2026-02-23 SLMATH SAULSCHLEIMER

FROM PSEUDO-ANOSOV FLOWS (WITHOUT PERFECT FITS) TO VEERING TRIANGULATIONS

① GRAND OVERVIEW

$$\begin{tikzcd}
\text{PSEUDO-ANOSOV FLOWS}^{(*)} \arrow[r, "\text{ORBIT}", "\text{SPACE}"'] & \text{BIFOLIATED PLANES}^{(*)} \arrow[d, "\text{MAXIMAL RECTANGLES}"', "\text{(GUERITAUD) (LANDRY-MINSKY-TAYLOR)}"] \\
\text{DYNAMIC PAIRS of BRANCHED SURFACES} \arrow[u, "\text{BOX DECOMPOSITION}", "\text{(MOSHER)}"'] & \text{VEERING TRIANGULATIONS}^{(*)} \arrow[l, "\text{(S.-SEGERMAN)}"', "\text{(AGOL-TSANG)}"]
\end{tikzcd}$$

(*) WITHOUT PERFECT FITS (UP TO ORBIT EQUIV.)

(*) WITH CONTROLLED PERFECT FITS

(*) WITH DEHN FILLING SLOPES

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② MOTAVATION FOR VEERING TRIANGULATIONS

(A) COMBINATORIAL INVARIANTS (AND CHARACTERISATIONS) of PSEUDO-ANOSOV FLOWS

(B) NOTION of "HORIZONTAL" VIA TAUT TRIANGULATION; CARRYING TRANSVERSE SURFACES / FOLIATIONS.

(C) CENSUS of ALL SMALL EXAMPLES.

③ AXIOMATIC APPROACH: LOOM SPACES

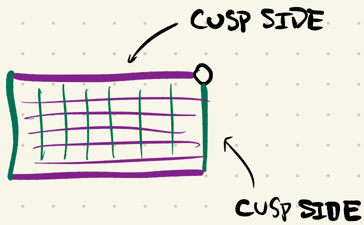
LET $\mathcal{L} = (\mathbb{R}^2, F^\pm)$ BE A BIFOLIATED PLANE, WHERE $F^\pm$ HAVE NO SINGULARITIES. DEFINE $R : (0,1)^2 \longrightarrow \mathcal{L}$ TO BE A RECTANGLE (SO LEAVES SENT TO LEAVES,

AND AN EMBEDDING). WE DEFINE THE FOLLOWING TYPES:

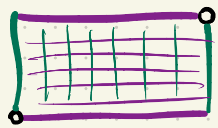
NAME	CLOSURE of R HOMEO. TO
CUSP	$[0,1]^2$ − { ONE CORNER }
EDGE	$[0,1]^2$ − { TWO NON-ADJ CORNERS }
FACE	$[0,1]^2$ − { ONE CORNER, TWO POINTS IN INTERIORS of THE NON ADJACENT SIDES }
TETRAHEDRON	$[0,1]^2$ − { FOUR PTS IN INTERIORS of SIDES, ONE PER }.

PICTURES

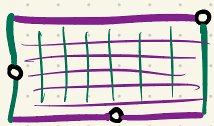
CUSP



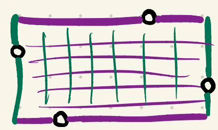
EDGE



FACE



TETRA



DEF: SAY $\mathcal{L} = (\mathbb{R}^2, F^+, F^-)$ IS A LOOM SPACE IF

(i) EACH CUSP SIDE of A CUSP RECTANGLE IS CONTAINED IN THE INTERIOR of SOME RECTANGLE AND

(ii) EVERY RECTANGLE $R \subset \mathcal{L}$ IS CONTAINED IN AT LEAST ONE TET. RECTANGLE.

