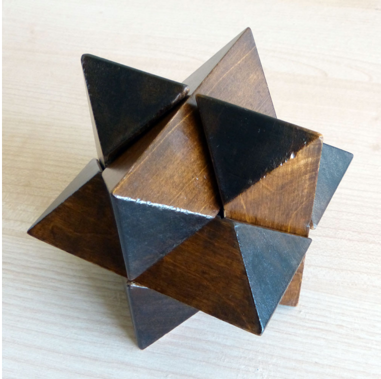
Saul Schleimer
University of Warwick

Henry Segerman
Oklahoma State University

Puzzling the 120–cell

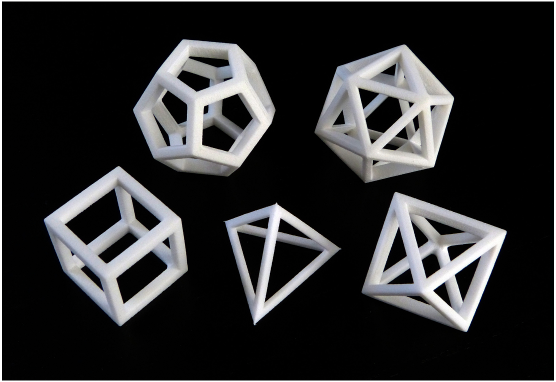
Burr puzzles

Burr puzzles, notched sticks.



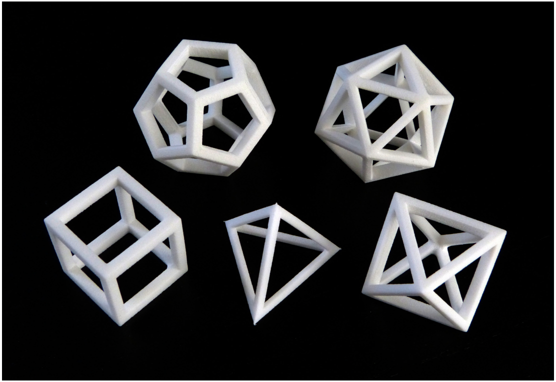
Quintessence

Platonic solids



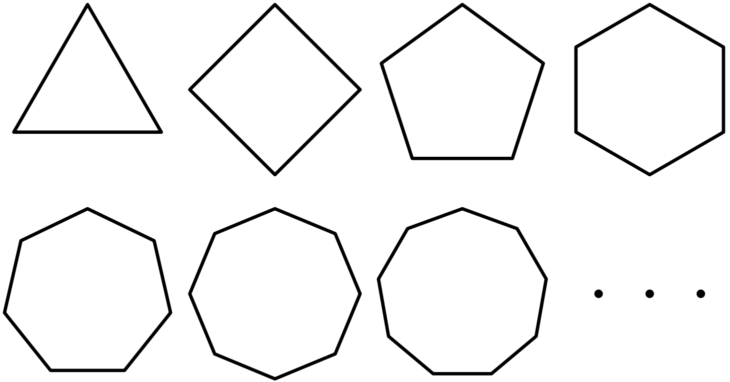
The Platonic solids

Platonic solids



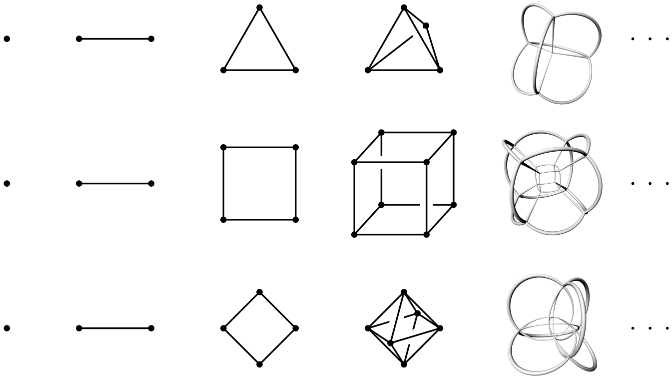
Regular polytopes in dimension three.

