(A)

Train tracks [2015-01-16 Warwick]

① **Surfaces**: Let $S = S_{g,n}$ be the connected, compact smooth, oriented surface of genus g with n boundary components ($|\partial S| = n$).

Picture

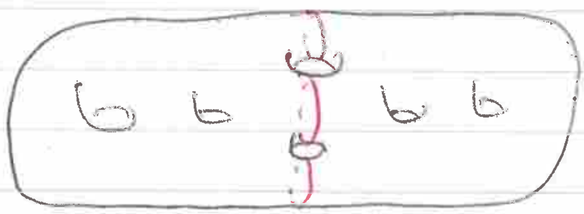
Recall that $\chi(S)$ = Euler characteristic of S

$$= 2 - 2g - n$$

We define the double, $D(S)$, of S as follows

$$D(S) = S \times \{0,1\} \Big/ (x,0) \sim (x,1) \text{ if } x \in \partial S.$$

Picture

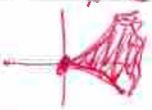


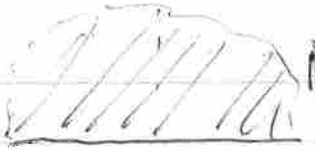
The genus of $D(S)$ is $2g + n - 1$ and so

Exercise: $\chi(D(S)) = 2 \cdot \chi(S)$.

② **Corners**: The points of ∂S are locally modelled on $\mathbb{R}^2_{\geq 0} = \{(x,y) \mid y \geq 0\}$

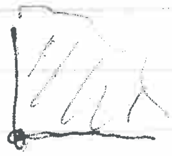
Known cusps are modelled on $x^3 \geq y^2$ (outward) or $x^3 \leq y^2$ (inward)



Picture:  $\mathbb{R}^2_{\geq 0}$

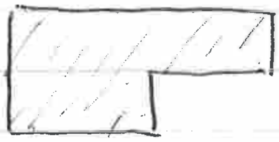
(outward / inward)

A corner point (or simply, a corner) of ∂S is instead modelled on a quadrant or its complement.



outward

inward.

Example:  } 5 outward corners, 1 inward corner.

3) index

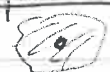
Definition: Suppose S has corners on ∂S.

Set $\text{index}(S) = \chi(S) - \frac{1}{4}(\#\text{ outward corners}) + \frac{1}{4}(\#\text{ inward corners})$

[Think: measures geodesic curvature.]

Simp.

Equivalently: we require every point of S to have

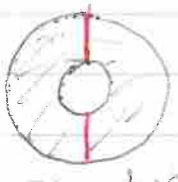
angle	name	picture
2π	interior	
$3\pi/2$	inward corner	
π	boundary	
$\pi/2$	outward corner	

B

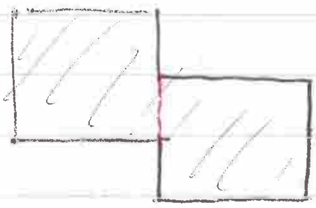
Examples	picture	name	index
		~~disk~~ nilgon	1
		bigon	1/2
		trigon	1/4
		rectangle	0
		once holed bigon	-1/2

Lemma: Index is additive (under gluing along boundary arcs).

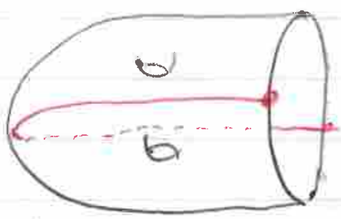
Picture



obtained as union of a pair of rectangles



index zero



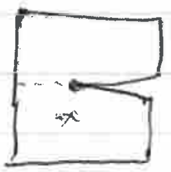
index $= \chi = -3$

obtained as union of

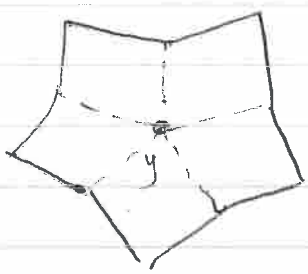
bigon with handle

index $= -3/2$.

Corollary. Any surface obtained by gluing rectangles has index zero.

NB: 

This is not allowed because x has angle 2π but is not an interior point.



Likewise y has angle $\frac{5\pi}{2}$ which is not allowed.

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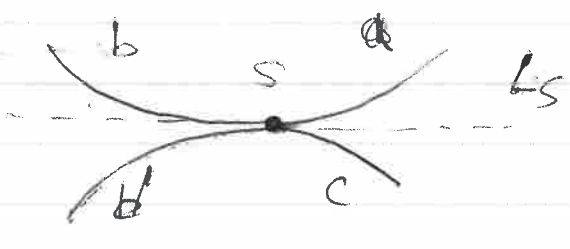
(4) Train tracks: Suppose $\tau \subset S$ is a smooth graph. We call the vertices of τ switches and the edges of τ branches. All branches are assumed to be smooth.

Smooth switches:
Every switch is equipped with a tangent ~~direction~~ line and adjacent branches asymptote to that tangent line.

Picture 

~~Nondegeneracy~~:

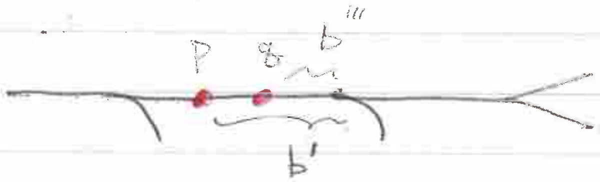
Move up

<u>Half branches</u>: Suppose b is a branch (open) and $p \in b$. Call the components b', b'' of $b - p$ <u>half branches</u>

<u>Picture</u>

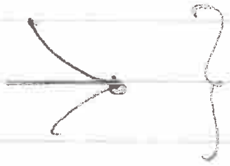
Say two half branches b', b''' (wrt $p, q \in b$) are equivalent if $b' \subseteq b'''$ or $b''' \subseteq b'$

<u>Picture</u>



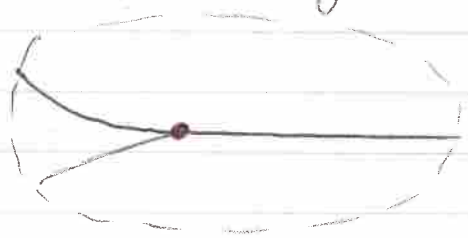
An equivalence class of half branches is an <u>end</u> of b.

<u>Nondegeneracy</u>: At every switch s there ~~are~~ is at least one end on each side of s. That is



not allowed.

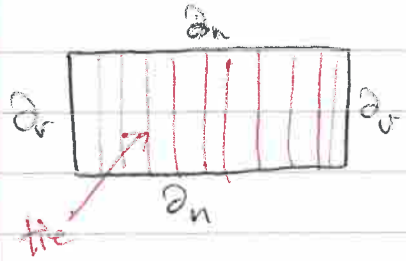
A switch is <u>generic</u> if it is modelled on



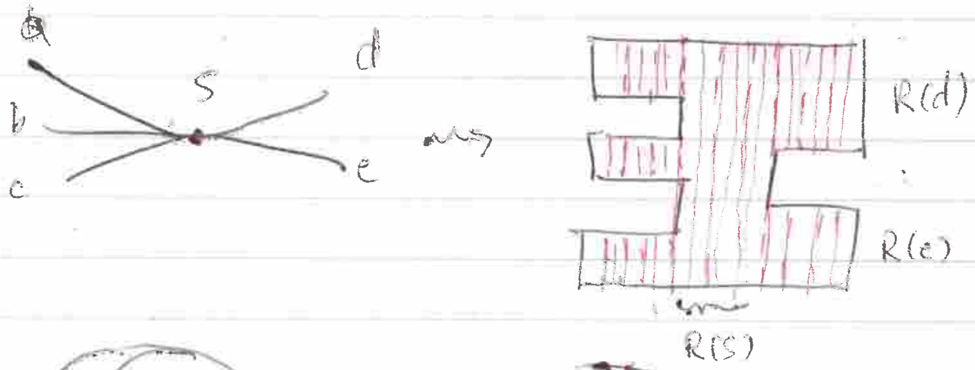
ie two ends on one side and one on the other

(5) Before we finish the definition, we will discuss <u>tie neighborhoods</u>

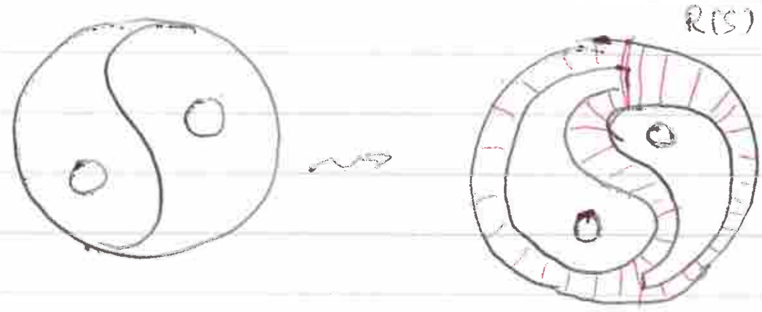
We start with the anatomy of a rectangle R:



the top and bottom of R is called $\partial_h R$ the <u>horizontal boundary</u>. The vertical sides are $\partial_v R$ the <u>vertical boundary</u>. We foliate R with copies of I (called <u>ties</u>) parallel to $\partial_v R$. We now define $N = N(\tau)$ the tie neighborhood. For each switch s and branch b we take a rectangle $R(s)$ and $R(b)$. [Define half rectangles and ends of rectangles as above, by cutting along an interior tie.] If an end of b is attached to s then we give $\partial_v R(b)$ to $\partial_v R(s)$. Note that if several branches are attached to s then we leave gaps in $\partial_v R(s)$ as shown in figure.



