2026-03-02 SAUL SCHLEIMER SLMATH

# FROM FLOWS TO TRIANGULATIONS II

## (1) THE PLAN

PSEUDO-ANOSOV FLOWS (*) —[ORBIT SPACE; GUERITAUD, AGOL (*)]→ VEERING TRIANGULATIONS (*)

VEERING TRIANGULATIONS —[DYNAMIC PAIRS; S.-SEGERMAN, TSANG; AGOL, MOSHER]→ PSEUDO-ANOSOV FLOWS

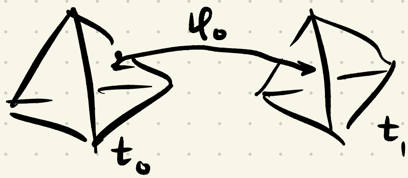
(*) WITHOUT PERFECT FITS, UP TO ORBIT EQUIV.

(*) WITH DEHN FILLING SLOPES.

(*) SEE ALSO LANDRY-MINSKY-TAYLOR

## (2) IDEAL TRIANGULATIONS [THURSTON]

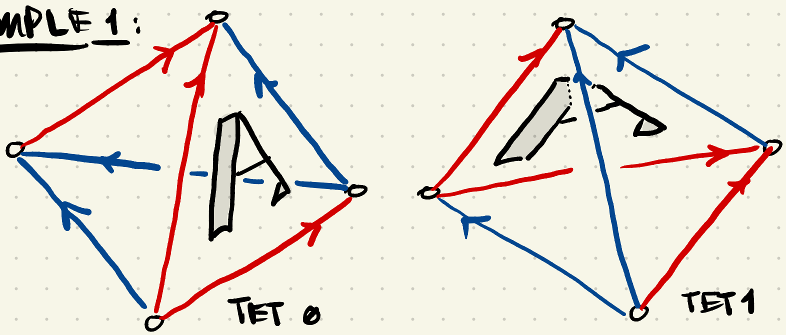
SUPPOSE $T = \{t_i, \varphi_j\}$ IS A COLLECTION of TETRAHEDRA AND FACE PAIRINGS. PICTURE




DEFINE $|T| = \coprod t_i \,/\, \varphi_j$

SUPPOSE $M$ IS A THREE MANIFOLD. SAY $T$ IS AN IDEAL TRIANGULATION of $M$ IF $|T| - T^{(0)}$ HOMEO TO $M - \partial M$.

EXAMPLE 1:



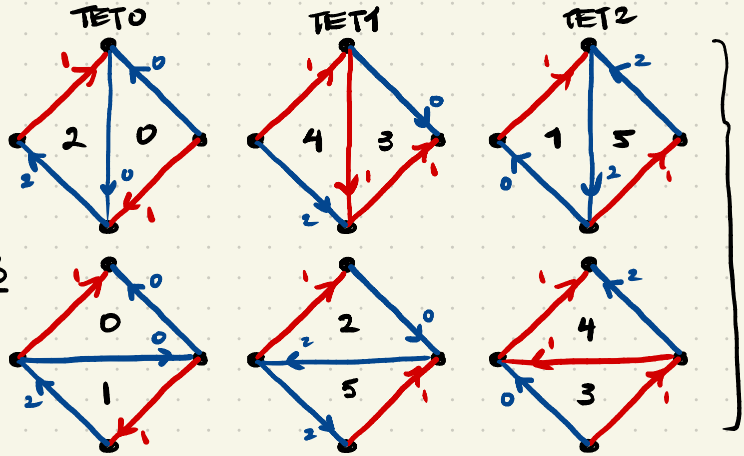

TWO TETS
FOUR FACES
TWO EDGES
ONE IDEAL VERTEX.

THURSTON: $|T| - T^{(0)} \cong S^3 - $ (FIGURE-EIGHT KNOT)

## EXAMPLE 2: dLQacccjsnk_200 FROM VEERING CENSUS.

TOPS

BOTTOMS




THREE TETS
SIX FACES
THREE EDGES
ONE IDEAL
VERTEX

WEEKS: $|T| - T^{(0)} \cong S^3 - ((-2,3,7)$ PRETZEL KNOT$)$.

