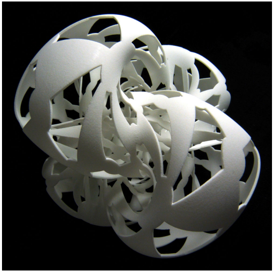
  

Symmetry and the Klein quartic

Saul Schleimer, University of Warwick

(Joint work with Henry Segerman)

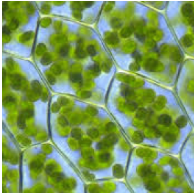
2017-06-08, 3D Printing Workshop, Durham

Tilings

Tilings everywhere

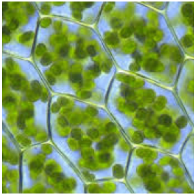
Frieze, British Museum

Tilings everywhere



Frieze, British Museum
Plant cells, Wikipedia

Tilings everywhere

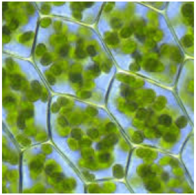




Frieze, British Museum
Plant cells, Wikipedia
Honeycomb, Wikipedia

Tilings everywhere

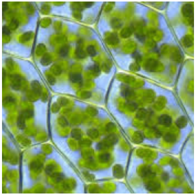
Frieze, British Museum
Plant cells, Wikipedia
Honeycomb, Wikipedia
Bricks, London

Tilings everywhere

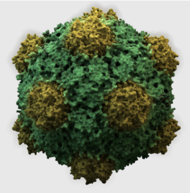




Frieze, British Museum
Plant cells, Wikipedia
Honeycomb, Wikipedia
Bricks, London
Soccer ball, Wikipedia

Tilings everywhere

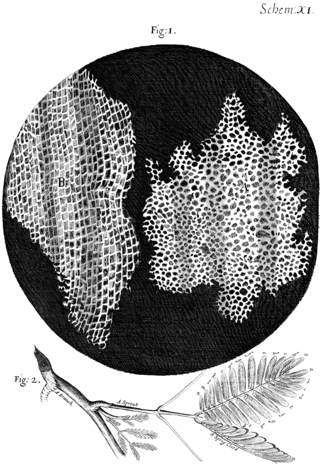







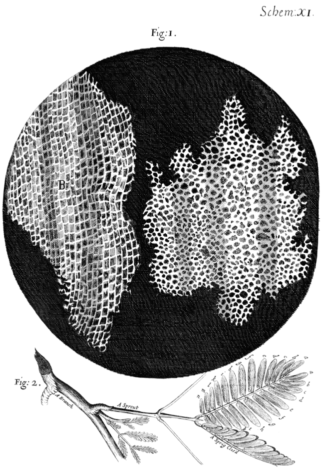
Frieze, British Museum
Plant cells, Wikipedia
Honeycomb, Wikipedia
Bricks, London
Soccer ball, Wikipedia
Virus, Wikipedia

Cells



From Robert Hooke's Micrographia (1664)

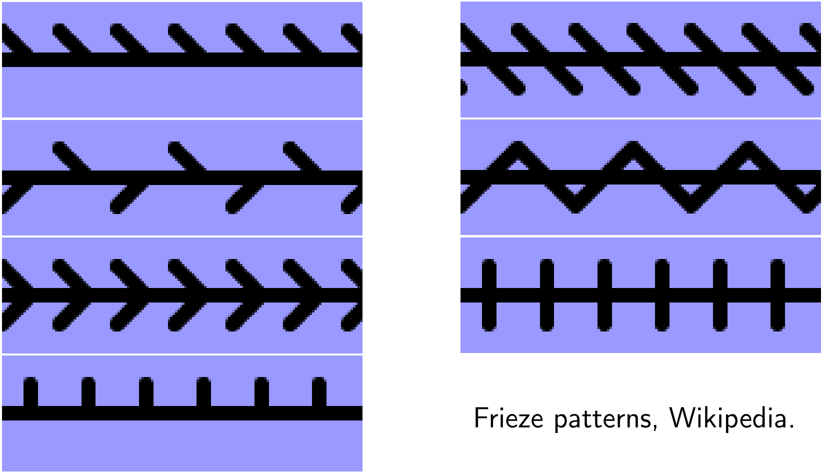
Cells



From Robert Hooke's Micrographia (1664)

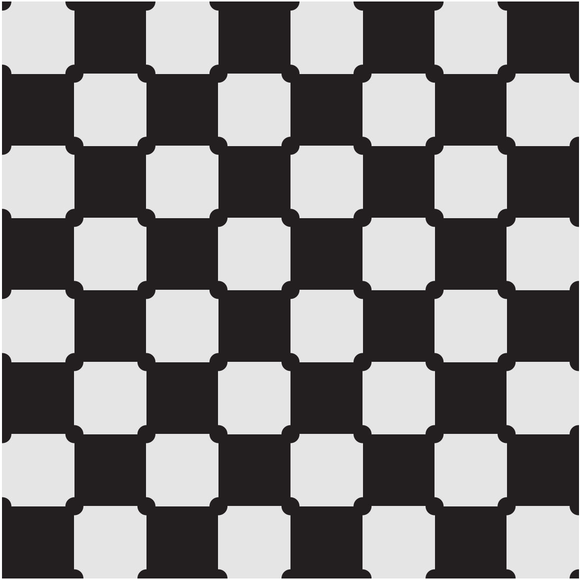
Observ. XVIII. *Of the Schematisme or Texture of Cork, and of the Cells and Pores of some other such frothy Bodies*.

Frieze patterns

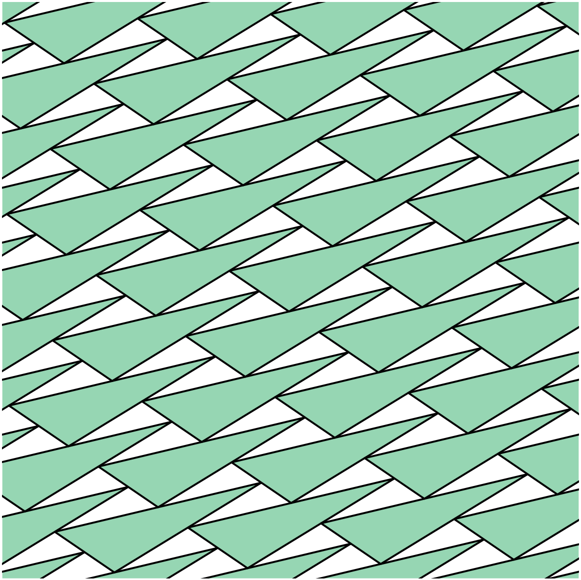


Frieze patterns, Wikipedia.