Example:



Here is the final requirement to be a trian track:

$\boxed{\text{Index}}$ Every component of $S - n(\tau)$ has negative index. [Here $n(\tau)$ is the interior of $N(\tau)$.]

That is, we rule out nullgons, bigons, rectangles and annuli. Note that any component $C \subset S - n(\tau)$ has boundary divided into vertical and horizontal parts, as dictated by $\partial N(\tau)$. [components of ∂S are horizontal by convention.]

Corollary: If $\tau \subset S$ is a train track then $\chi(S) \leq -1$ and $S \not\cong S_{0,3}$.

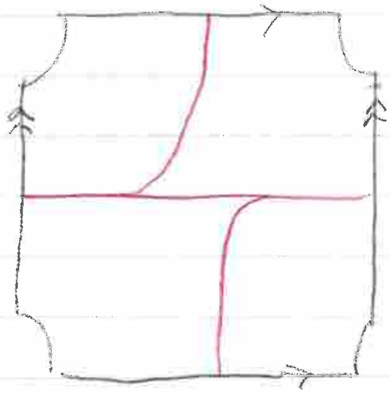
Pf: Index is additive. If $S - n(\tau)$ has at least three components then $\chi(S) \leq -3/2$.

Exercise

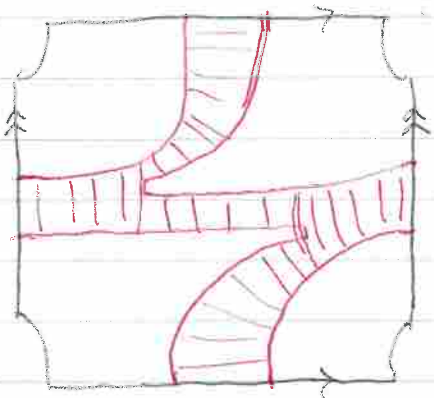
If $\tau \subset S = S_{0,3}$, and $|S - n(\tau)| \le 2$ then there $\exists$ one component of $S - n(\tau)$ containing at least two components of ∂S. Thus $\tau \subseteq A \subseteq S$ some annulus. So $\chi(A) \le -1$ a contradiction. //

(6) An example:

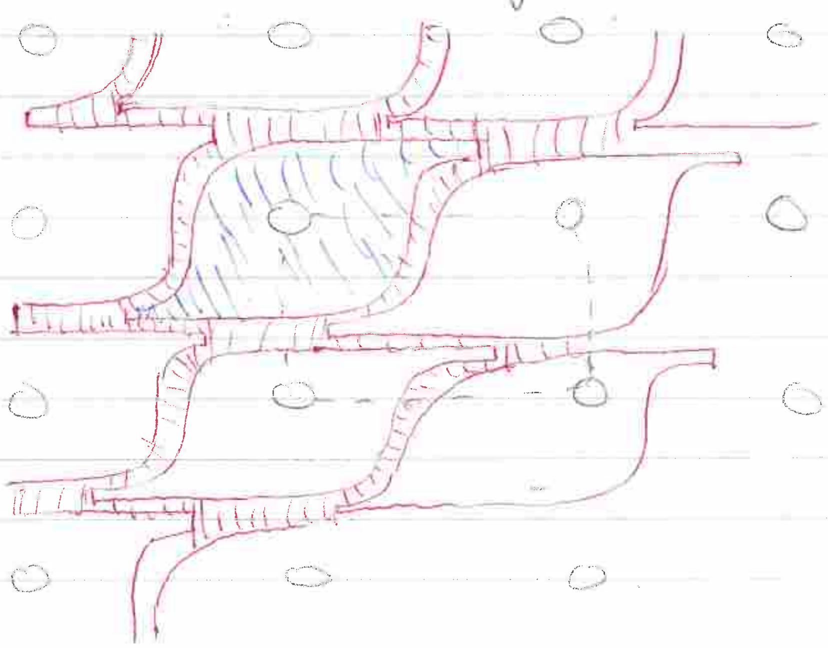
we cut $S = S_{1,1}$ along two arcs to get an octagon



We now take the tie neighborhood to get



we use copies of this to tile the $\mathbb{Z}^2$ cover of S



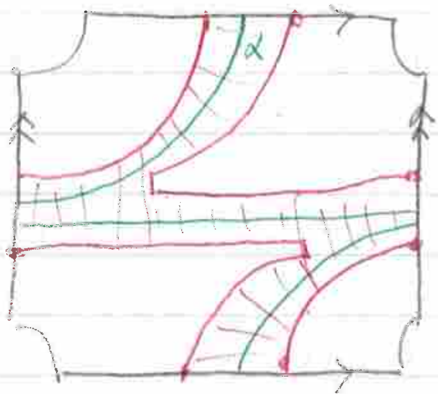
We have shaded in blue the complem. component, a once-holed rectangle.

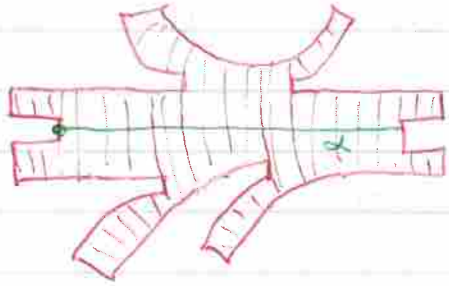
E

7) <u>Carrying</u>: Fix $\tau \subset S$ a track and $N = N(\tau)$ a tie neighborhood. A curve or arc $\alpha \subset S$ is <u>carried</u> by τ (written $\alpha < \tau$) if (i) α is prop. emb. in N (with $\partial\alpha \subset \partial_v N$) and

(ii) α is transverse to the ties of N.

<u>Examples</u>





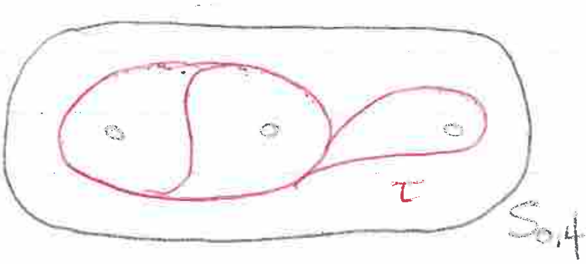
<u>Exercise</u>: Suppose $\alpha < \tau$ is a simple closed curve. Show α is <u>essential</u> (does not bound a disk) and <u>non-peripheral</u> (does not cobound an annulus with ∂S).

8) <u>Recurrence</u>: We say a branch $b \subset \tau$ is <u>recurrent</u> if there is a carried curve $\alpha < \tau$ so that $\alpha \cap R(b) \neq \emptyset$.

Def: A track τ is <u>recurrent</u> if all of its branches are recurrent. [Exercise: It suffices to show the small half-branches are recurrent].

Def: $\sigma = \bigcup_{b \text{ recurrent}} b \subset \tau$ is the <u>recurrent part</u> of τ

<u>Example</u>:



τ is <u>not</u> recurrent.

Here is the tool necessary for testing recurrence.

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(9) <u>Transverse measures</u>: (aka weightings on τ)

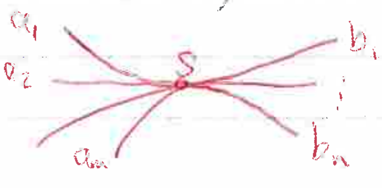
Recall $B(\tau) = \{ b \subset \tau \mid b \text{ a branch of } \tau \}$.

Suppose $\mu : B(\tau) \longrightarrow \mathbb{R}_{\geq 0}$ is a function and suppose, for every switch $s \in S(\tau)$ if $\{a_i\}$ and $\{b_j\}$ are the half branches on the two sides of s then μ satisfies the <u>switch equation</u>

$$\sum_i \mu(a_i) = \sum_j \mu(b_j)$$

Then we call μ a <u>transverse measure</u> on τ.

<u>Picture</u>



(F)

If $\mu, \nu : B(\tau) \to \mathbb{R}_{\geq 0}$ are transverse measures,
Note that we may scale μ or add $\mu + \nu$.

Thus let $ML(\tau) = \{ \mu : B(\tau) \to \mathbb{R}_{\geq 0} \mid \text{transverse measure on } \tau \}$.
be the cone of transverse measures on τ.

Define $P(\tau) = \dfrac{ML(\tau) - \{0\}}{\mathbb{R}_{>0}}$

This is the polytope of projective measured laminations carried by τ. [we will return to these ideas later!]

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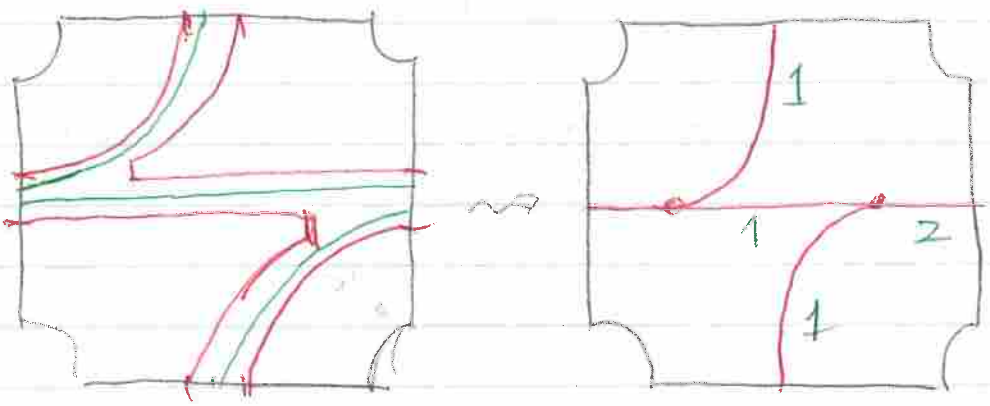
(10) Integral weightings:

Note that if $\alpha < \tau$ is a curve then we may define $\mu_\alpha \in ML(\tau)$ via

$$\mu_\alpha(b) = |\alpha \cap R(b)| = \#\text{components } \alpha \cap R(b).$$

Example:



Conversely: If $\mu \in ML(\tau)$ is integral then we may take $\mu(b)$ horizontal arcs in $R(b)$

and so form a multicurve $\alpha_\mu < \tau$.