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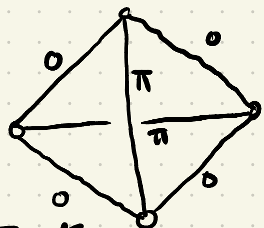
## (3) TRANSVERSE TAUT TRIANGULATIONS [LACKENBY]

A TRANSVERSE TAUT TET HAS

(i) COORIENTATIONS ON MODEL FACES

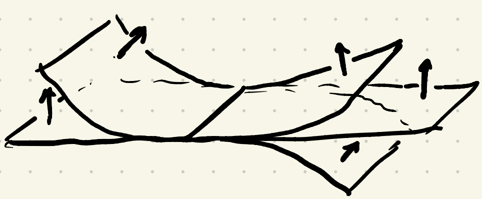
(ii) ANGLES ON MODEL EDGES




THINK: THE TET IS FLATTENED INTO THE PLANE AND COORIENTATIONS POINT AT US.

A TRANSVERSE TAUT TRIANGULATION IS BUILT of TRANS. TAUT TETS ① OBEYING CO-ORIENTS. AND SO THAT ② EVERY QUOTIENT EDGE GETS TOTAL ANGLE $2\pi$.

PICTURE: ABOUT EDGE



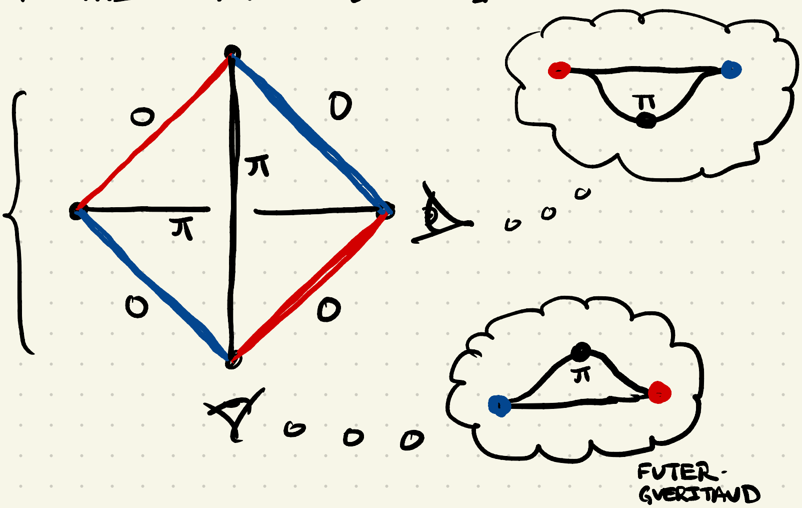
SO THE TWO-SKELETON IS A "HORIZONTAL" BRANCHED SURFACE (WITHOUT VERTICES).

**EXERCISE:** SUPPOSE $T$ IS A TTT of $M$. THEN $\partial M \neq \emptyset$ AND $\partial M$ IS A DISJOINT UNION of TORI.

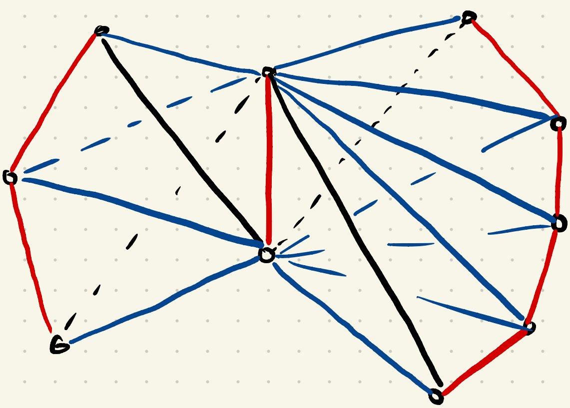
## (4) VEERING TRIANGULATIONS [AGOL]

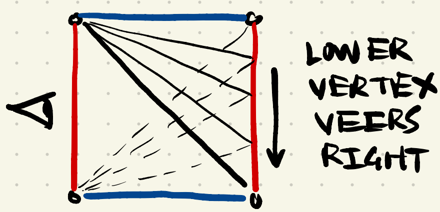
SUPPOSE $T$ IS A TRANSVERSE TAUT TRIANGULATION. SAY $T$ IS **VEERING** IF WE CAN COLOUR EDGES of $T^{(1)}$ BY {RED, BLUE} SO THAT ALL EQUATORS ALTERNATE COLOUR MATCHING THE FIGURE [HRST]:

UPPER/LOWER EDGES ARE RED OR BLUE




AGOL'S PICTURE of NEIGHBOURHOOD of VEERING EDGE [OMITTING TETS ABOVE AND BELOW]





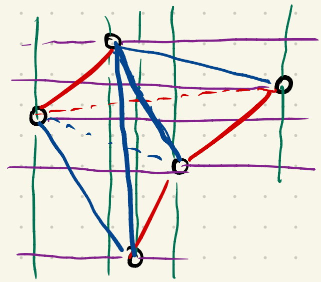
## ⑤ FROM FLOWS TO TRIANGULATIONS:

SUPPOSE $M$ IS CLOSED THREE-MFD. SUPPOSE $\varphi : M \times \mathbb{R} \to M$ IS A PSEUDO-ANOSOV FLOW, WITHOUT PERFECT FITS.

**EXERCISE**: $\varphi$ HAS SINGULAR ORBITS.

LET $M^\circ = M -$ SING. ORBITS. LET $\varphi^\circ = \varphi | M^\circ \times \mathbb{R}$. LIFT TO TO UNIVERSAL COVER TO OBTAIN $\tilde{\varphi}^\circ$ FLOW ON $\tilde{M}^\circ$. QUOTIENT TO GET $L(\varphi^\circ)$ ORBIT SPACE.

**EXERCISE**: $L(\varphi^\circ)$ IS A LOOM SPACE



**DEFINE** $V(\varphi^\circ) = V(L(\varphi^\circ))$ TO BE THE RESULTING VEERING TRIANGULATION.

**THEOREM**: $V(\varphi^\circ)$ IS A VEERING TRIANGULATION of $\mathbb{R}^3$.

**THEOREM**: THE NATURAL HOMOMORPHISM

$$\mathrm{AUT}(L(\varphi^\circ)) \longrightarrow \mathrm{AUT}(V(\varphi^\circ))$$

IS AN ISOMORPHISM. [PROOF: A POINT $p \in L(\varphi^\circ)$ IS DET. BY THE SET of TET. RECTS CONTAINING IT. □]

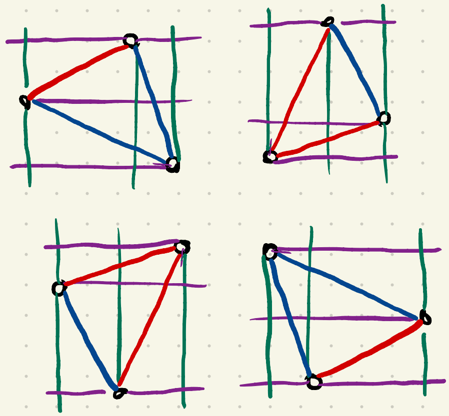
**NOTE**: $\pi_1(M^\circ)$ EMBEDS INTO $\mathrm{AUT}(L(\varphi^\circ))$.

**THEOREM** (*): $V(\varphi^\circ) / \pi_1(M)$ IS A **FINITE** VEERING TRIANGULATION of $M^\circ$.

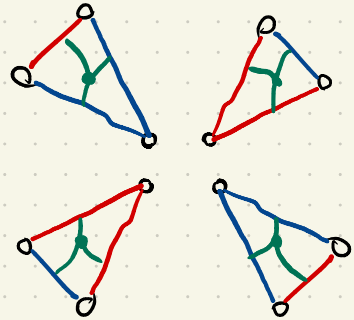
[PROOF: WALDHAUSEN AND ACTION ON CUSP NEIGHBOURHOODS □]

(*) DUE TO AGOL-GUERITAUD, BUT SEE LANDRY-MINSKY-TAYLOR

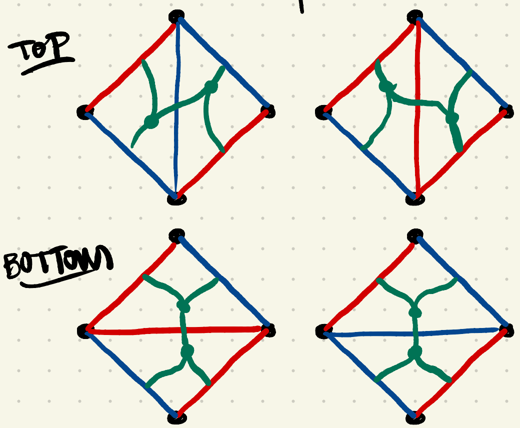
⑥ <u>(UN)STABLE BRANCHED SURFACES</u> : WE RECALL THE FOUR FACE RECTANGLES