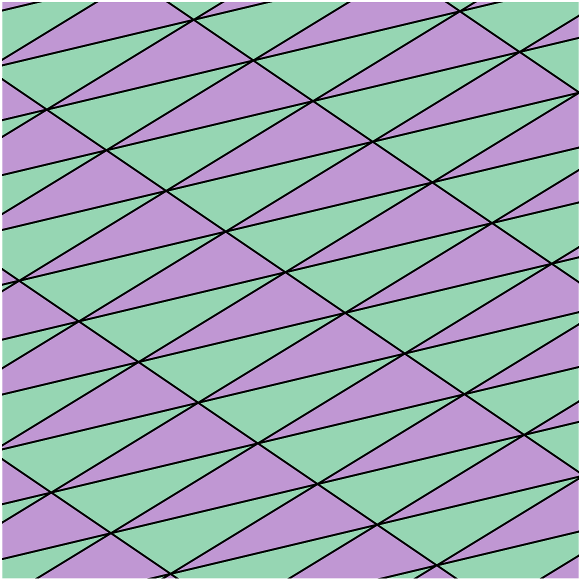
Wallpaper groups



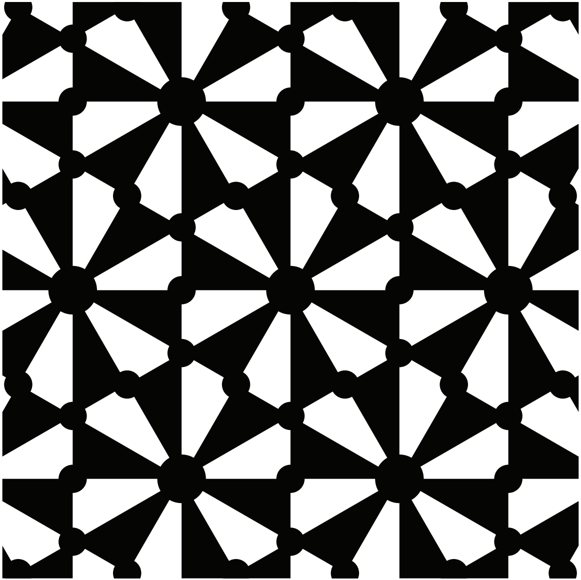
Triangles do not tile



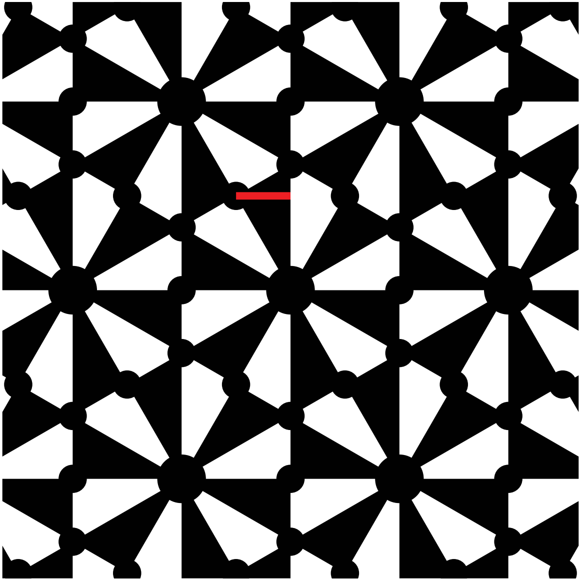
Triangles do tile!



