_X$. So $X \to Y[i-k] \cdot w \cdot Z[l:]$ is in $NF(n)$.

Picture:




Rmk: may have to move end pts linear distance away from $x$ -- //