Regular polygons



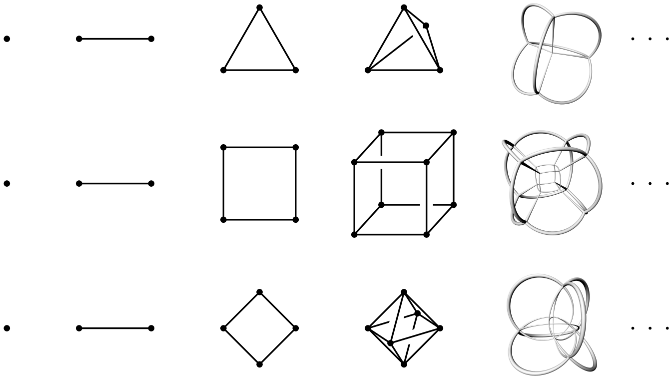
First infinite family of regular polytopes. Polygons.

Regular polytopes



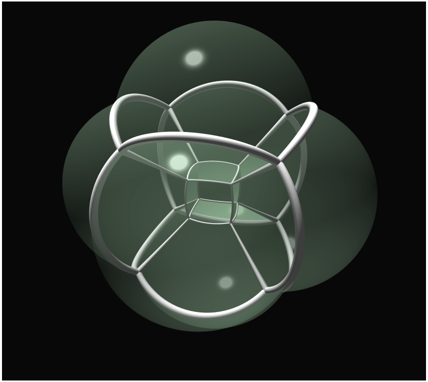
The other three families: simplices, cubes, cross-polytopes. Tilings.

Regular polytopes



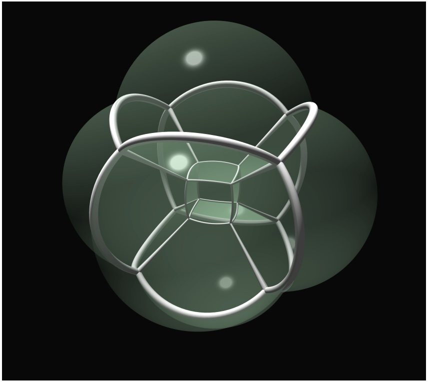
Odd-balls.

Hypercube



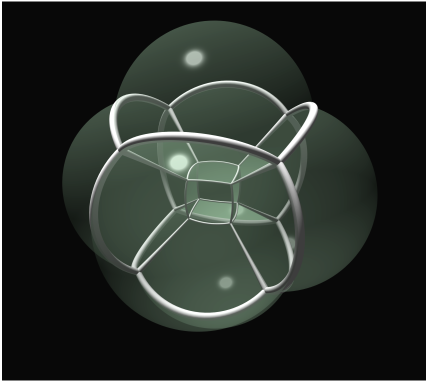
The 4–cube (or 8–cell, hypercube, tesseract, unit orthotope).
F-vector.

Hypercube



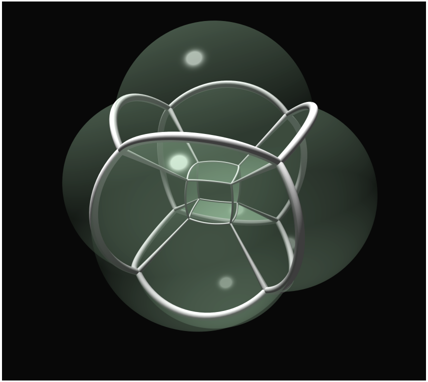
Not a hypercube! Boundary...

Hypercube



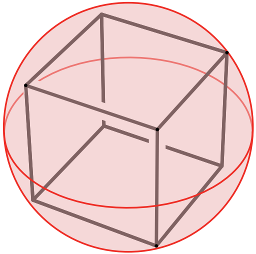
... missing a point. And projected.

Hypercube



Curvy, dimensionality.