The two procedures

$$\alpha \longmapsto \mu_\alpha$$
$$\alpha_\mu \longleftarrow\!\shortmid \mu$$

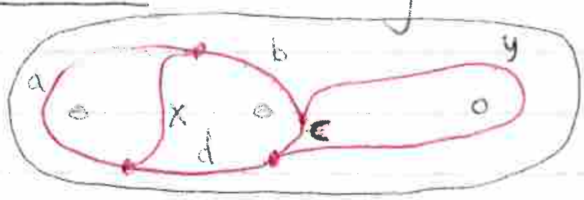
are inverse to each other. In fact more is true.

<u>Lemma</u>: The maps above induce a bijection

$$\left\{ \begin{array}{c} \text{integral transverse} \\ \text{measures} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{isotopy classes of multicurve} \\ \alpha \subset S \text{ st. some rep is} \\ \text{carried by } \tau \end{array} \right\}$$

As a consequence: If a curve α is carried by τ, the carrying is unique (up to isotopy preserving $S-N$ pointwise).

(ii) <u>Detecting</u> <u>non recurrence</u>: We may now revisit our example



Using the labels shown the switch equations are:

$$\left.\begin{array}{l} a+x=b \\ d+x=a \\ b+y=c \\ c+y=d \end{array}\right\}$$

Add these to find

$$2(x+y)=0$$

so $x=y=0$.

(12) <u>Efficient position</u>: Suppose $\alpha \subset S$ is a properly embedded arc or curve. We say α is in <u>efficient position</u> with respect to τ (written $\alpha \pitchfork \tau$) if

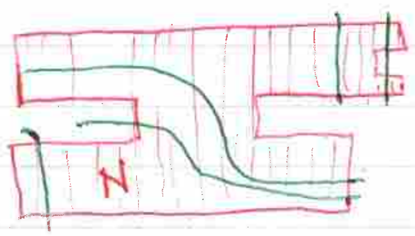
(i) α is transverse to ∂N (so is disjoint from the corners of N),

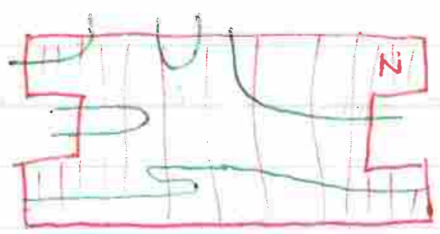
(ii) every arc of $\alpha \cap N$ is either carried or is a tie, and

(iii) every component of $S - (\alpha \cup N)$ has non-positive index.

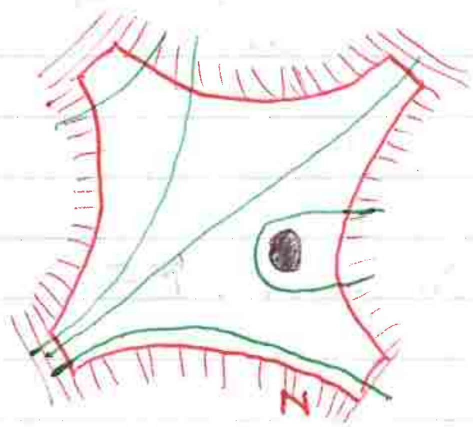
<u>Pictures</u> | Allowed | Not allowed

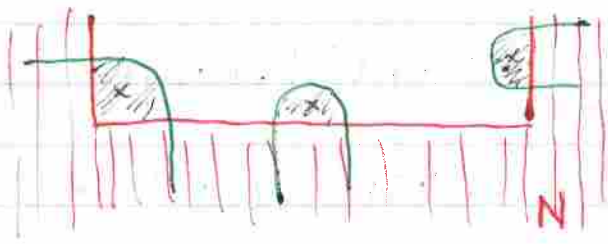
In $N(\tau)$



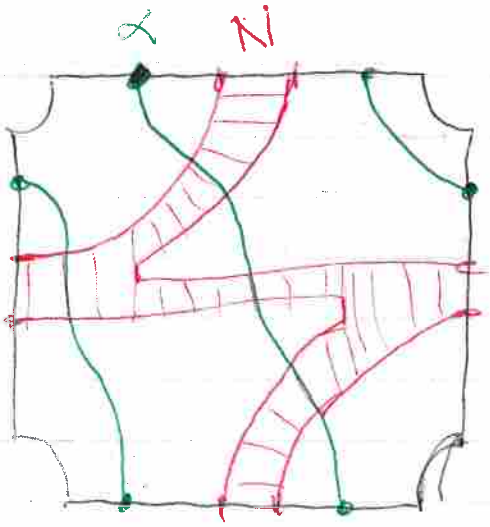


In $S-N$



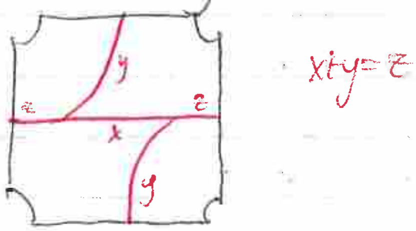


Example:



Check this is in efficient position

[Hard]
Exercise: Show that α is not isotopic to a carried curve. [The curve α has positive geometric intersection number with every $\beta < \tau$; in fact if β has weights then $i(\alpha,\beta) = 2x+3y$.]



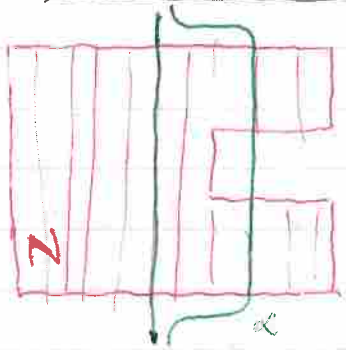
[Easy]
Exercise: Suppose $\tau \subset S$ is a track, and $\alpha \subset S$ is in efficient position. Then α is essential [and non-peripheral] ← This is wrong! [Perhaps require that $\alpha \cap N \neq \emptyset$?] [In paper rule out annuli explicitly]

This exercise has a converse!:

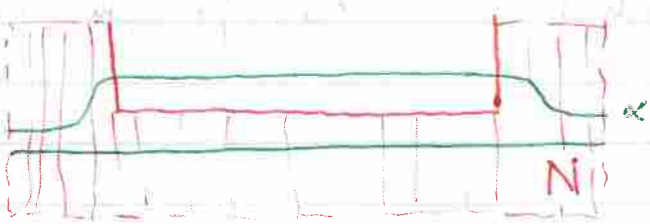
Theorem: [Takarajima, Masur-Mosher-S.] Fix $\tau \subset S$. Any essential, nonperiph. $\alpha \subset S$ can be isotoped into efficient position. Moreover, efficient position is unique up to rectangle and annulus swaps.

H

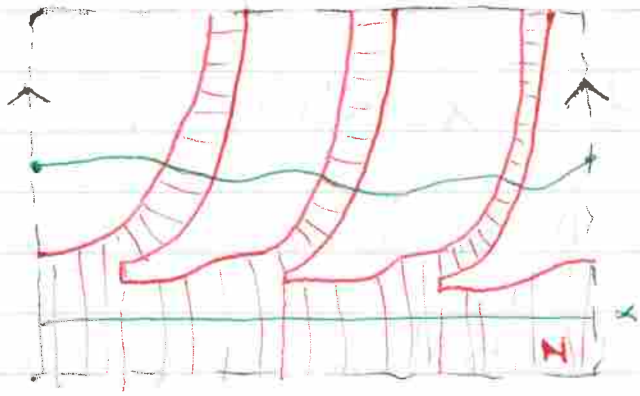
Pictures Vertical rectangle swap



Horizontal rectangle swap



Annulus swap



Rmks: Takarajima proves existence [redacted] algorithmically. The running time is not analyzed. MMS does not give an algorithm and additionally requires $\tau \subset S$ be transversely recurrent.

Questions: (1) Analyze the algorithm in [T].

(2) How does effici. pos change under train track splitting?

⑬ Dual curves: The example above was quite special.

Definition: A properly embedded curve or arc $\alpha \subset S$ is dual to τ (written $\alpha \pitchfork \tau$) if α is in efficient position and all arcs of $\alpha \cap N$ are ties.

[Thus dual curves are essential and non-peripheral.]

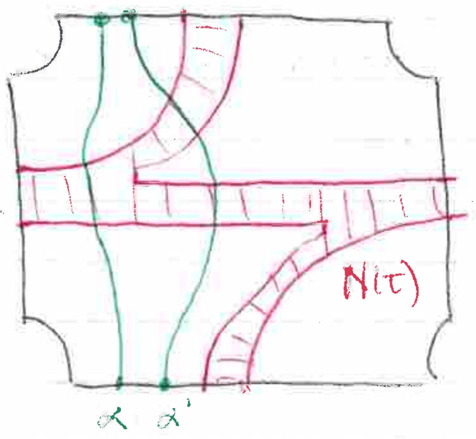
Definition: We say a branch $b \subset \tau$ is transversely recurrent if there is a dual curve $\alpha \pitchfork \tau$ so that $\alpha \cap R(b) \neq \emptyset$.

Definition: A track τ is transversely recurrent if all of its branches are transversely recurrent.

[Question: Does τ have a "trans. recurrent part"?]

[Exercise: It suffices to check the large half branches.]

