Curves in the Masur domain

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[Book, paper ⇝ work of Aaron, Karen, Kevin, myself]

Thm [S-] A curve α lies in $\Omega(V)$ [the Masur domain] iff $d_S(\alpha, D(V)) \geq 3$. [Souto, Luo]

Thm [Moore-Campisi-Rathbun, S-] If $d_{Haus}(D(V), D(W)) < \infty$ then $D(V) = D(W)$ [Abrams-Masur]

Thm: [S-] All dead-ends in $\mathcal{C}(S)$ rel $D(V)$ come from I-bundle structures on V.

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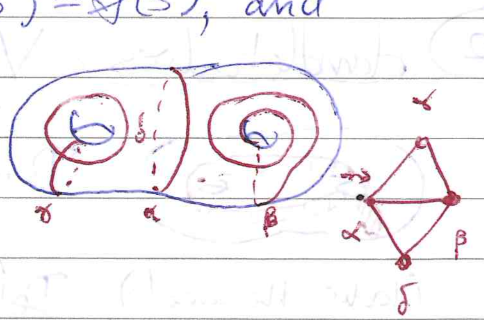
(1) Surfaces: Recall S_g  is the closed orientable surface of genus g. Fix a hyp. metric on S and define $\mathcal{S}(S) = \{\alpha \subset S \mid \alpha \text{ simple closed geodesic}\}$

(A) Define a graph $\mathcal{C}(S)$ where $\mathcal{C}^0(S) = \mathcal{S}(S)$, and $\{\alpha, \beta\}$ is an edge iff $\alpha \cap \beta = \emptyset$



Notation: $d_S(\alpha, \beta)$ is the number of edges in a minimal edge path from α to β.

(B) Laminations $\mathcal{GL}(S) = \{\lambda \subset S \mid \lambda \text{ is a geodesic lamination}\}$

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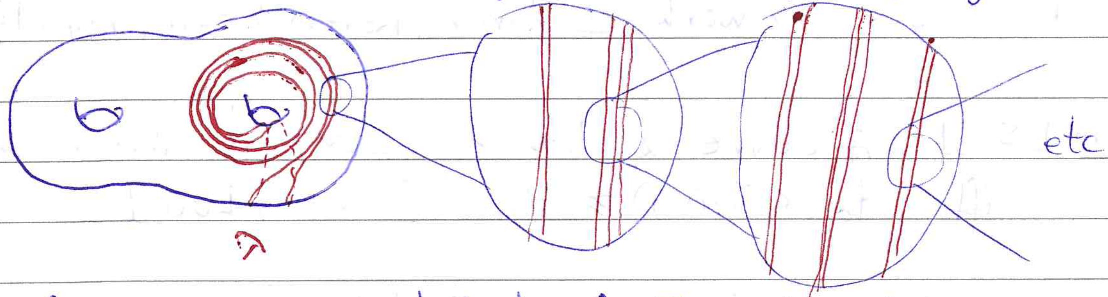
That is λ is a closed, disjoint union of simple geodesics.

Picture:



Def: A leaf $\alpha \subset \lambda$ is isolated if $\exists$ ε-ball $B \subset S$ so that $B \cap \lambda = B \cap \alpha$. [Eg any $\alpha \in \mathcal{S}(S)$ isolated in itself.

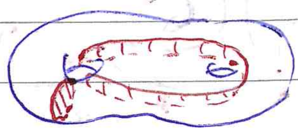
Def: $\mathcal{GL}_0(S) = \{ \lambda \in \mathcal{GL}(S) \mid$ all isolated leaves of λ are loops$\}$.

Rmk: We actually work with $\mathcal{PML} \xrightarrow[\text{measures}]{\text{forget}} \mathcal{GL}_0$ but the above is enough to go on.

Def: $\mathrm{Supp}(\lambda) \subset S$ is the smallest subsurface $X \subset S$ s.t. (i) $\lambda \subset X$ (ii) $\partial X \subset \mathcal{S}(S)$. Call X the support of λ.

[Annoying special case: The Annulus.]

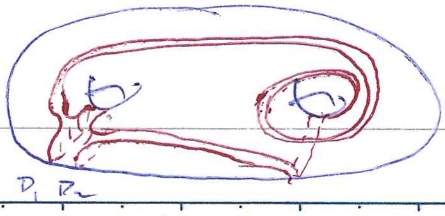
(2) Handlebodies: V_g = handle body of genus g. $\partial V_g \cong S_g$



Define $D(V) = \{ \alpha \in \mathcal{S}(S) \mid \alpha$ bounds a disk in $V\}$

(looks the same!) Define $\overline{D(V)}$ to be the closure of $D(V)$, taken in $\mathcal{PML}(S)$.