[EXERCISE: AXIOMS (i), (ii) ARE INDEPENDENT]

(4) EXAMPLES

(A) S CLOSED CONNECTED, ORIENTED SURFACE, $g(S) \geq 1$
$f: S \to S$ PSEUDO-ANOSOV. $F^{\pm} \subset S$ STABLE/UNSTABLE FOLIATIONS. $\Sigma(f)$ = SINGULARITIES of $F^{\pm}$. NOTE $\Sigma(f) \neq \emptyset$ [OR $g(S)=1$ AND SET $\Sigma(f) = \{0\}$.] SET $S^{\circ} = S - \Sigma(f)$. LIFT $F^{\pm}$ TO UNIV. COVER of S°.

EXERCISE: VERIFY AXIOMS.

(i) FOLLOWS FROM FACT S° IS A SURFACE AND STRUCTURE of $F^{\pm}$.

(ii) FOLLOWS FROM FACT ALL LEAVES ARE DENSE (AND THE INVARIANT MEASURES).

(B) M^3 CLOSED MANIFOLD. $\varphi: M \times \mathbb{R} \to M$ A PSEUDO-ANOSOV FLOW (WITHOUT PERFECT FITS). LET $\Sigma(\varphi)$ BE THE SET of SINGULAR ORBITS. DEFINE $M^{\circ} = M - \Sigma(\varphi)$. LIFT TO $\widetilde{M^{\circ}}$ AND QUOTIENT BY THE FLOW. THE WEAK (UN)STABLE FOLIATIONS GIVE $F^{\pm}$.

EXERCISE: VERIFY AXIOMS.

(i) FOLLOWS FROM NO PERFECT FITS AND EXISTENCE of LOCAL SECTIONS.

(ii) NO PERFECT FITS IMPLIES DENSITY of LEAVES.

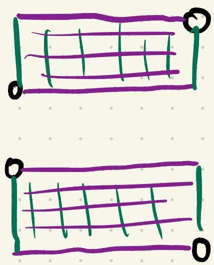
(C) S AS ABOVE, EQUIP IT WITH A QUADRATIC DIFFERENTIAL WITHOUT VERTICAL OR HORIZONTAL SADDLE CONNECTIONS.

EXERCISE: VERIFY AXIOMS.

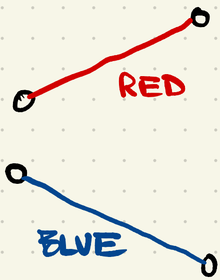
⑤ VEERING TRIANGULATIONS:

FOR EVERY KIND OF RECTANGLE WE BUILD A CELL

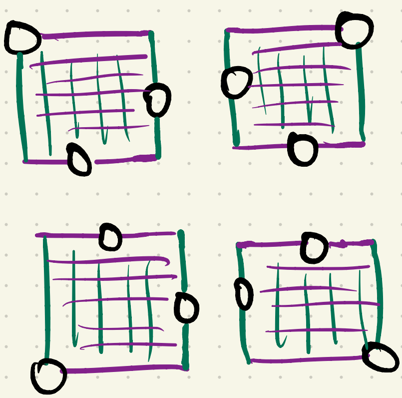
EDGE RECTANGLES



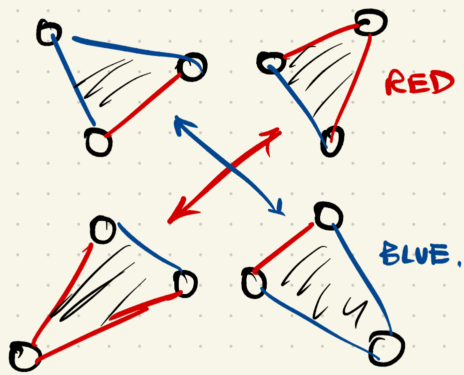
VEERING EDGES



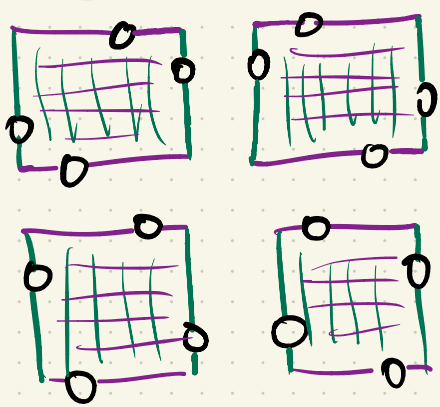
FACE RECTANGLES



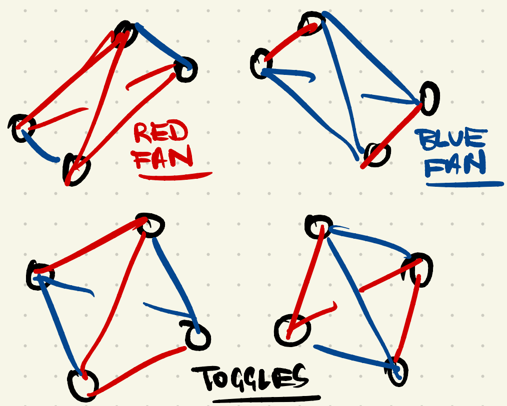
VEERING TRIANGLES



TETRAHEDRON RECTANGLES.



VEERING TETRAHEDRA

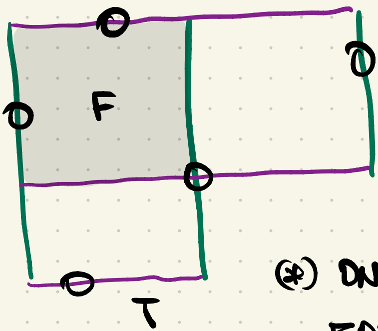


NOTE: ① EVERY TET. RECT CONTAINS FOUR FACE RECTS.

② EVERY FACE RECT IS CONTAINED IN TWO TET. RECTS.

③ ANY PAIR of TET RECTS IS CONN. BY SOME CHAIN of TET, FACE, TET, FACE, ... RECTS.

PICTURE:



(*) EVERY (EDGE) RECT IS CONTAINED IN ONLY FINITELY MANY TET RECTANGLES.

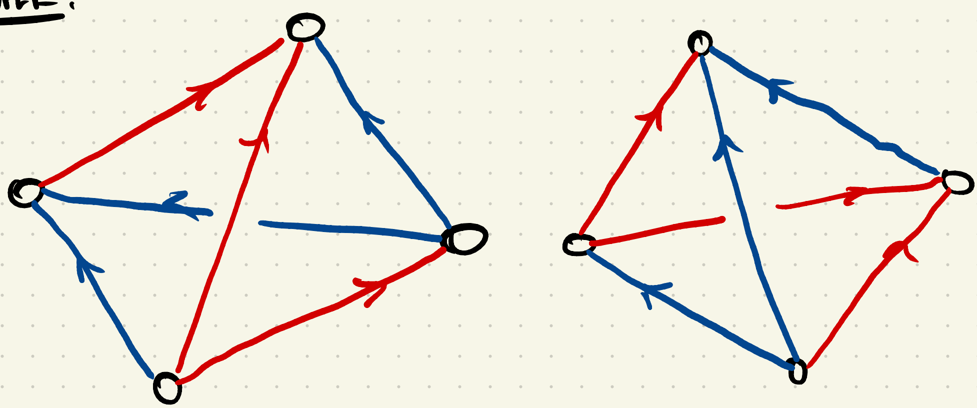
(*) ONLY COUNTABLY MANY EDGE, FACE, TET RECTANGLES.

GUERITAUD: BUILD $V(\mathcal{L})$, TRIANGULATION of $\mathbb{R}^3$, BY GLUING MODEL VEERING CELLS ACCORDING TO OVERLAPPING of RECTANGLES.

(*) MUCH MORE: FACES TRANSVERSE TO THE ORIGINAL FLOW, CARRY SURFACES.

NOTE: $AUT(\mathcal{L})$ ACTS VIA ISOMORPHISMS ON $V(\mathcal{L})$. IN PARTICULAR $AUT(\mathcal{L})$ COUNTABLE AND $V(\mathcal{L})/AUT(\mathcal{L})$ IS A THREE-DIMENSIONAL ORBIFOLD.

EXAMPLE:



VEERING TRIANGULATION of $(S^3 - \text{FIG } 8)$ MADE of TWO TOGGLES