SNAPPY TUTORIAL

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Problem 1. Install (or upgrade) **snappy** on your laptop from the following webpage.

`https://www.math.uic.edu/t3m/SnapPy/installing.html`

In this tutorial, we will assume that you are running the application version of snappy (the "app"). In particular we will be using the graphical interface. To check that the app is working, open it and type `T = Manifold("t12067")` at the prompt `In[1]`. Now type `T.volume()` at the prompt `In[2]`. You should see `Out[2]: 7.32772475342`.

You can get the documentation of a *method* like `volume` by typing a question mark after it: in this case type `T.volume?` to see the documentation and a helpful example. You can get a list of all of the methods of the manifold T by typing `T`, then a period, and then a tab. Other interesting methods include `T.browse()` and `T.identify()`.

Problem 2. Here we explore the canonical $\mathrm{SL}(2, \mathbb{C})$ representation of the fundamental group of T. Type `G = T.fundamental_group()` and then type `G`. We see that G is a three-generator, two-relator group. Typing `G.SL2C("a")` will give us the image of a under the representation. Is this matrix elliptic, parabolic, or hyperbolic? Typing `G.complex_length("a")` may help. In general, any word in $\{a, b, c, A, B, C\}$ gives an element of $G \cong \pi_1(T)$. Which elements of the image of the representation have the shortest (real part of their) complex length? How about the second shortest? Now explore the method `length_spectrum`.

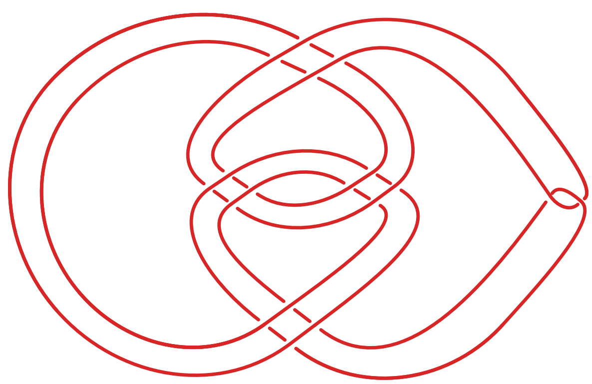

FIGURE 1. The Whitehead double $D = D(F)$ of the figure-eight knot F, equipped with a blackboard framing.

Problem 3. Type `XD = Manifold()`. This will open a drawing window (called **plink**). Draw the knot from Figure 1 in the plink window. Double- and triple-check that all of the crossings are correct. Now go to the **Tools** menu and click the **Send to SnapPy** button.

Date: December 15, 2018.

You can now ask snappy for the hyperbolic volume of X_D, or other hyperbolic invariants. However, you should not trust the answer... why not?

If you type `XD.splitting_surfaces()` then snappy should show you exactly one normal torus inside of D. You can cut D along this surface by typing `X0, X1 = XD.split(0)`. The manifolds X_0 and X_1 are necessarily knot and/or link complements in S^3. Use `identify` to find their snappy names. It is interesting to compare the sum of the volumes of X_0 and X_1 to the volume of X_D.

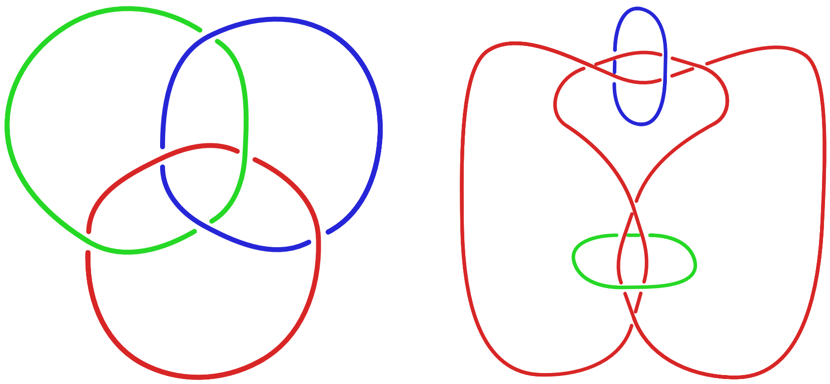

FIGURE 2. On the left we have the Borromean link B. On the right we have A, an *augmentation* of the figure-eight knot.

Problem 4. Consider the links B and A shown in Figure 2. Prove that the link complements $X_B = S^3 - B$ and $X_A = S^3 - A$ are both homeomorphic to T (from Problem 1) using the method `is_isometric_to`. Recall that you can ask for documentation by typing `?` after a method. Challenge: Give a computer-free proof that X_B and X_A are homeomorphic to each other.

Problem 5. With A and B the links shown in Figure 2: prove that there is no isotopy of S^3 taking A to B. That is, there is no continuous motion of S^3 that takes A onto B.