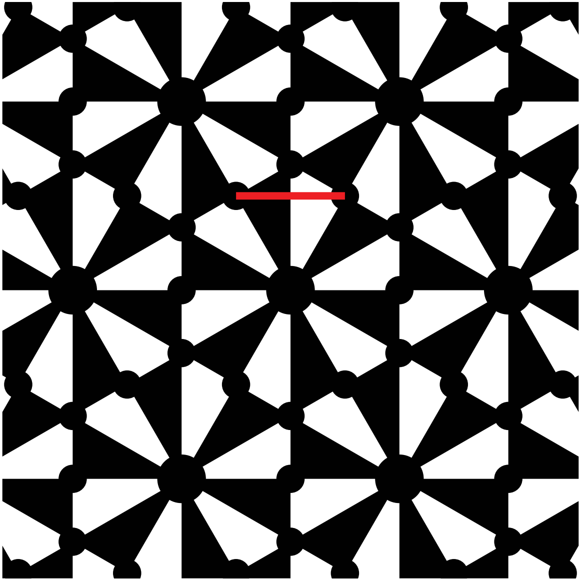
Reflections



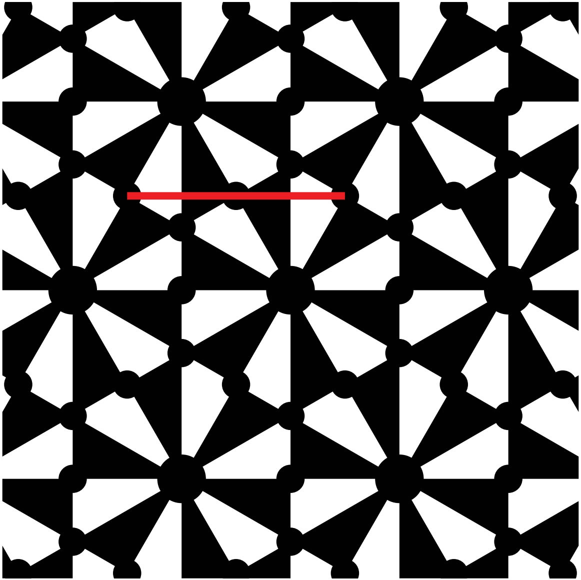
The kite path



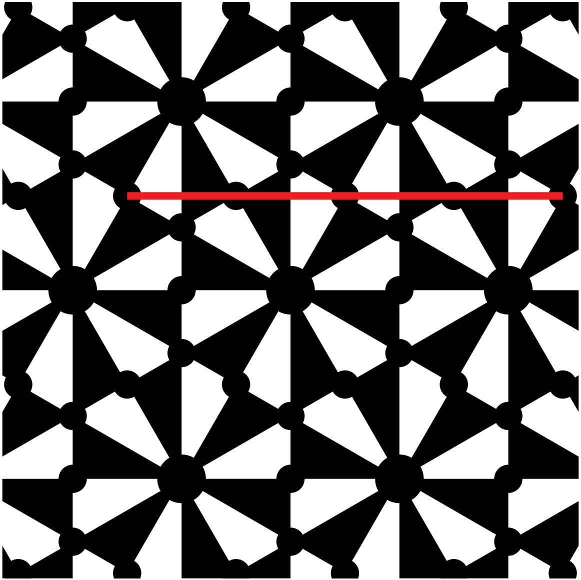
The kite path



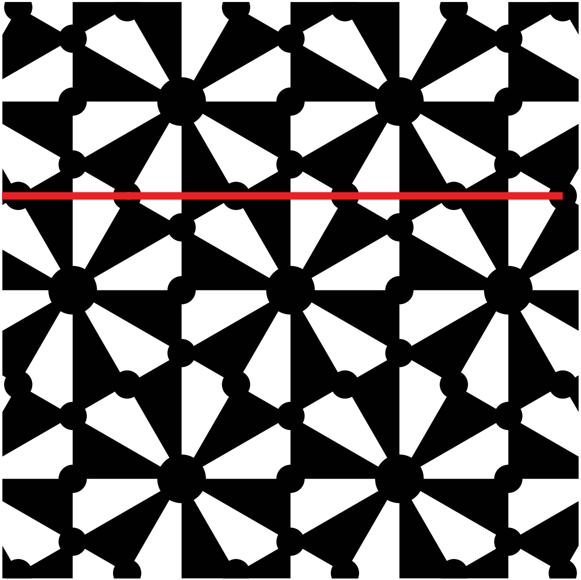
The kite path



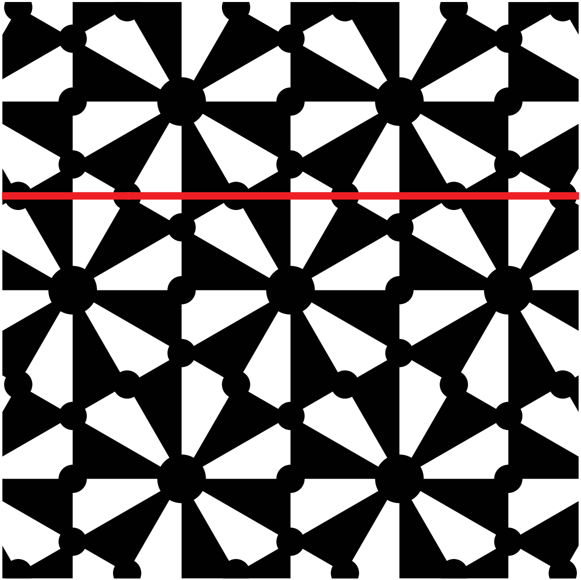
The kite path



The kite path

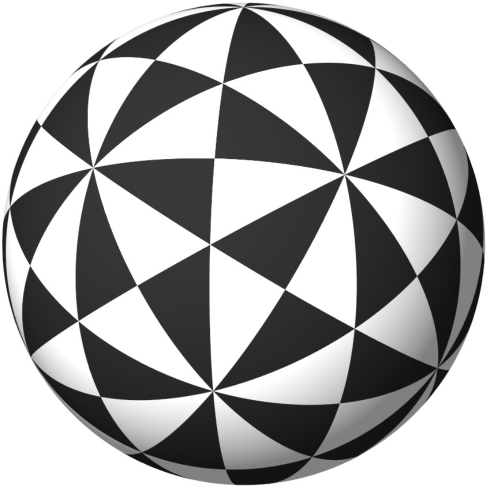


The kite path



Non-euclidean geometry, I

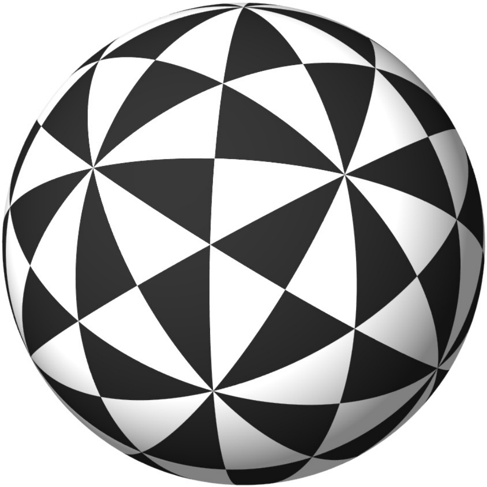
Non-euclidean geometry, I



Non-euclidean geometry, I

Non-euclidean geometry, I

Non-euclidean geometry, I



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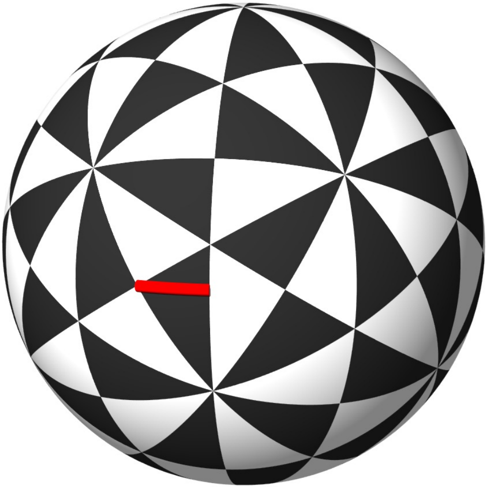

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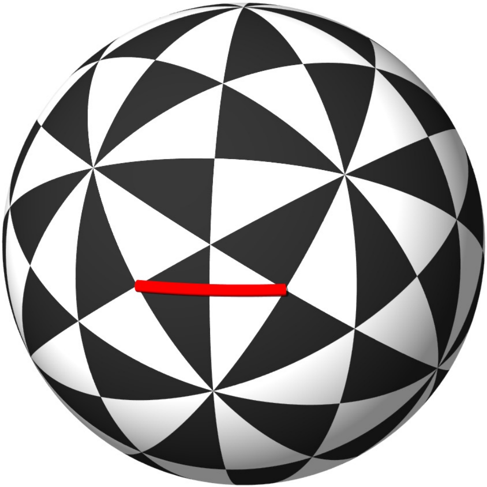
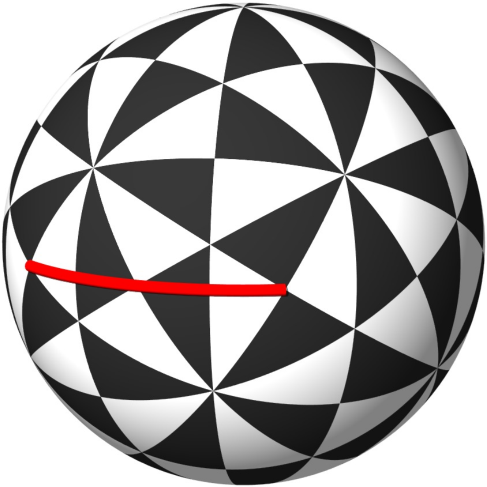


Non-euclidean geometry, I



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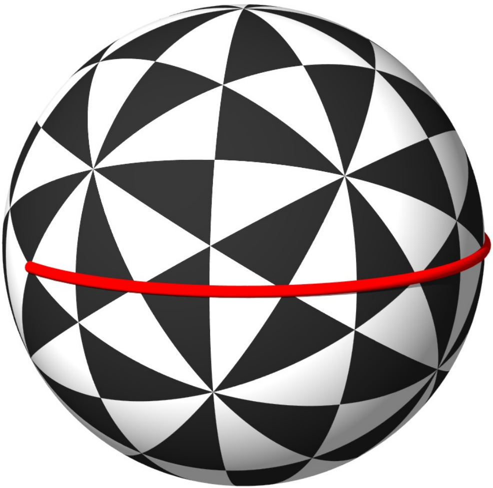
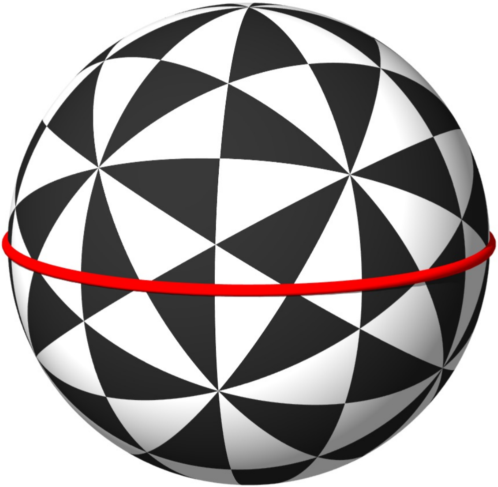
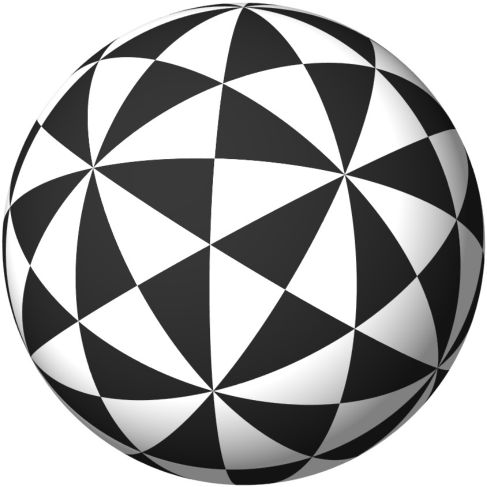