Projecting a cube from $\mathbb{R}^3$ to S^2 to $\mathbb{R}^2$



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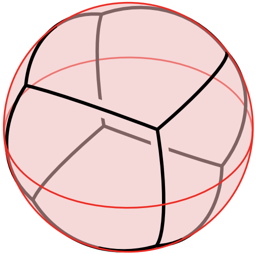

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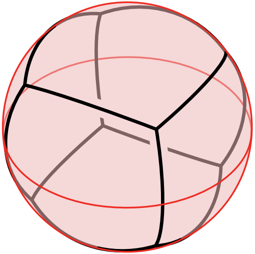

Radial projection

$$\mathbb{R}^3 \setminus \{0\} \to S^2$$

$$(x, y, z) \mapsto \frac{(x, y, z)}{|(x, y, z)|}$$

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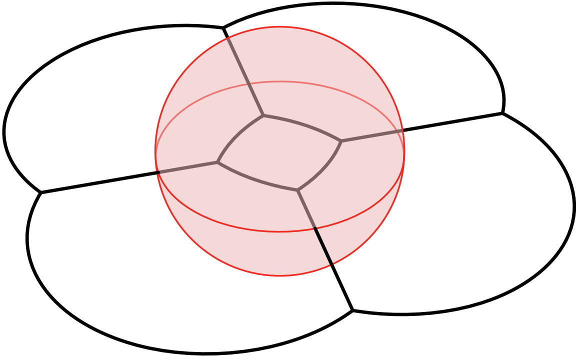

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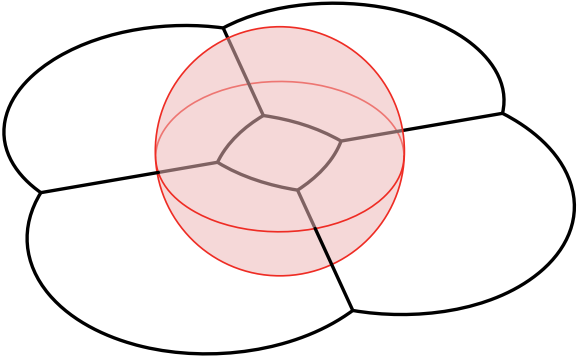

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$\mathbb{R}^3 \setminus \{0\} \to S^2$

$$(x, y, z) \mapsto \frac{(x, y, z)}{|(x, y, z)|}$$

Stereographic projection

$S^2 \setminus \{N\} \to \mathbb{R}^2$

$$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

Stereographic projection

In general, stereographic projection maps from $S^n \setminus \{N\}$ to $\mathbb{R}^n$.

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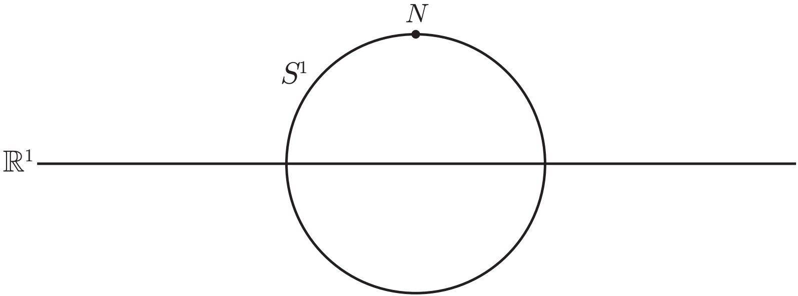

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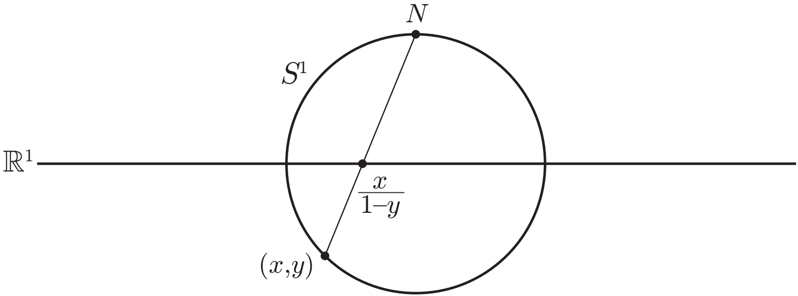