Picture

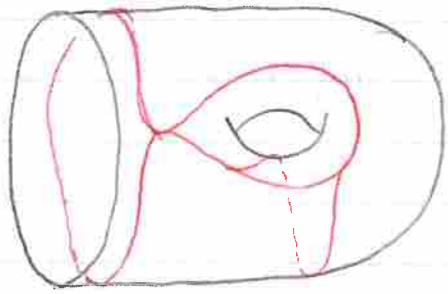


Note: α and α' differ by a vertical rectangle swap.

Remark: We may change (weaken) the definition of transverse recurrence by allowing α to be a dual arc as well [cf. annulus tracks]

14 Tangential measures

Picture:



This is a version of Figure 1.3.2 from [PH]. Check that τ is a track (ie satisfies the index condition) and that τ is recurrent. We wish to prove that τ is not transversely recurrent.

Suppose $\gamma \subset \partial_h N(\tau)$ is a component of the horizontal boundary of $N(\tau)$. } call γ a horizontal side

Suppose $\nu : B(\tau) \longrightarrow \mathbb{R}_{\geq 0}$ is a function [NB: we do not assume ν satisfies the switch equations.]

Define $\nu(\gamma) = \sum_{b \in B(\tau)} |\gamma \cap R(b)| \cdot \nu(b)$

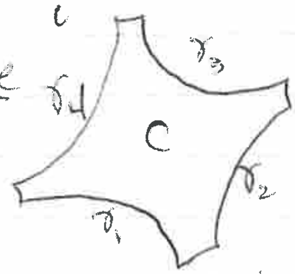
Of course: $|\gamma \cap R(b)| = 0, 1$, or 2. $\nu(\gamma)$ is the total measure of ν coming through γ.

Definition: ν is a <u>tangential measure</u> if for every component $C \subset S - n(\tau)$ we have the following:

(i) if C is a disk with horizontal sides $\gamma_1, \gamma_2, \dots, \gamma_m$ then for all i

$$\nu(\gamma_i) \le \sum_{j \neq i}{}' \, \nu(\gamma_j)$$

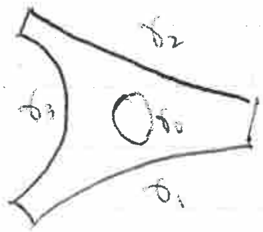
Picture



(ii) if C is an annulus with one smooth side γ_0 and other sides $\gamma_1, \gamma_2, \dots, \gamma_m$ then

$$\nu(\gamma_0) \le \sum_{i=1}^{m} \nu(\gamma_i)$$

Picture



(iii) if C is any other region we impose no condition.

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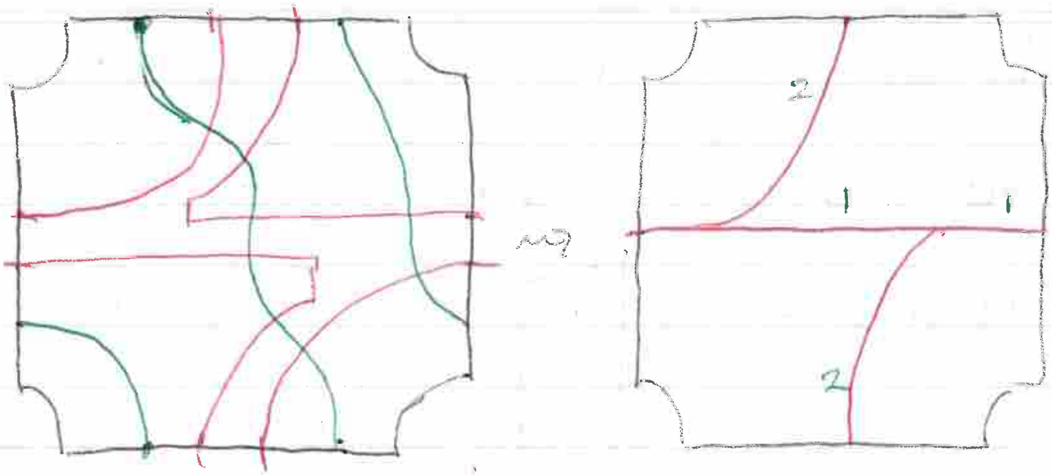
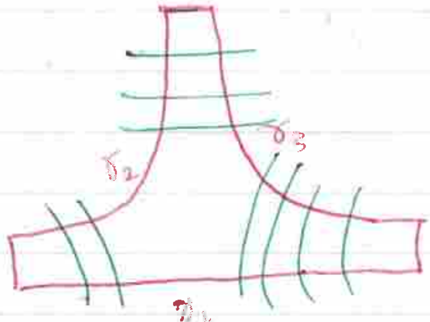
(15) <u>Integral transverse measures.</u>

If $\alpha \pitchfork \tau$ is a dual curve then we may define ν_α via $\nu_\alpha(b) = |\alpha \cap R(b)|$

Picture



More generically: $\nu(\gamma_1) \leq \nu(\gamma_2) + \nu(\gamma_3)$

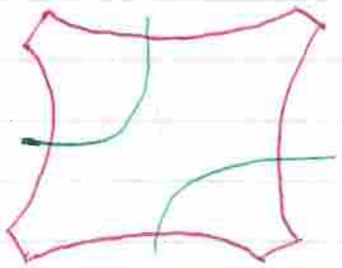


Note that if C is a component of $S - n(\tau)$ then $\sum_{\gamma \subset \partial_n C} \nu_\alpha(\gamma)$ is even.

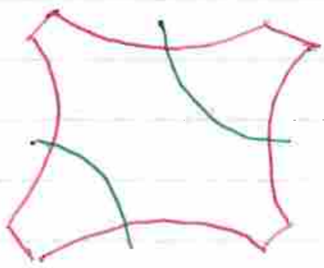
The converse also holds: if $\nu : B(\tau) \to \mathbb{Z}_{\geq 0}$ is an <u>even</u> integral measure then there is a multicurve $\alpha_\nu \pitchfork \tau$ s.t. $\nu(b) = |\alpha_\nu \cap R(b)|$.

~~However: α_ν is only well defined~~

However α_ν may not be unique. For example

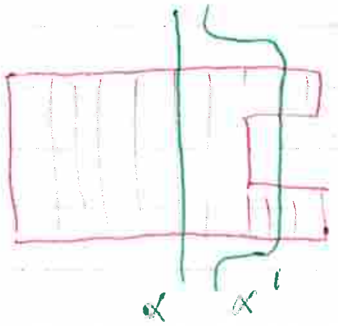


versus



Like wise, isotopic dual curves α, α' may induce distinct tangential measures.

Picture

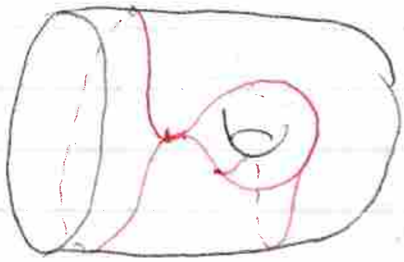


Thus the theory of tangential measures is not a clean as that of transverse measures.

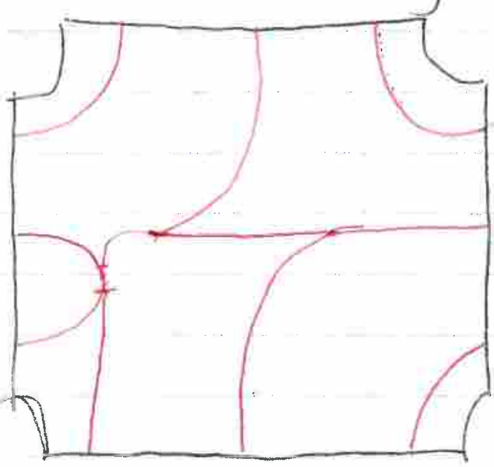
---

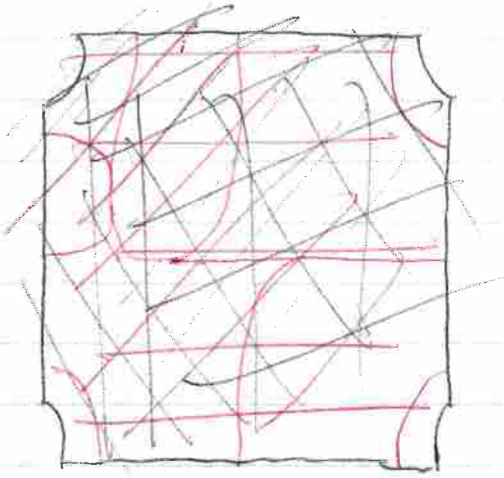
(16) Detecting non-transverse recurrence

Recall the track $\tau \subset S = S_{1,1}$



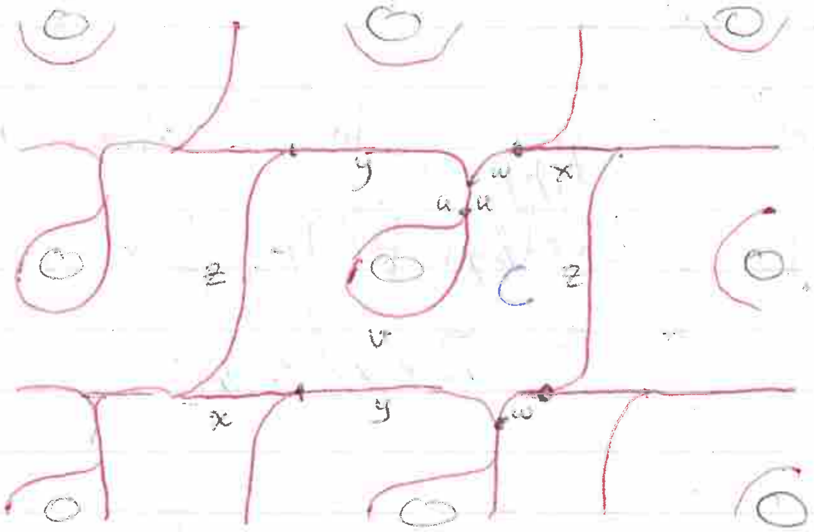
We cut S open in the usual way to get





We use copies of this to tile the plane (giving the $\mathbb{Z}^2$ cover of S)

Picture:



Consider the shaded region C, giving a hexagon in $S - n(\tau)$. If α is a dual curve then ν_α satisfies the three triangle inequalities given by C. So

$$z + y + u + v + u + w + x \le x + y + w + z$$

and thus $2u + v \le 0$ so $u = v = 0$.