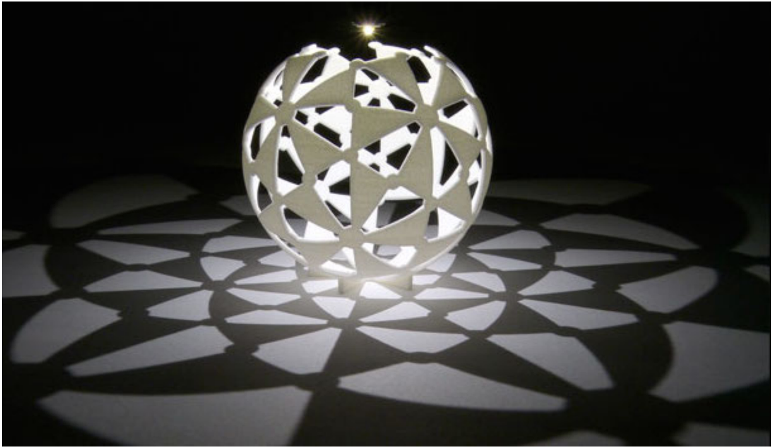
Non-euclidean geometry, I



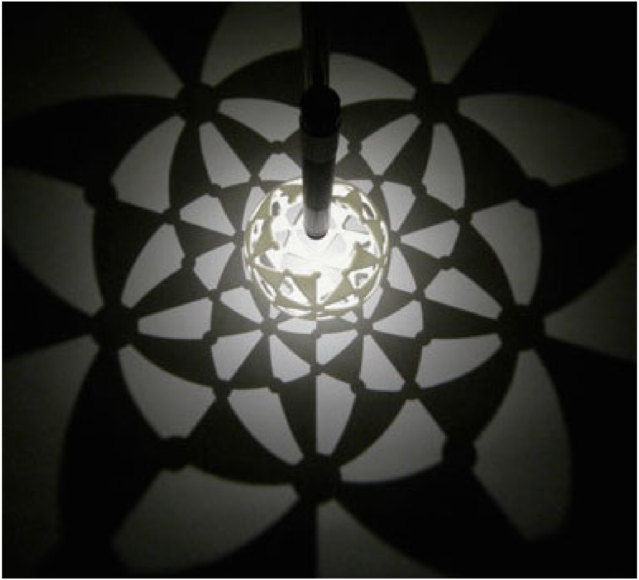
Stereographic projection



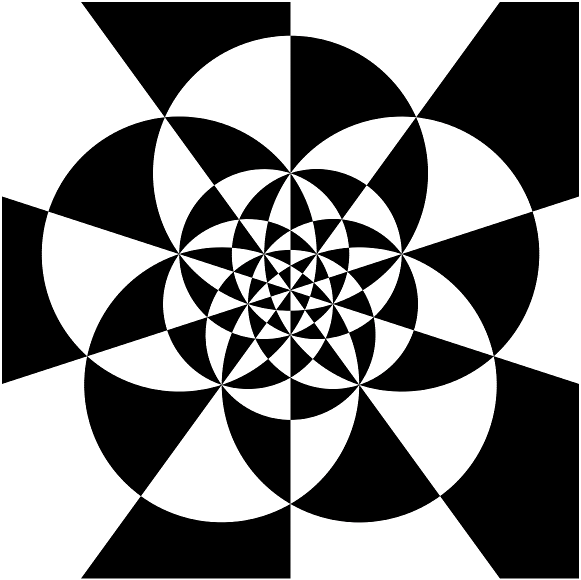
Stereographic projection



Stereographic projection



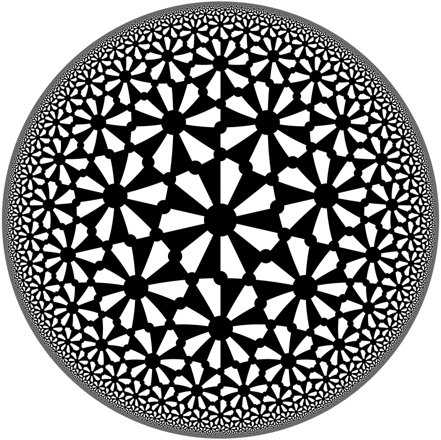
Stereographic projection



Non-euclidean geometry, II

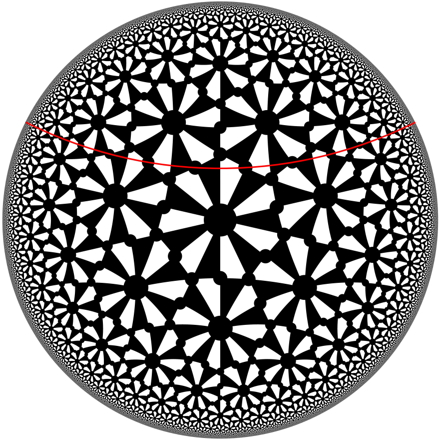
M.C. Escher, Circle Limit III

Non-euclidean geometry, II



Roice Nelson, $(2, 3, 7)$ tiling

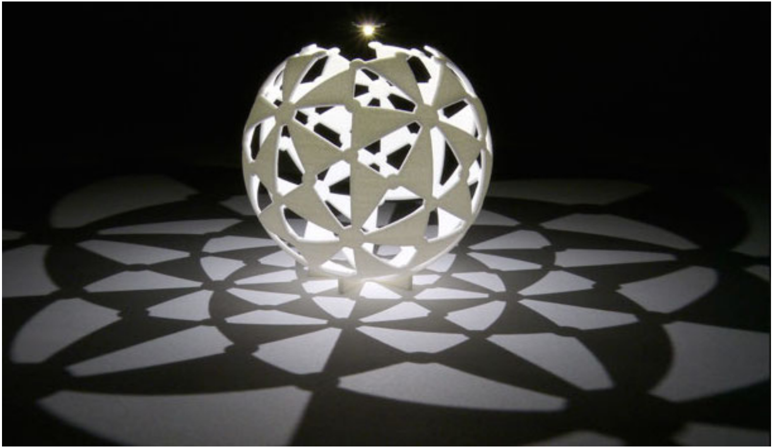
Non-euclidean geometry, II



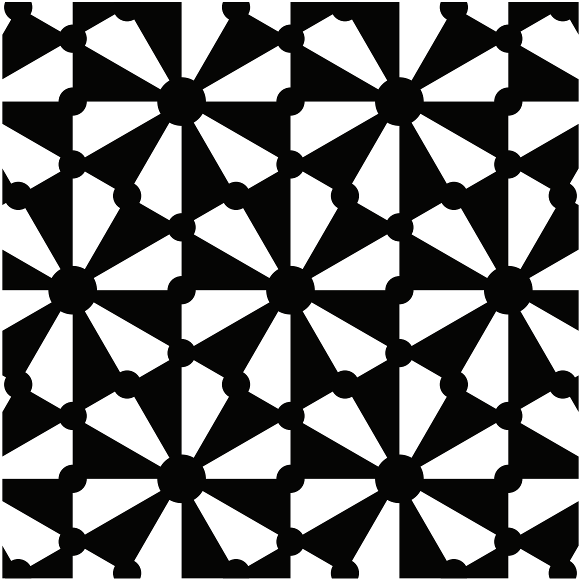
Roice Nelson and Henry Segerman, $(2, 3, 7)$ tiling with kite path

Covers and quotients

Finite versus infinite



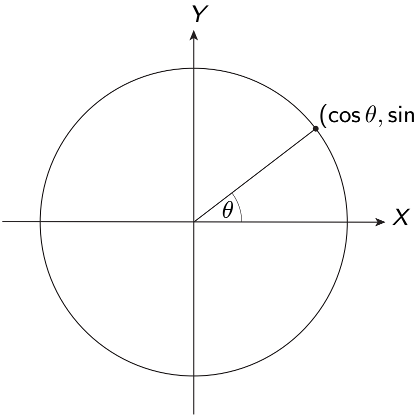
Finite versus infinite



Cylinder seals

Late Urak cylinder seal, about 3300-3000 BC. British Museum.

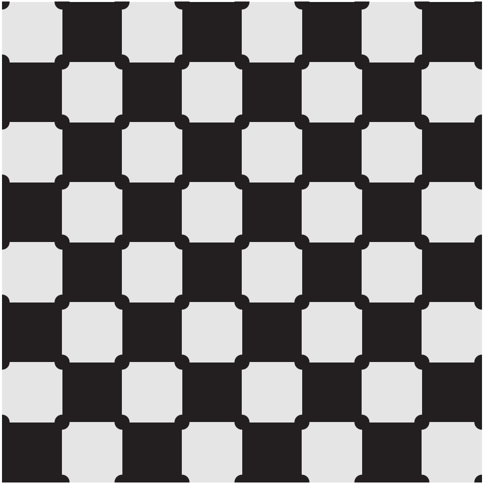
$X^2 + Y^2 = 1$



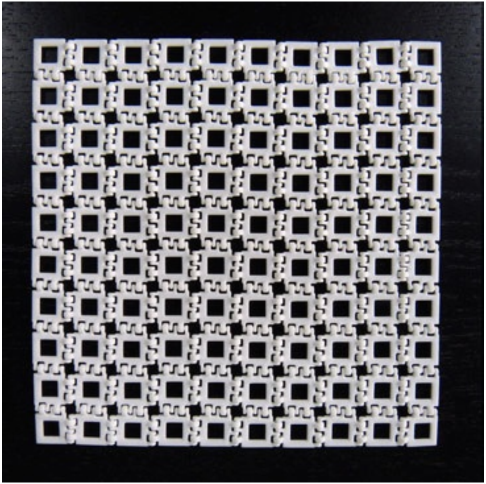
$$\cos(\theta) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$
$$= \sum (-1)^k \frac{\theta^{2k}}{(2k)!}$$