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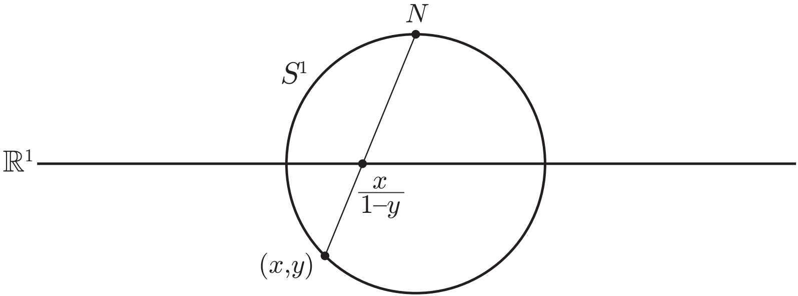
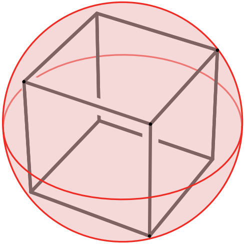


This is *also* a cross-section of stereographic projection for $n > 1$.

Thickening the edges



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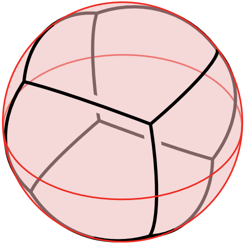

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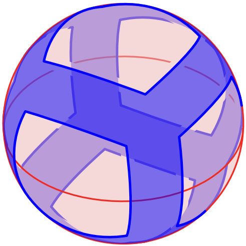
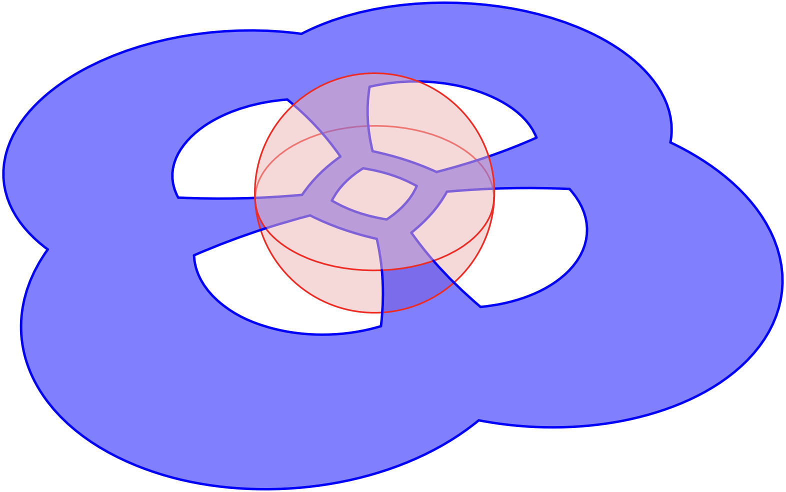
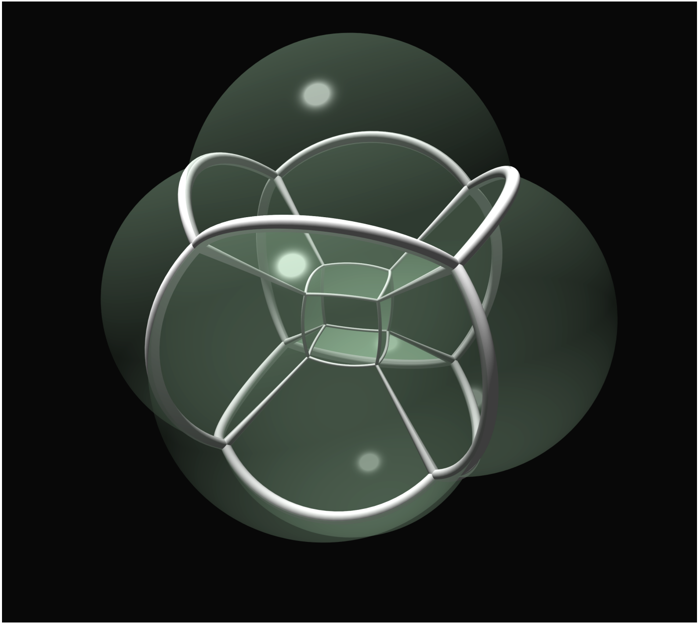