Example: If $D \subset V$ is nonsep and $\lambda \cap \partial D = \emptyset$ then $\lambda \in \overline{D(V)}$



Picture:

Take two copies of D and band sum along an approximation to λ. Take a limit and discard isolated leaves.

Define: $Z(V) = \{\lambda \in PML(S) \mid \exists \kappa \in \overline{D(V)} \text{ s.t. } i(\lambda, \kappa) = 0\}$
= the zero set of V.

Define: $MCG(X) = Homeo(X)/Homeo_0(X)$
$(= \{f \in Homeo(X) \mid f \simeq Id_X\})$

Diagram:

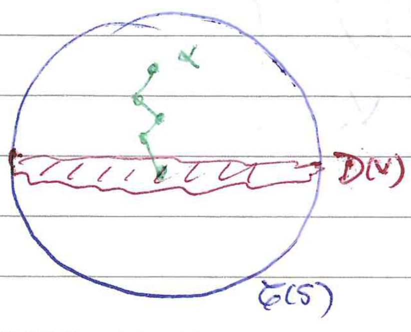
$$\begin{array}{ccccc} MCG(S) & : & PML(S) & \supset & \mathcal{C}(S) \\ \vee & & \cup & & \cup \\ MCG(V) & : & Z(V) & \supset & D(V) \end{array}$$

[Masur]
Thm: $MCG(V)$ acts prop discontinuously on (the open nonempty set) $\Omega(V) = PML(S) - Z(V)$
= the Masur Domain of V

[Think: $Z(V)$ is a "limit ~~set~~" for the action of $MCG(V)$ on $PML(S)$]

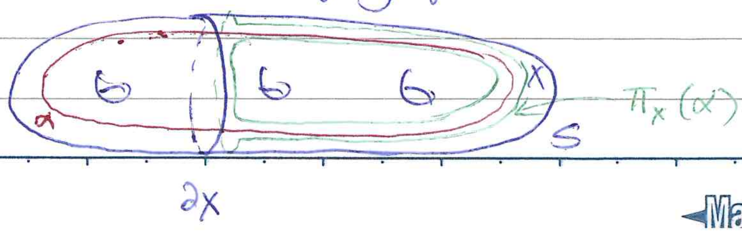
(3) Thm [S.] $\alpha \in \mathcal{C}(S)$ lies in $\Omega(V)$ iff $d_S(\alpha, D(V)) \geq 3$

Picture



(4) Subsurface projection [Ivanov, Masur-Minsky]

$\pi_X : \mathcal{C}(S) \longrightarrow \mathcal{C}(X)$
$\quad\ \ \psi \qquad\qquad\qquad \psi$
$\quad\ \ \alpha \longmapsto$ surgery of $\alpha \cap X$



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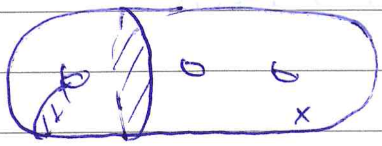
⑤ <u>Holes</u>: Say $X \subset S = \partial V$ is a <u>hole</u> for $\mathcal{D}(V)$ if

(∗) $\forall \alpha \in \mathcal{D}(V)$ α <u>cuts</u> X

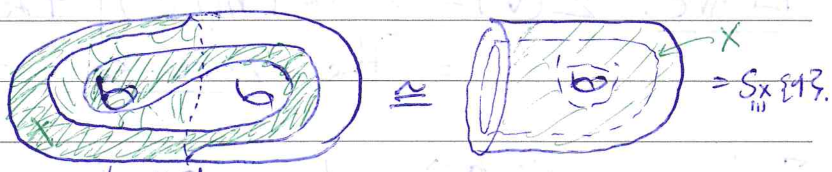
i.e. $\alpha \cap X \neq \emptyset$, i.e. $\pi_X(\alpha) \neq \emptyset$ ie X is <u>disk busting</u>

<u>Ex</u>:



Not a hole

<u>Ex</u>



A hole $S_{1,1} \times I$

<u>Define</u>: $\mathrm{diam}(X) = \mathrm{diam}_X(\pi_X(\mathcal{D}(V)))$ diameter of X.