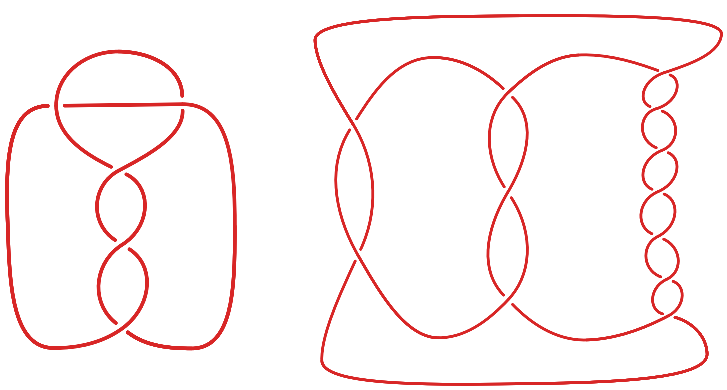

FIGURE 3. On the left we have T_3, the third *twist knot*. The trefoil and the figure-eight are the first and second twist knots, respectively. On the right we have $P = P(-2, 3, 7)$, a *pretzel knot*.

Problem 6. In the diagrams for T_3 and P in Figure 3, find all of the *bigons*: components of the complement (of the diagram in the plane) which are bounded by exactly two arcs. Use this to justify the naming schemes in Figure 3.

Problem 7. Give a rigorous, computer-aided proof that that $X_T = S^3 - T_3$ is *not* homeomorphic to $X_P = S^3 - P$. (There is an important warning at the end of the documentation for `is_isometric_to`. So you will have to find a different way to solve this problem.)

Problem 8. Consdier the triangle in $\mathbb{C}$ with vertices at 0, 1, and z, shown in Figure 4. Check that its three complex dihedral angles are $z_0 = z$, $z_1 = 1/(1-z)$, and $z_2 = (z-1)/z$. Now verify the following relations:

$$z_0 z_1 z_2 + 1 = 0$$
$$z_0 z_1 + z_1 z_2 + z_2 z_0 - z_0 - z_1 - z_2 + 3 = 0$$
$$z_0^2 z_1 + z_1^2 z_2 + z_2^2 z_0 + 3 = 0$$

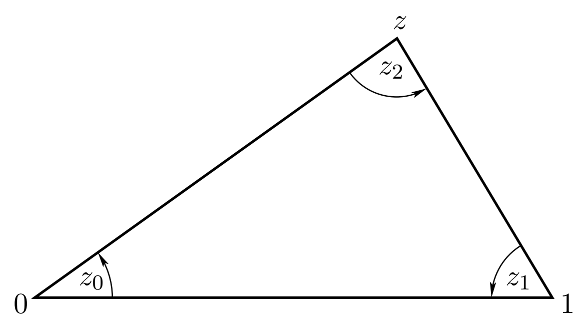

FIGURE 4. A triangle in the complex plane.

Problem 9. [Medium.] Give a rigorous, computer-aided proof that $X_T = S^3 - T_3$ and $X_P = S^3 - P$ have identical volumes. The methods `tetrahedron_shapes` and `gluing_equations` will be helpful, as will some messing about with algebraic numbers.

Problem 10. List all manifolds in the `OrientableCuspedCensus` with volume in the range $[2.0, 2.2]$.

Problem 11. Let T_k be the k^{th} twist knot (as defined in Figure 3). Compute the volume of $X_k = S^3 - T_k$ for values of k as large as your computer can handle. The command `RationalTangle` may be useful. Graph the volume of X_k as a function of k. Guess the asymptotic behaviour including the rate of convergence.

Problem 12. [Hard.] Compute the volume of X_W, the complement of the Whitehead link (shown in Figure 5). Give a plausability argument that the volumes of the twist knot complements (as in Problem 11) converge to the volume of X_W from below. Understanding the tetrahedra shapes may be useful; you can do this via the `browse` method and by looking at the *cusp neighborhood*.

Problem 13. [Medium.] Set `XF = Manifold("m004")`.

(1) Check that X_F is a triangulation of the figure-eight knot complement.
(2) You can see a list of *drillable* curves using the method `dual_curves`. Check that the first of these is a *systole* of X_F.

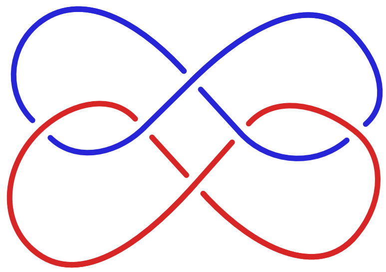

FIGURE 5. The Whitehead link W.

(3) Drill one of the systoles out of X_F by typing `X1 = XF.drill(0)`. Identify X_1 and draw a picture of the resulting link (where one component is the figure-eight knot).
(4) Drill the third shortest curve in X_1 by typing `X2 = X1.drill(2)` and identity the result. Again, draw a picture.
(5) What do you get if you instead drill a systole of X_1?

Problem 14. The method `dehn_fill` can be used to fill a torus boundary component of a snappy triangulation. Set `XW = Manifold("m129")` and check that this is homeomorphic to the complement of the Whitehead link. Show that there is a sequence of fillings on one component of ∂X_W that recovers the twist knot complements.

Problem 15. [Medium.] Find as many non-hyperbolic Dehn fillings of X_F, the complement of the figure-eight knot, as you can. Identify these non-hyperbolic manifolds.

Problem 16. Snappy is also available as a **python** module. Again, instructions are available at the URL given in Problem 1. When running inside of **sage** (the computer algebra system) additional functions are available. If you do not have sage already installed, then you can run snappy inside of an on-line sage notebook, using **CoCalc**.

`https://cocalc.com/`

For example, inside of sage the method `isometry_signature` becomes rigorous, and gives a very quick solution to Problem 7.