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$
$$= \sum (-1)^k \frac{\theta^{2k+1}}{(2k+1)!}$$

Wrapping up a chequerboard

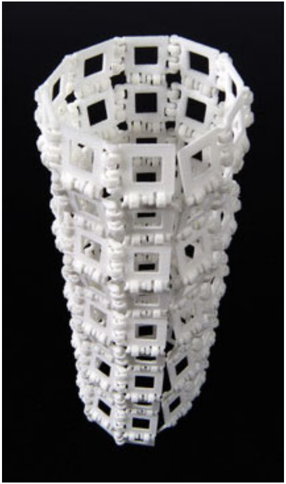


Wrapping up a chequerboard

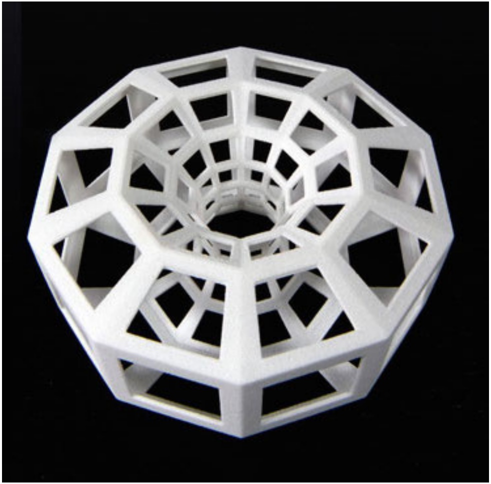


Wrapping up a chequerboard

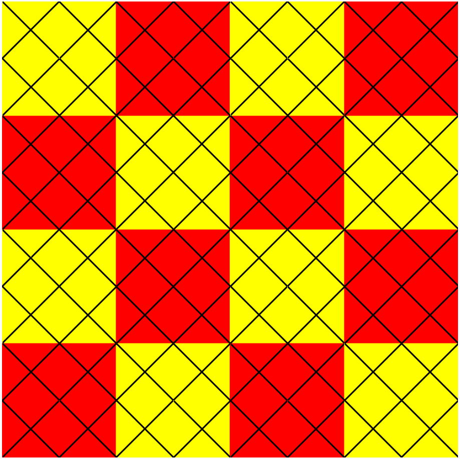
Wrapping up a chequerboard



Wrapping up a chequerboard

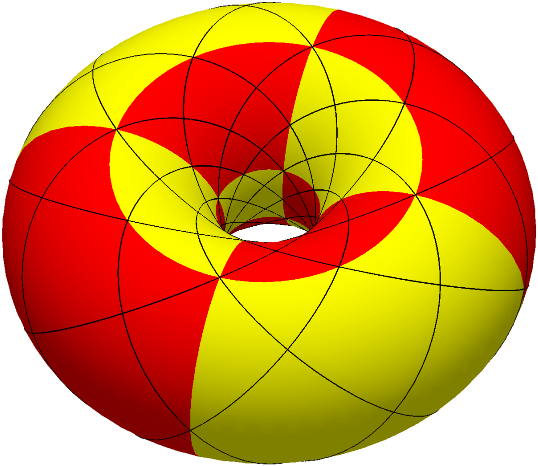


Wrapping up a chequerboard



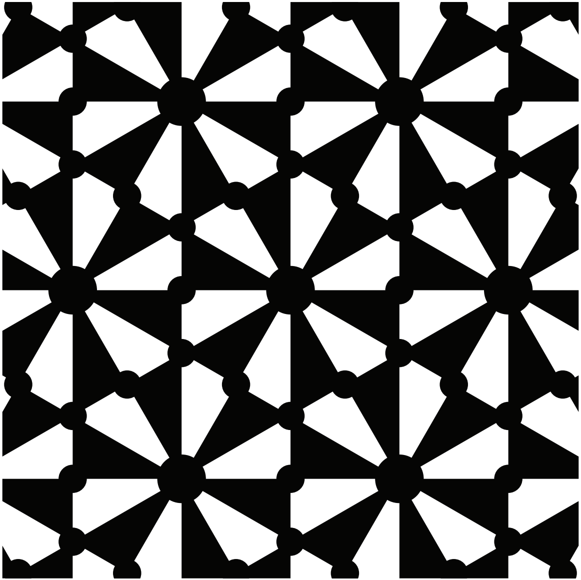
John Sullivan, Conformal tiling on a torus, Figure 1

Wrapping up a chequerboard

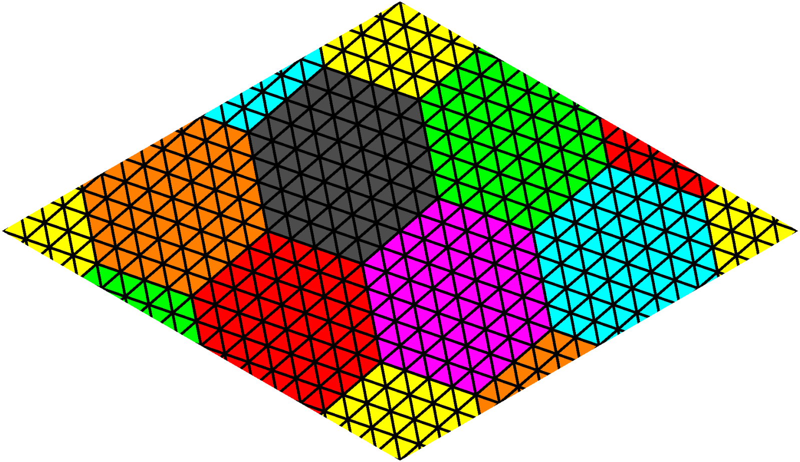


John Sullivan, Conformal tiling on a torus, Figure 1

Wrapping up the $(2, 3, 6)$ tiling

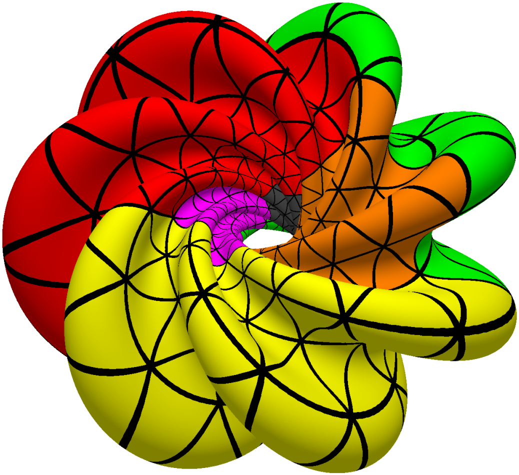


Wrapping up the $(2, 3, 6)$ tiling



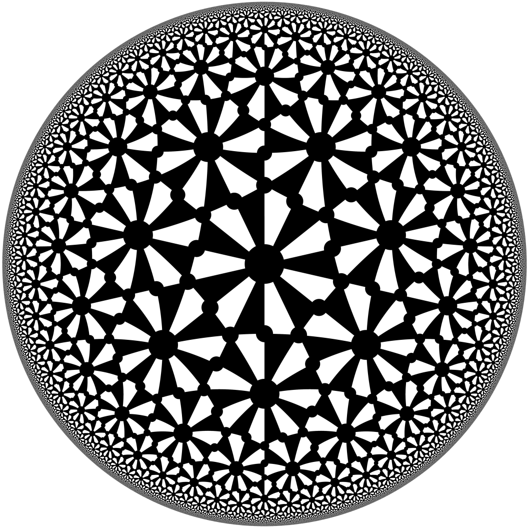
John Sullivan, Conformal tiling on a torus, Figure 4