Thus τ is not trans. recurrent. //

---

(17) Birecurrence: ~~A track $\tau \subset S$ is birecurrent if it is both recurrent and transversely recurrent. [Examples appear in the theory of pA mapping classes]~~

~~Def: A track $\tau \subset S$ is maximal if —~~
is large if —

(17) **Birecurrence:**

A track $\tau \subset S$ is *birecurrent* if it is recurrent and transversely recurrent. [Rmk: These were introduced by Thurston in the middle of his study of pseudo-Anosov maps. It is an interesting question (at least in my mind!) to see if the hypotheses of recurrence / transverse recurrence can be excised from the theory of pA maps...]

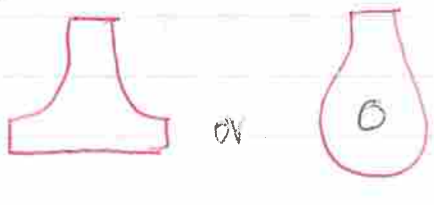
A few more definitions:

Def: A track $\tau \subset S$ is *large* if all ~~regions~~ components of $S - n(\tau)$ are either disks or peripheral annuli.

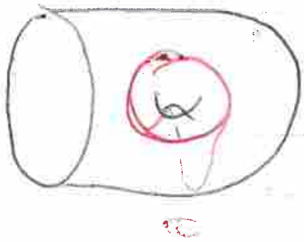
Ex: , as usual.

Def: A track $\tau \subset S$ is *maximal* ~~if all regions of $S - n(\tau)$ have index $\geq \frac{1}{2}$~~ if it is large and all components of $S - n(\tau)$ have index $-1/2$.

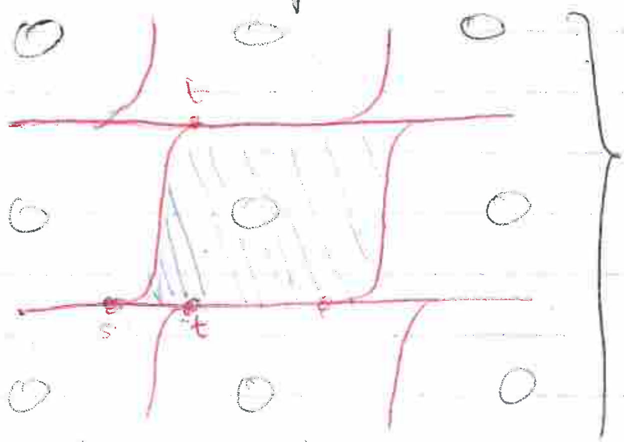
[So they are hexagons or once-holed bigons

]

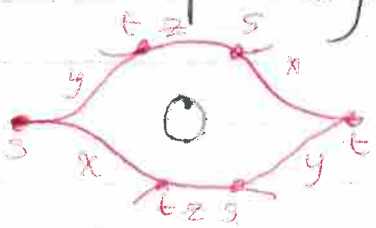
Definition: A birecurrent track $\tau \subset S$ is complete if it is not a proper subtrack of any birecurrent track.

Exercise: 

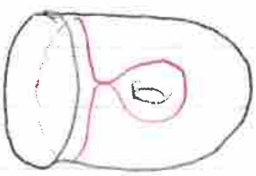
The track $\tau \subset S$ is the only complete track in $S = S_{1,1}$, up to homeomorphism.



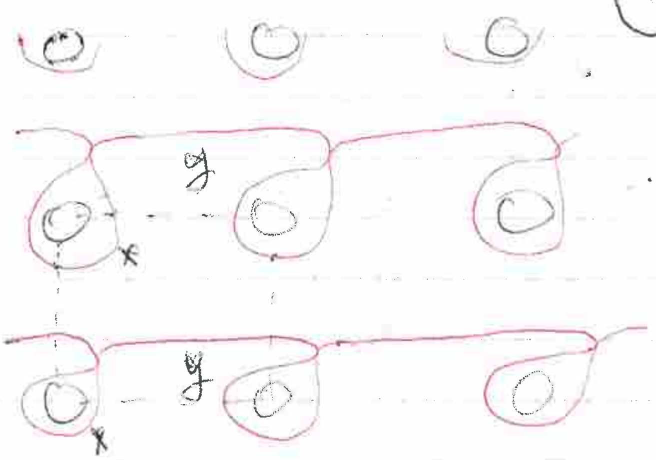
It may help to consider the complementary region



adding a branch, and checking recurrence. You will also need to use the fact that



is not recurrent.



Exercise: If $\tau \subset S$ is trans. recurrent then so is any subtrack $\sigma \subset \tau$.

tangential: $x + y \leq y \Rightarrow x = 0$

M

<u>Theorem 1.3.6</u> [PH] Suppose $\chi(S) \le -2$.
If $\tau \subset S$ is complete, then τ is maximal.

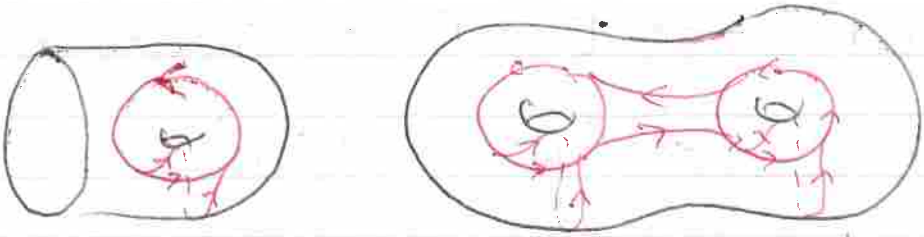
<u>Exercise</u>: The converse is false.
[even ignoring $S = S_{1,1}$]

Before we begin the proof of theorem 1.3.6 we will discuss

18 Orientability.

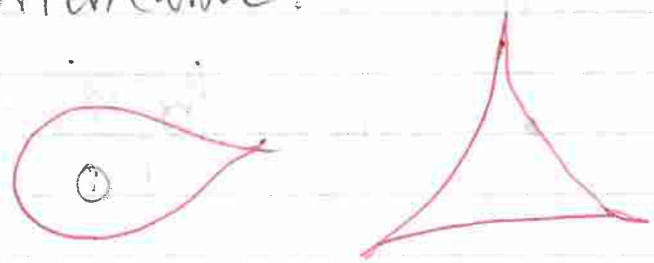
<u>Definition</u>: A track $\tau \subset S$ is <u>orientable</u> if there is a orientation of every branch that extends across switches.

<u>Examples</u>:



<u>Remark</u>: If $C \subset S - n(\tau)$ has an odd number of ~~horizontal~~ vertical sides then τ is not orientable.

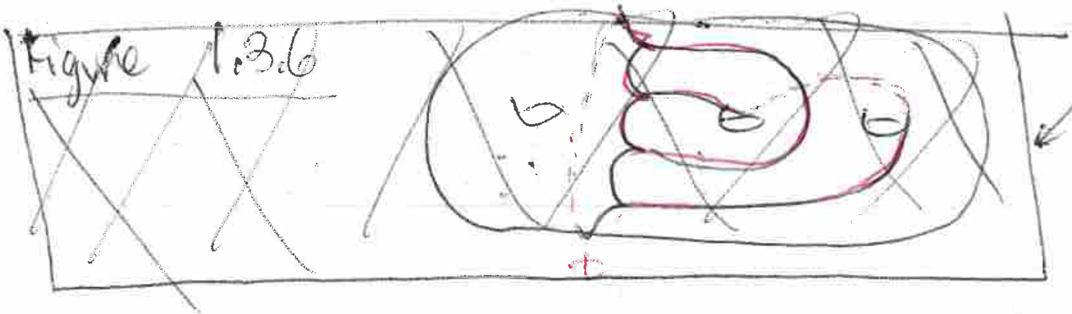
<u>Pictures</u>:



Thus maximal tracks are never orientable.

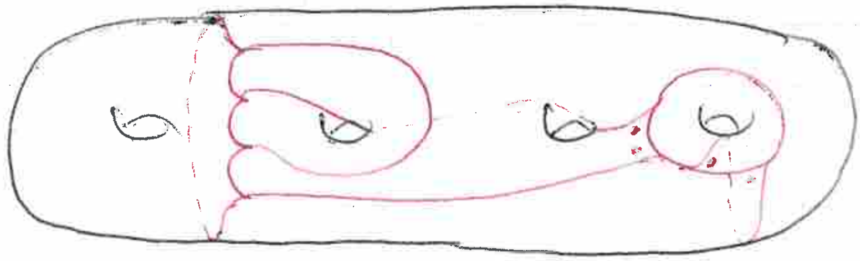
Note that the converse statement is not true.

Figure 1.3.6



works if we count vertical sides.

Figure 1.3.6 [Corrected]



[or perhaps we should ~~[illegible]~~ vertical sides?
[Yes, that works.]

Def: Suppose $m < n$ are integers. A train route $\rho : [m, n] \to \tau$ is a smooth map (immersion) so that $\rho^{-1}(S(\tau)) = [m, n] \cap \mathbb{Z}$.

switches

[that is, integers and only integers are sent to switches].