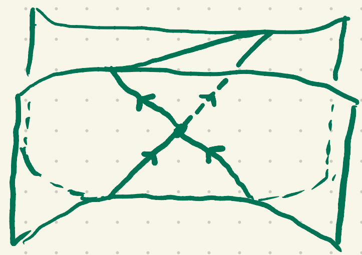
COLLAPSE THE STABLE FOLIATION TO GET TRAIN TRACKS



ON BOUNDARY of TETRAHEDRA



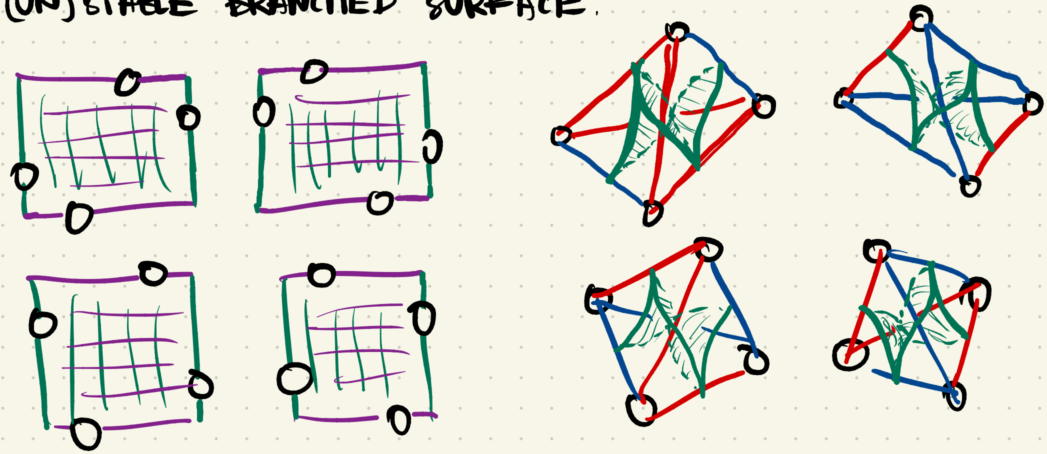

INSIDE TETRAHEDRA GET

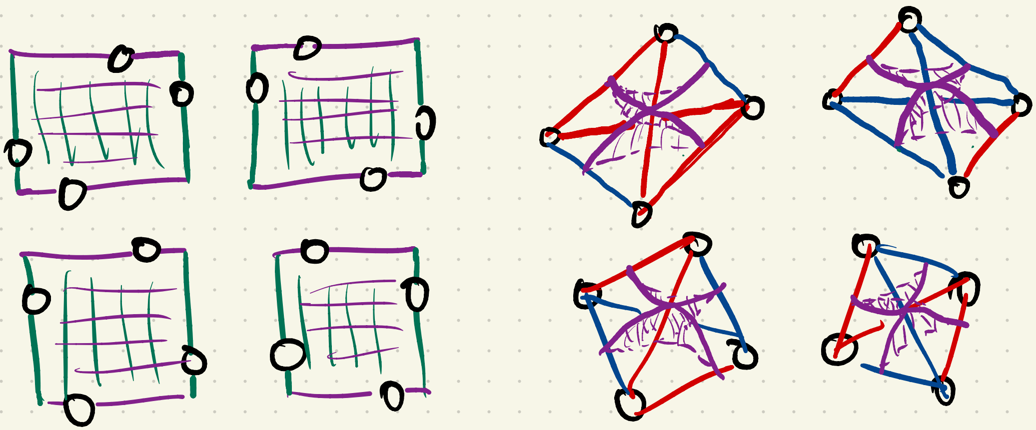


THE VEERING (STABLE) BRANCHED SURFACE IS ISOTOPIC TO THE DUAL TWO-SKELETON. ☐

## BONUS FIGURES [NOT USED]

(UN)STABLE BRANCHED SURFACE.





THE BRANCHED SURFACES SHOWN CARRY THE (UN)STABLE FOLIATIONS. BUT THEY ARE NOT YET TRANSVERSE (WITH GOOD COMPLEMENTARY REGIONS). SEE [SS4].

# PICTURE OF NORMAL BRANCHED SURFACE