Wrapping up the $(2, 3, 6)$ tiling



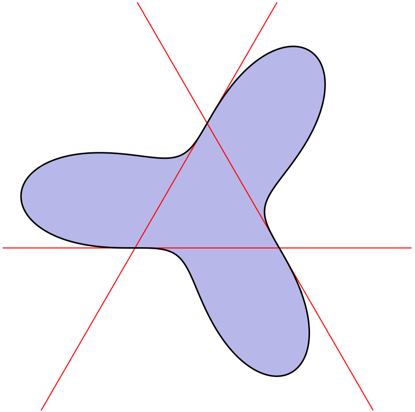
John Sullivan, Conformal tiling on a torus, Figure 5

Wrapping up the $(2, 3, 7)$ tiling



The Klein quartic

$Q : X^3Y + Y^3Z + Z^3X = 0$

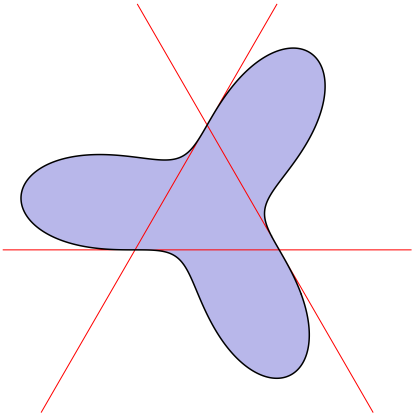


The real points of Q in the plane $X + Y + Z = 1$. The defining equation is degree four and is *homogeneous*.

Genus

Genus formula: A smooth curve X in $\mathbb{CP}^2$ of degree d has genus $g(X) = (d-1)(d-2)/2$. [So $g(Q) = (4-1)(3-1)/2 = 3$]

Symmetries of $Q : X^3Y + Y^3Z + Z^3X = 0$



The origin is a point of order three. Also, there are three reflection lines.

Symmetries of $Q : X^3Y + Y^3Z + Z^3X = 0$

$$r' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad t = \begin{bmatrix} \omega^4 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}$$

$$s = \frac{-2}{\sqrt{7}} \begin{bmatrix} \sin 2\alpha & \sin 3\alpha & -\sin \alpha \\ \sin 3\alpha & -\sin \alpha & \sin 2\alpha \\ -\sin \alpha & \sin 2\alpha & \sin 3\alpha \end{bmatrix}$$

Symmetries of $Q : X^3Y + Y^3Z + Z^3X = 0$

$$r' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad t = \begin{bmatrix} \omega^4 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}$$

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Here $\alpha = \pi/7$ and $\omega^7 = 1$ is a primitive root of unity.

Symmetries of $Q : X^3Y + Y^3Z + Z^3X = 0$

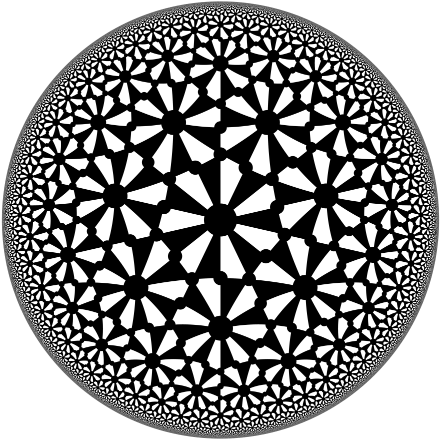
$$r' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad t = \begin{bmatrix} \omega^4 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}$$

$$s = \frac{-2}{\sqrt{7}} \begin{bmatrix} \sin 2\alpha & \sin 3\alpha & -\sin \alpha \\ \sin 3\alpha & -\sin \alpha & \sin 2\alpha \\ -\sin \alpha & \sin 2\alpha & \sin 3\alpha \end{bmatrix}$$

Here $\alpha = \pi/7$ and $\omega^7 = 1$ is a primitive root of unity.

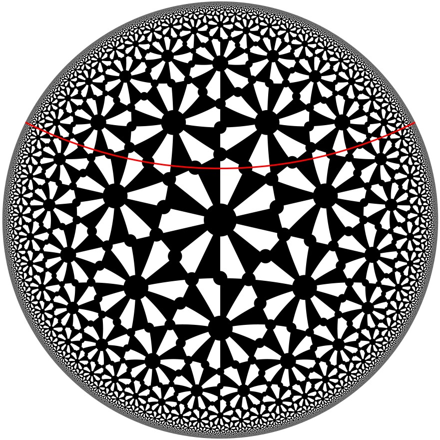
So t has order seven, s has order two, and $r = TS$ has order three (and is conjugate to r'). However, we also have $(tsTS)^4 = 1$.

Symmetries of the $(2, 3, 7)$ tiling



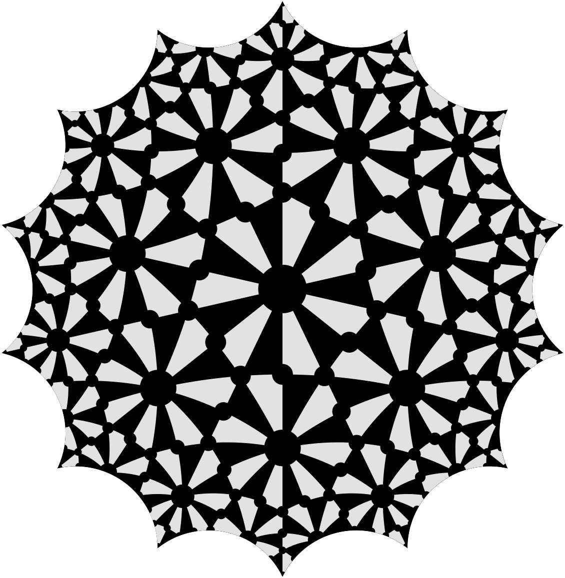
Note s and t have orders two and seven. The product $r = TS$ is a rotation of order three. However, the element $tsTS$ is not finite order.

Symmetries of the $(2, 3, 7)$ tiling

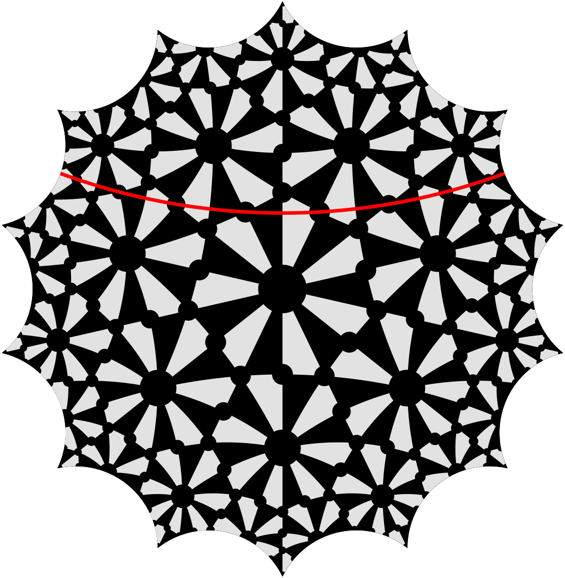


Note s and t have orders two and seven. The product $r = TS$ is a rotation of order three. However, the element $tsTS$ is not finite order. It is the kite path!