Proposition 1.3.7 Suppose $\tau \subset S$ is a connected ~~[illegible]~~ recurrent track. Suppose ~~b, b'~~ b, c $\subset \tau$ are branches.

[better notation] a, b, c are branches etc.

N

Orientable case If τ is orientable then there is a train route $\rho: [0,n] \to \tau$ so that $\rho([0,1]) = b$, $\rho([n-1,n]) = c$ and ρ everywhere obeys the orientation of τ.

Nonorientable case: Suppose τ is non-orientable. Pick any orientations on b, c. Then there is a train route $\rho: [0,n] \to \tau$ so that $\rho([0,1]) = b$, $\rho([n-1,n]) = c$, and ρ obeys the chosen orientations on those two intervals (but not necessarily in the middle of ρ).

Proof of 1.3.7: Orientable case: Define the set E ⊂ $B(\tau)$ to be $\{b' \in B(\tau) \mid \exists$ such

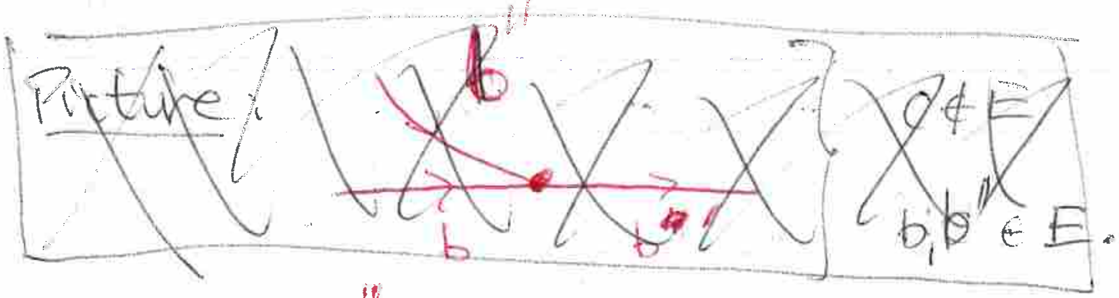
E_a

etc route ρ connecting b^a to $b'\}$.

Note that $b \in E$. Suppose $c \notin E$.

Fix attention on a branch $b'' \notin E$ adjacent to E along switch s.

Picture:



Picture:

Note b'' exists because τ is connected.

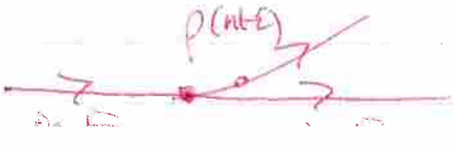
Since τ is recurrent, there is a carried curve $\alpha < \tau$ s.t. $\alpha \cap R(b'') \neq \emptyset$. We may consider α as a periodic train route $\rho : \mathbb{R} \to \tau$. ~~[illegible]~~

Suppose $\rho((0,1)) = b''$. Note ρ obeys the orientations of τ. Since $\rho(1/2) \notin E$, and $\rho(1) \in E$, there is a time $n \in \mathbb{Z}$ s.t. (i) $\rho(n+\varepsilon) \notin E$

(ii) $\rho([1,n]) \subseteq E$.

Thus the ~~edge~~ branch $\rho([n,n+1])$ is not in E, but is reachable from b, ✱.

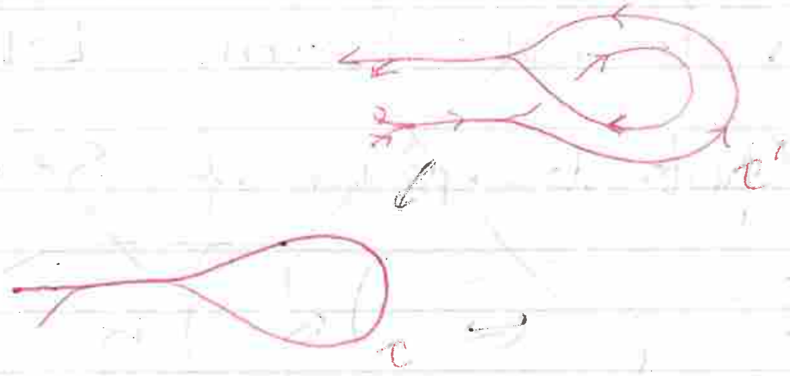
Simpler: α is periodic so $b'' \in E$ //



Nonorientable case:

Consider $N(\tau)$ as a surface. As τ is nonorientable we may form $N'(\tau)$, the double cover of $N(\tau)$ [take two rectangles for each one of N and give them opposite orientations, etc.]

Picture:



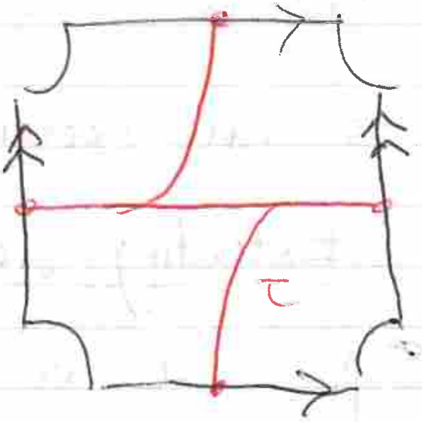
Observe that because τ is non-orientable τ' is connected. Lifting weights proves τ' is recurrent. Now the orientable case (applied to τ'), followed by the covering map, finishes the proof //.

Rmk: The nonorientable case arises when computing the dimension of $P(\tau) \subseteq PML(S)$.

We now begin the proof of

<u>Theorem 1.3.6</u>: Suppose S is oriented, connected, compact, with $\chi(S) \leq -1$.

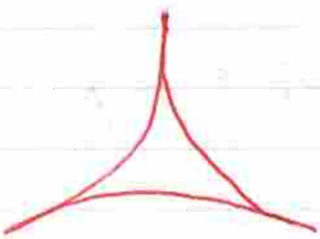
(i) If $S \cong S_{1,1}$ then ~~the only~~ any complete track $\tau \subset S$ is homeomorphic to the standard track

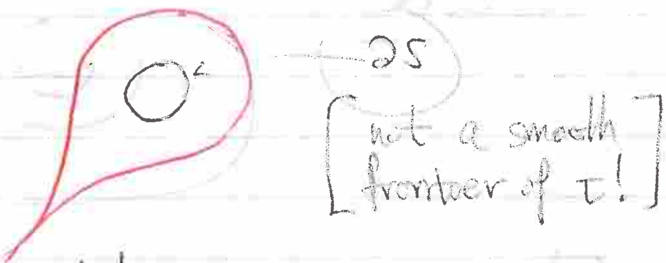


(ii) If $S \not\cong S_{1,1}$ then any complete track is maximal.

Recall the definitions:

<u>Def</u>: A track $\tau \subset S$ is <u>maximal</u> if all regions of $S - \tau$ are either ideal triangles or ideal once-holed monogons





<u>Def</u>: A track $\tau \subset S$ is <u>complete</u> if τ is birecurrent and there is no

supertrack $\sigma \supset \tau$ that is also birecurr.

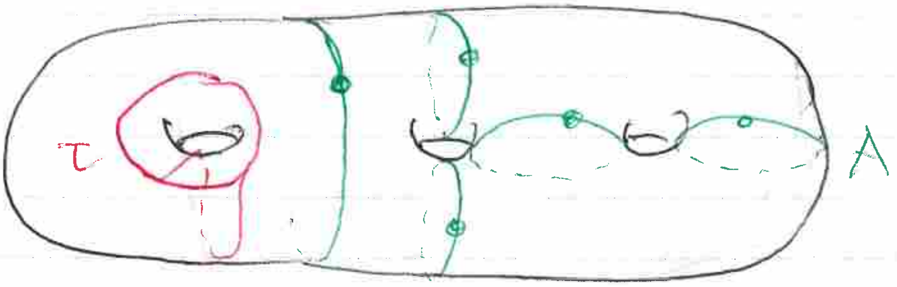
Thus the proof of 1.3.6 involves <u>extending</u> tracks while preserving birecurrence.

(20) Case (i) of 1.3.6 is deferred to the very end. Perhaps it could be considered as an exercise.

(21) <u>Extending trans. recurr. tracks</u>

Suppose τ is trans. recurrent track. Fix $\nu : B(\tau) \longrightarrow \mathbb{N}_{>0}$ a <u>positive</u> <u>even</u> tangential measure. [<u>Corollary 1.3.5</u> such ν exists iff τ is trans. recurrent.] Let A be a pants decomposition of $S - \tau$.

<u>Picture</u>



Place one switch on each component of A

Form $\tau' = \tau \cup A$. Extend ν to ν' via

$$\nu'(b) = \begin{cases} \nu(b), & \text{if } b \in \tau \\ 2, & \text{if } b \in A \end{cases}$$

<u>Exercise</u> ν' is ~~an~~ positive even tangential measure.

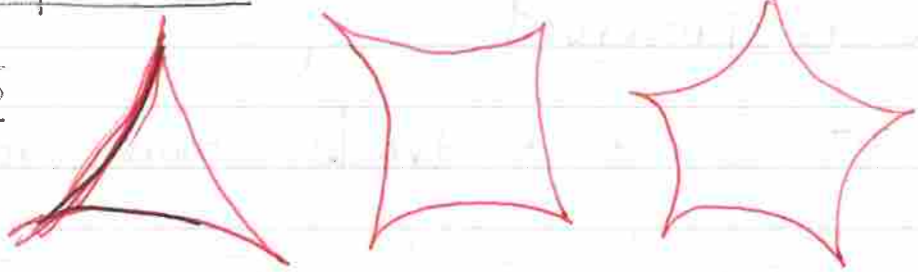
Thus by Corollary 1.3.5, τ' is trans. recurrent.