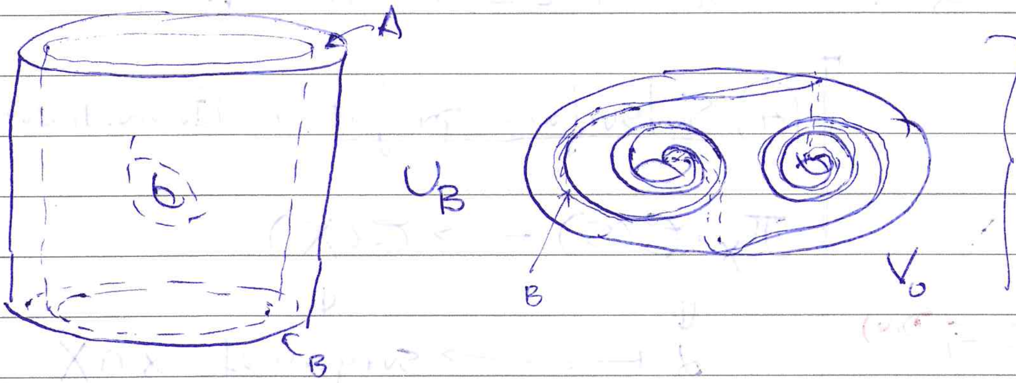
Classification Thm [Masur-S-] Suppose $X \subset S = \partial V$ is a hole with $\mathrm{diam}(X) \geq 57$. <u>Then</u>

① X is not an annulus <u>and</u>

② if X is compressible then it contains a pair of filling disks.

③ if X is incompressible then there is an I-bundle $\rho: T \to F$ s.t.
(i) X is a comp't of $\partial_h T$
(ii) $\partial_h T \subset S$
(iii) Some comp't $A \subset \partial_v T$ is in S
(iv) F admits a pA map.

<u>Ex</u>:

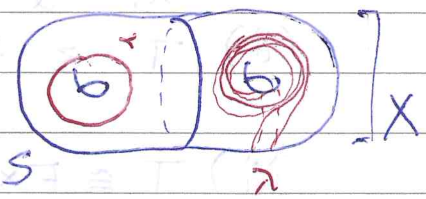


$V = T \cup_B V_0$.

(6) Thm : $\alpha \in \Omega(V)$ iff $d_S(\alpha, \mathcal{D}(V)) \geq 3$.

Pf: Easy direction : If $d(\alpha, \mathcal{D}(V)) \leq 1$ then $\alpha \in \overline{\mathcal{D}(V)}$ as in the examples above. If $d(\alpha, \mathcal{D}(V)) = 2$ then $\exists \beta$ st. $d(\beta, \mathcal{D}(V)) = 1$, $d(\alpha, \beta) = 1$. So $\beta \in \overline{\mathcal{D}(V)}$ and $\alpha \in Z(V)$. ✓

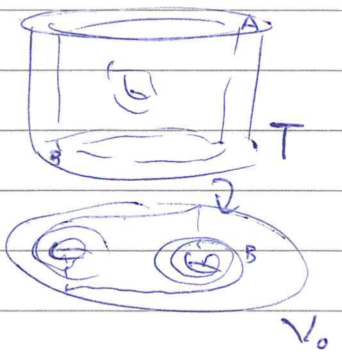
Hard direction : Suppose $\alpha \in Z(V)$, α is a curve. Fix $\lambda \in \overline{\mathcal{D}(V)}$ st. $\alpha \cap \lambda = \phi$. Set $X = \mathrm{supp}(\lambda)$. Picture



If $S - X$ compresses then $d_S(\alpha, \mathcal{D}(V)) \leq 2$ ✓
If X " " " ≤ 1 ✓

So we may assume that X is an incomp hole. As λ is a limit of disks, and as λ is ending in X, we find [Kobayashi] that $\mathrm{diam}(X) = \infty$. So apply Part (3) of the classification. We are given $p : T \to F$, $T \subset V$ as above.

Cartoon



Note $\alpha \cap X = \phi$ and $\exists$ vertical disk $D \subset T$ meeting A exactly twice.

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Thus: If $\alpha \cap T = \phi$ we have $d(\alpha, D(V)) \leq 1$.

If α is parallel to a comp't of ∂X then $i(\alpha, \partial D) \leq 2$ and $d(\alpha, D(V)) \leq 2$.

If α is contained in $Y = F \times \{0\}$ $(X = F \times \{1\})$ then $d(\alpha, D(v)) = 1$.

Cases (i) X has at least two boundary compts. Then α misses $\delta \subset \partial X - A$ so $d(\alpha, D(v)) \leq 2$.

(ii) T is twisted. So $\alpha \subset A$ and $i(\alpha, D) = 2$ so $d(\alpha, D(v)) \leq 2$.

(iii) $T \cong F \times I$ and $\alpha \subset Y$ so $d(\alpha, D(v)) = 1$. //