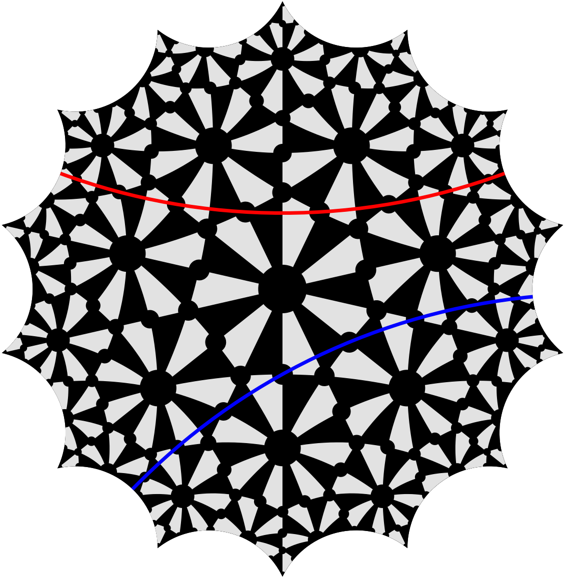
Fundamental domain



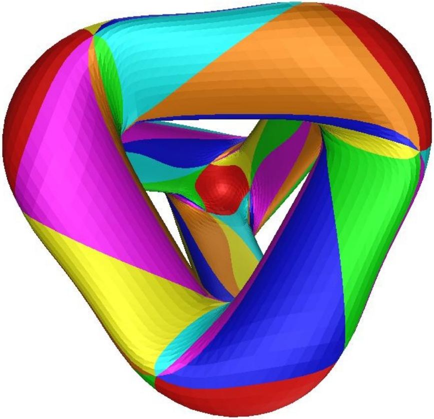
Fundamental domain



Fundamental domain

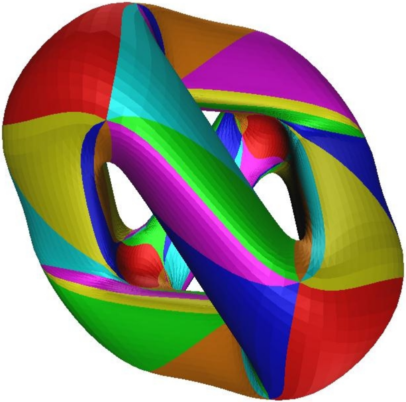


Topological models



Joe Christy

Topological models

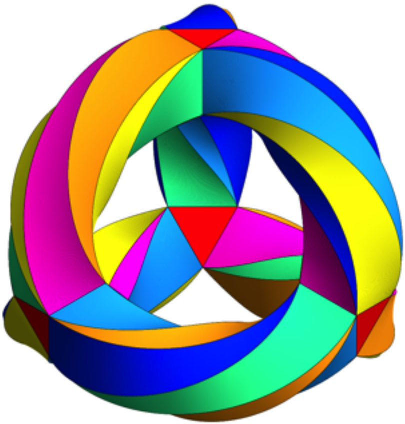


Joe Christy

Topological models

Carlo Séquin, Patterns on the genus-three Klein quartic

Topological models



Greg Egan, Klein's quartic curve

Topological models

Helaman Ferguson, The eightfold way

Ramanujan's q–series

$$a = \sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{(14n+5)^2}$$

$$b = \sum_{n=-\infty}^{\infty} (-1)^{n} q^{(14n+3)^2}$$

$$c = \sum_{n=-\infty}^{\infty} (-1)^{n} q^{(14n+1)^2}$$

Here $z = x + iy$ is a point in the upper-half plane ($y > 0$) and $q = \exp(2\pi iz/56)$. The q–series a, b, and c satisfy the quartic equation! [Lachaud, Berndt, Ramanujan, Klein] This gives a parametrisation of Q.

Extracting Q from $\mathbb{CP}^2$

Extracting Q from $\mathbb{CP}^2$

mathoverflow

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Map of the Klein quartic from CP^2 to R^3

4

2

The Klein quartic Q is cut out of $\mathbb{CP}^2$ by the homogeneous equation

$$x^3y + y^3z + z^3x = 0.$$

It has 168 orientation preserving automorphisms and includes several copies of the tetrahedral group (with twelve elements).

> Is there a nice way to take the points of Q in $\mathbb{CP}^2$, map them to $\mathbb{R}^3$ (preserving one of the tetrahedral symmetry groups) and so produce an embedded, compact, genus three surface?

There are already a number of models of the Klein quartic in $\mathbb{R}^3$. So far we've found the two by Joe Christy and Greg Egan (see this webpage by John Baez) and also a version by Carlo Sequin. As far as we (Saul Schleimer and I) can tell, these are all "topological" models and not obtained by mapping from $Q \subset \mathbb{CP}^2$ in some sensible way.

ag.algebraic-geometry algebraic-curves

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edited 1 min ago

asked Mar 16 at 20:02

Henry Segerman

956 • 14 • 21

Bihomogeneous polynomials

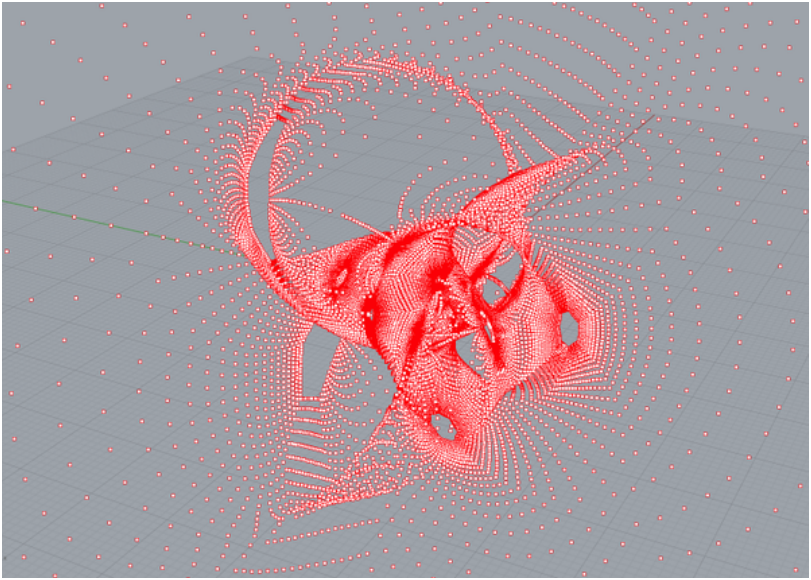
Noam Elkies says to look for degree $2d$ bihomogeneous polynomial functions that are equivariant with respect to the A_4 action. Here are a few examples:

$$(Y\bar{Z}, Z\bar{X}, X\bar{Y})/(X\bar{X} + Y\bar{Y} + Z\bar{Z})$$

$$(YZ\bar{X}^2, ZX\bar{Y}^2, XY\bar{Z}^2)/(X\bar{X} + Y\bar{Y} + Z\bar{Z})^2)$$

We found all such for $d = 1, 2, 3$. Next we took linear combinations, searching for an embedding.

Progress



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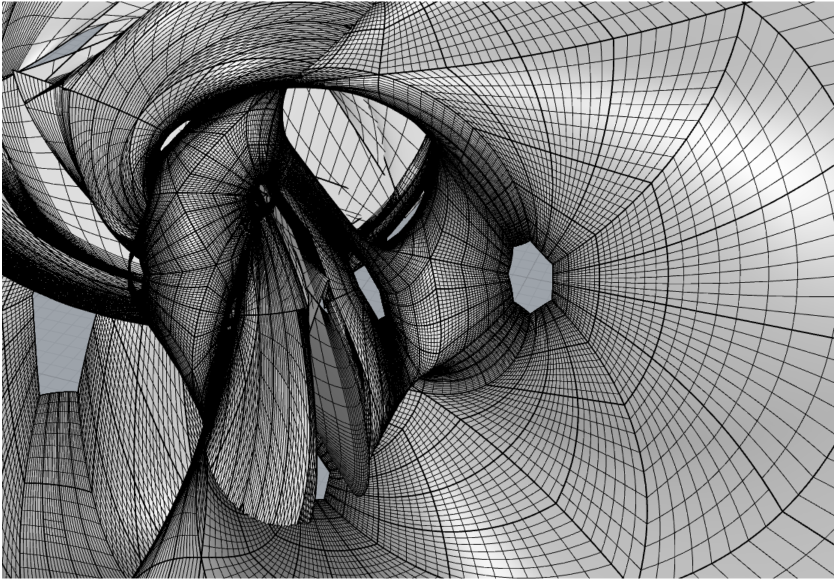