Deduce that the components of $S - \tau'$ are disks, annuli, or pairs of pants.

<u>Some pictures</u>:

<u>Disks</u>

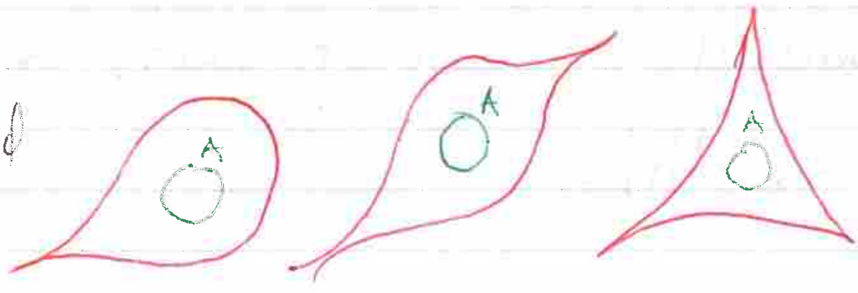
ideal n-gon



n 3   4   5

<u>Annuli</u>

ideal once holed n-gon



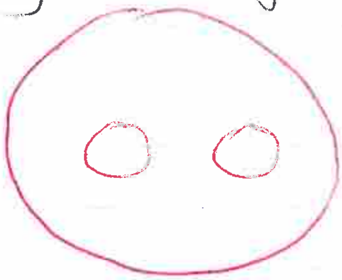
n 1   2   3

<u>Better</u>: only consider $\rightsquigarrow$ fewer cases!

$\tau' = \tau \cup A$

Rmk: There is a second family, where the inner boundary belongs to ∂S.

Pair of pants:



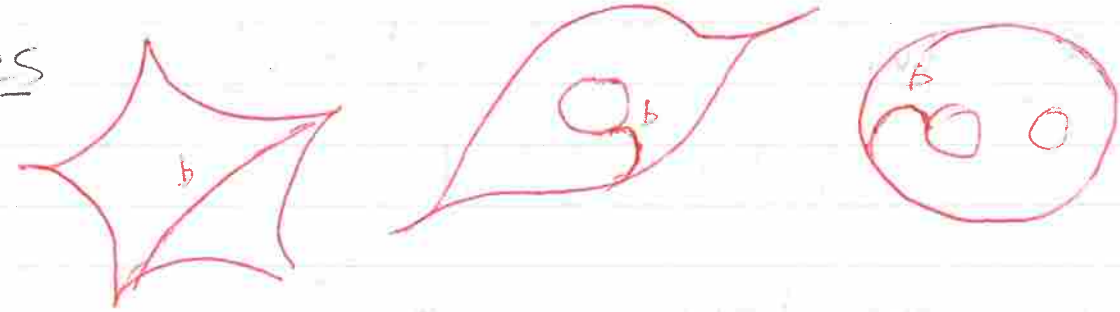
[Any number of cuffs belong to τ or to A.
At most two cuffs belong to ∂S.]

~~Penner gives six "moves": each move involves adding a single branch in a component, enlarging τ! In each case he claims transverse recurrence is preserved.~~

Suppose $\sigma \subset S$ is a track (trans. recurrent) and the components of $S - \sigma$ are as listed above (disks, annuli with exactly one smooth boundary, or pair of pants with all cuffs smooth).

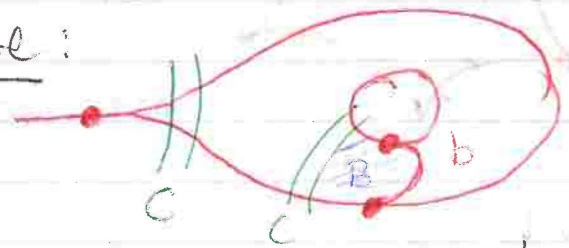
Let $\sigma' = \sigma \cup b$ be ~~the~~ any track obtained by adding exactly one branch to σ.

Pictures



Penner claims that σ' is transversely recurrent. This is false.

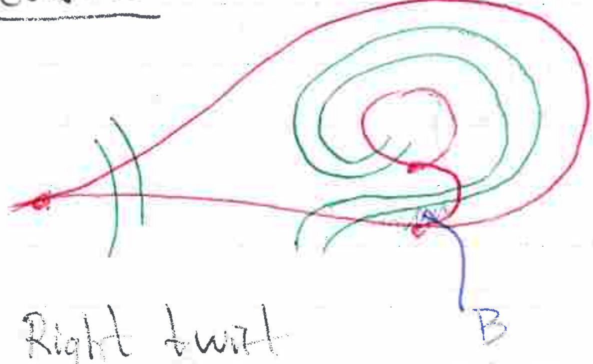
Example:



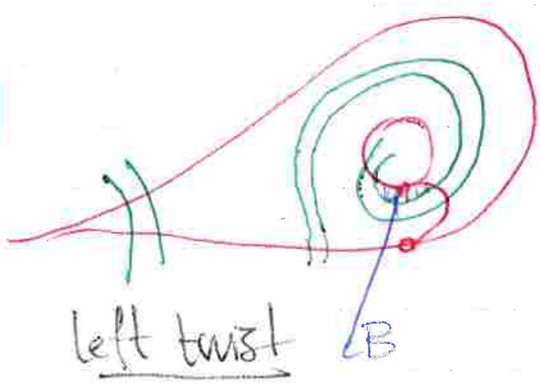
The green multi-curve C proves that σ is transversely recurrent.

After adding b, the multicurve C is no longer in efficient position (because of the shaded bigon B). Penner (page 30, line 7) says we may apply a Dehn twist to C as needed. However this doesn't save us:

Pictures



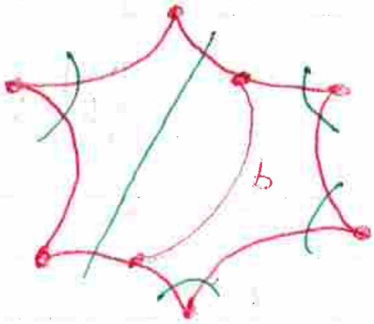
Right twist



left twist

Once you start looking for it, this problem occurs many times on page 30

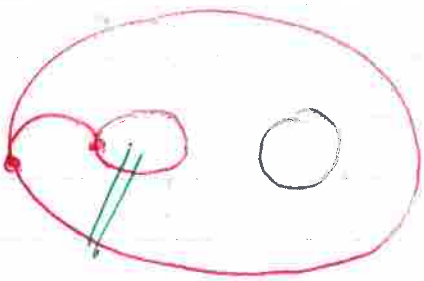
Example in a disk



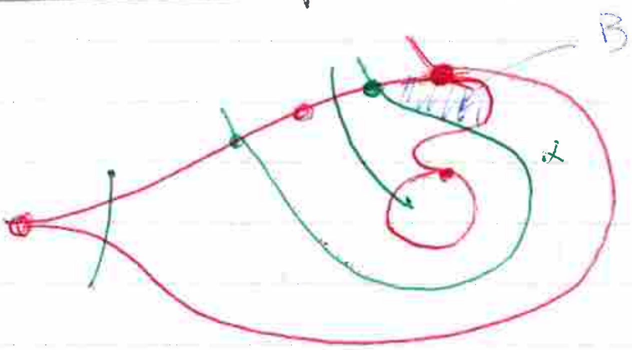
Here no Dehn twists are possible

Example in a pair of pants

No single Dehn twist salvages this situation



Another example in annulus:



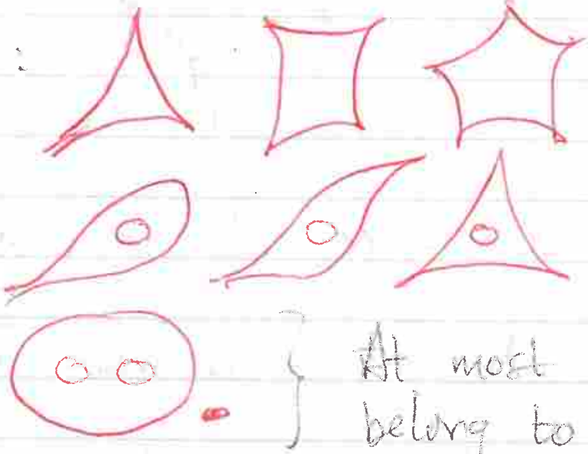
Now Dehn twists do not move the arc α, so the bigon B cannot be removed.

S

(22) Extending trans. recurrent tracks II

Suppose $\sigma \subset S$ is trans recurrent and all components of $S - \sigma$ are n-gons, once-holed n gons, or pairs of pants

Pictures:



The inner boundary may belong to ∂S.

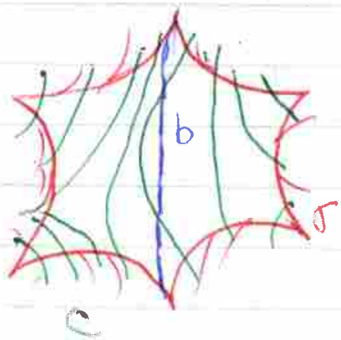
At most two boundary components belong to ∂S

Fix a component R of $S - \sigma$.

Claim (Disk): If R is a disk and $b \subset R$ is a branch connecting non-adjacent cusps then $\sigma \cup b$ is trans. recurrent.

Proof: Since σ is trans recurrent we are given a dual multicurve C hitting all branches of σ

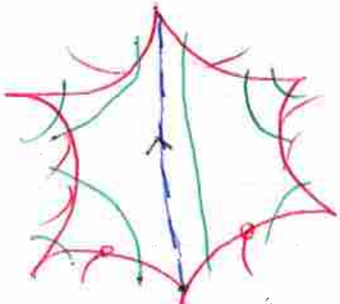
Picture



Any bigon between b and C occurs in the interior of R (because C avoids the switches of σ)

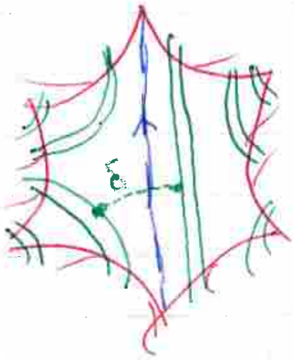
So we may isotope C (rel $S-R$) to remove all bigons. If $c\cap b\neq\emptyset$ then we are done.

Suppose $c\cap b=\emptyset$. Picture:



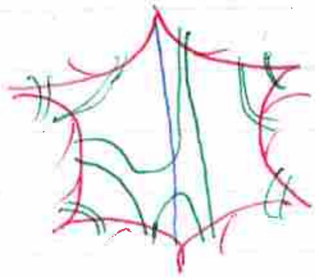
Orient b and double c to get

Picture:



Pick $\gamma, \gamma' \subset 2c\cap R$ the arcs of $2c\cap R$ nearest the beginning of b, on either side. Pick δ an arc from γ to γ' crossing b once.

Surger $2c$ along δ Picture:



[Move (4) in Penner]

Claim (Annulus I)

If R is an annulus (once-holed n-gon, $n\geq 2$) and $b\subset R$ is a branch connecting non adjacent cusps (i.e. $R-b$ has no ideal bigon component) then $\sigma\cup b$ is trans recurrent.

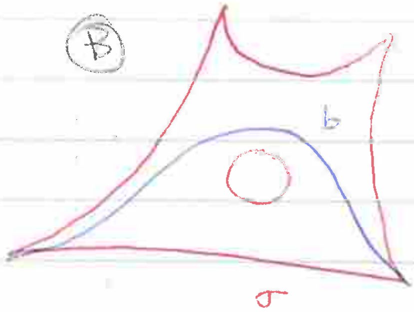
II

Proof:

Pictures

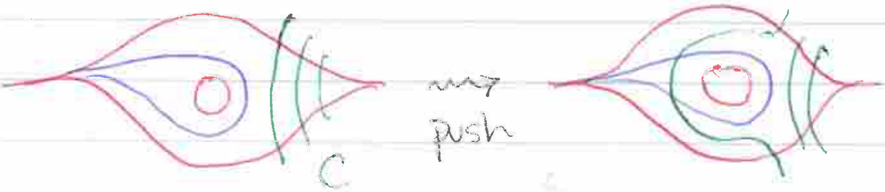
(A)

(B)



Pf: Again all bigons (w/ 2 corners) must lie in interior of R so can be isotoped away. If $c \cap b \neq \phi$ we are done. If not; in (A) push an arc of c across the inner boundary.

Picture:



In (B), if we can't push, we surger (after doubling)

Picture

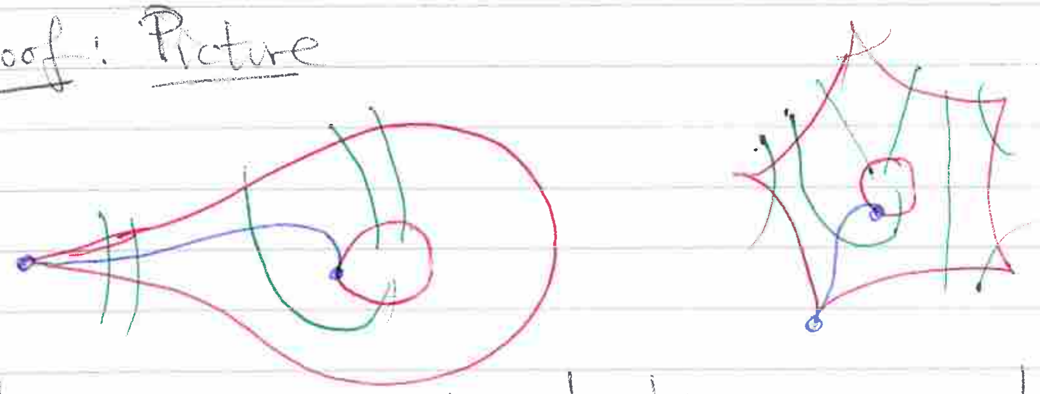
//

[Moves (2) and (6) in Penner]

Claim (Annulus II)

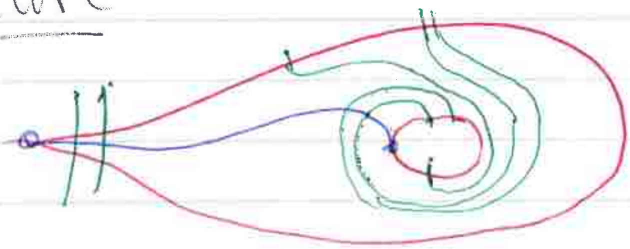
If R is an annulus (once holed n-gon $n \geq 1$) and $b \subset R$ is a branch connecting a cusp to the smooth boundary, then $\sigma \cup b$ is trans recur.

Proof: Picture



Here removing internal bigons may not produce efficient position. So, in addition Dehn twist C in the opposite direction of the way b meets the inner boundary.

Picture

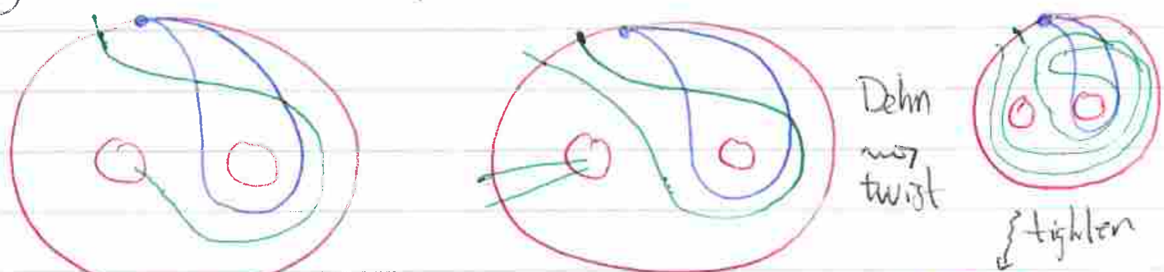


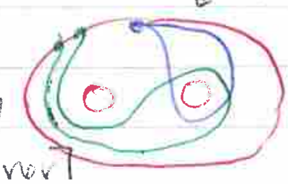
[Replaces Move (3) in Penner] //

Claim (Pants I)

If R is a pants and $b \subset R$ is a branch meeting only one cuff, then $\sigma \cup b$ is trans. recurr.

Proof:





Again, the Dehn twist opposite the direction of b saves us. // [Move (5) of Penner]

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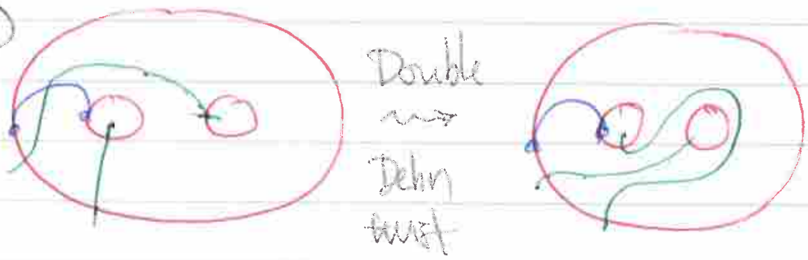
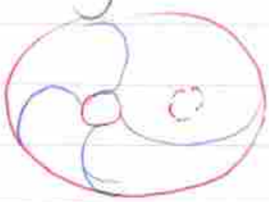
Claim (Pants II)

If R is a pants and $b \subset R$ is a branch meeting two cuffs, then $\tau \cup b$ is trans recur.

Proof: Picture (A)



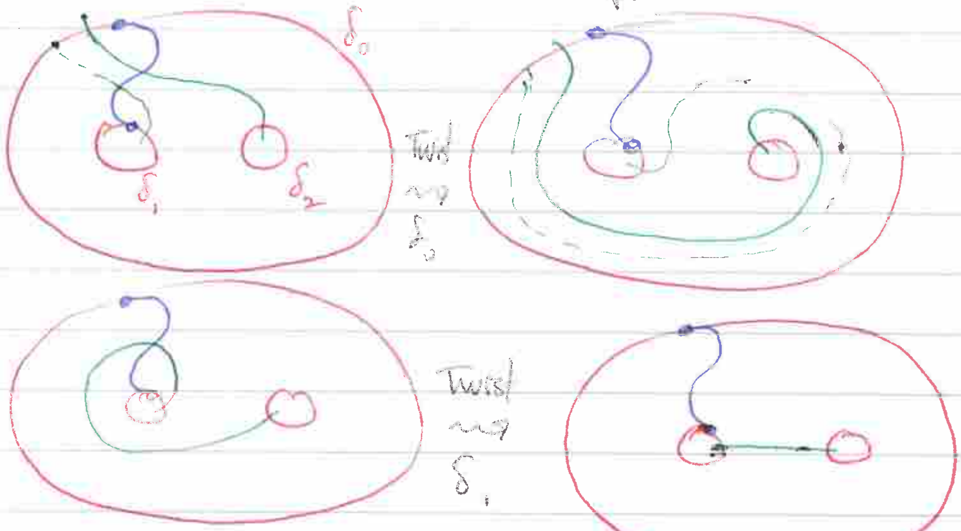
The double Dehn twist (aka point pushing) is in the direction opposite that given by b. [Think of b determining a "Reeb foliation" on the annulus



and we point push in the opposite direction.]

Picture (B)

Here the twists agree with each other



[Fixes Move (4) of Penner]

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