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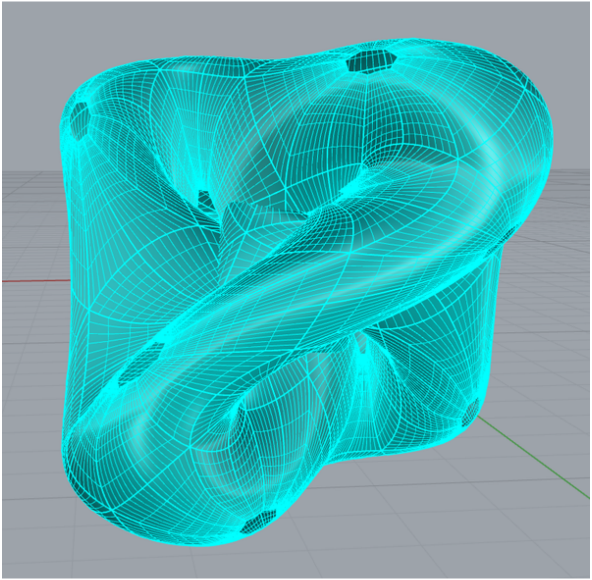

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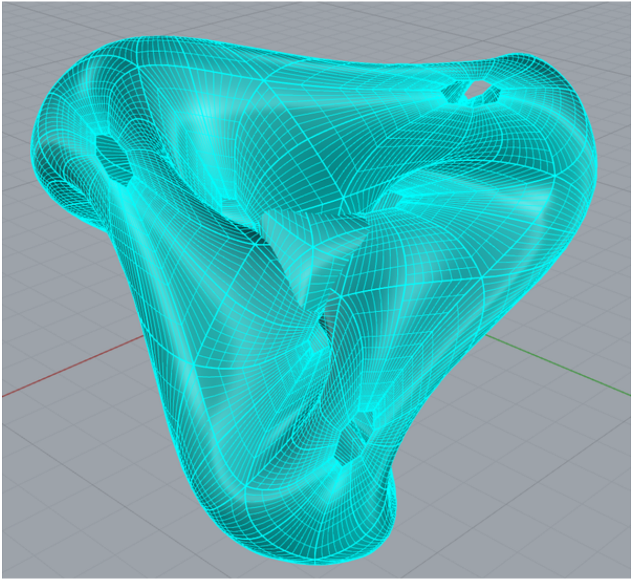

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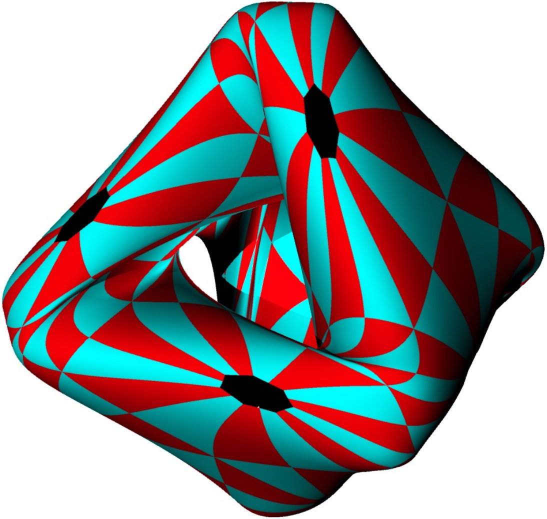

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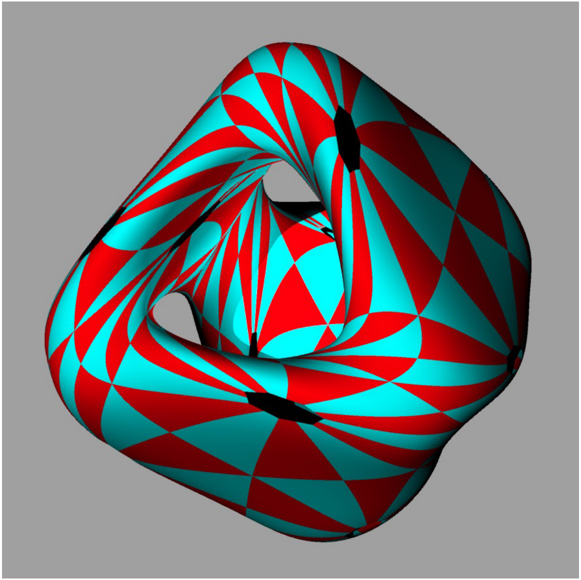

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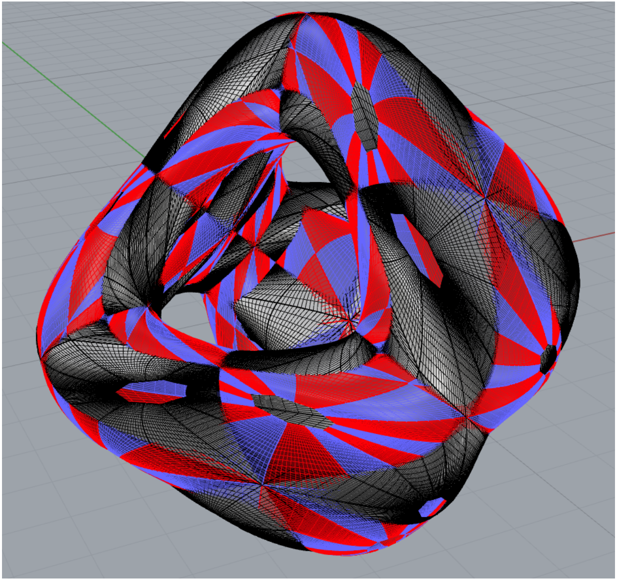

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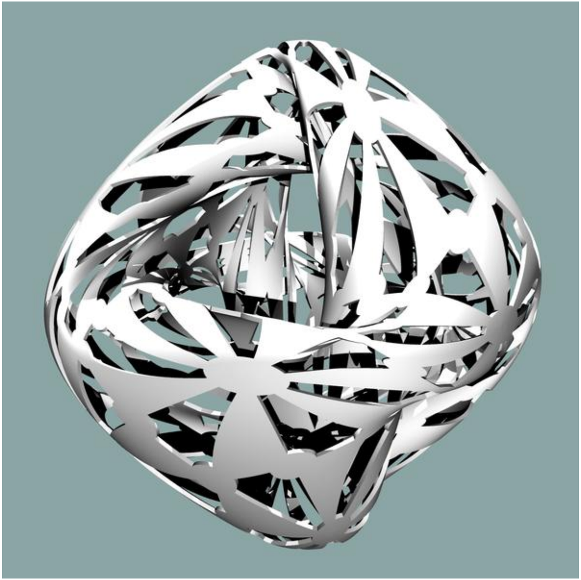
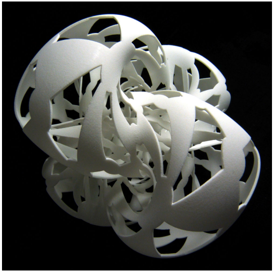


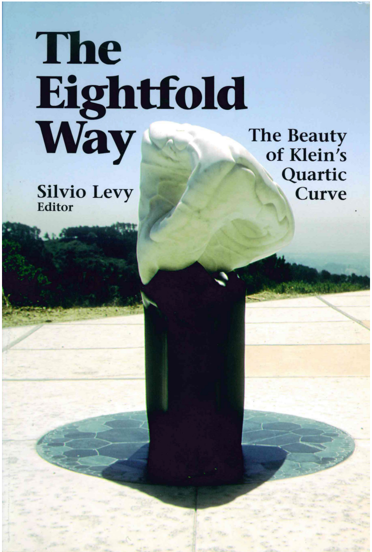
Hill climbing

[Video]

Hill climbing

Thank you!



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