Thickening the edges

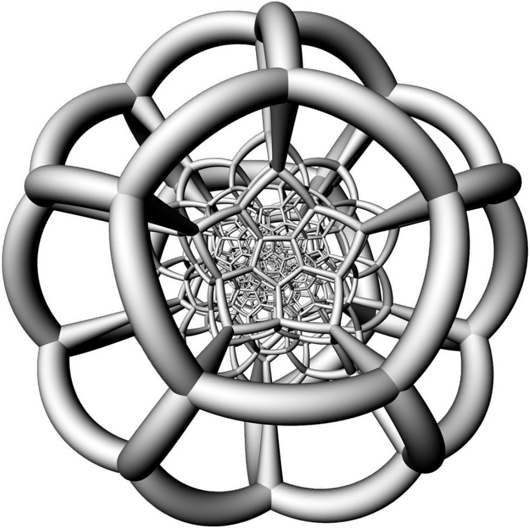


Hypercube, redux



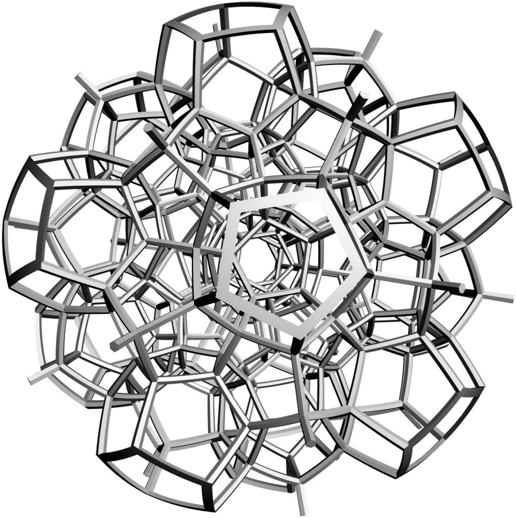
120–cell

Cell-centered stereographic projection of the 120–cell. Dodecahedral symmetry in $\mathbb{R}^3$.



Cut-away

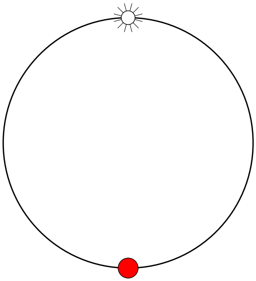
The one-half 120–cell.
Dodecahedral
symmetry in $\mathbb{R}^3$.



Spherical layers in the 120–cell

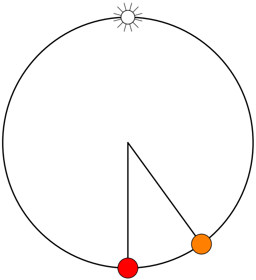
Spherical layers in the 120–cell

- 1 central dodecahedron



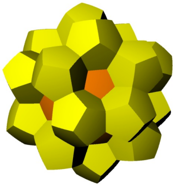
Spherical layers in the 120–cell

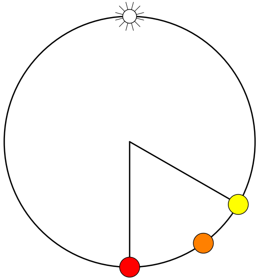
- 1 central dodecahedron
- 12 dodecahedra at distance $\pi/5$



Spherical layers in the 120–cell

- 1 central dodecahedron
- 12 dodecahedra at distance $\pi/5$
- 20 dodecahedra at distance $\pi/3$





Spherical layers in the 120–cell

- 1 central dodecahedron
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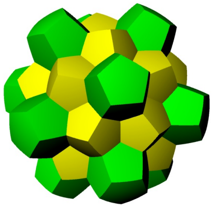
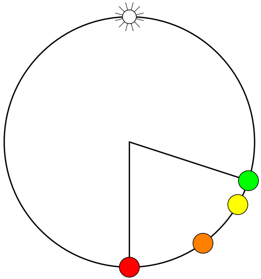
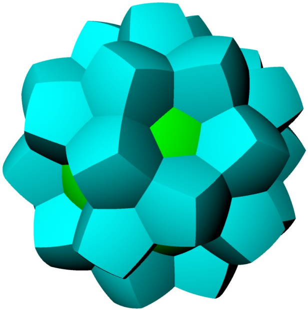
- 20 dodecahedra at distance $\pi/3$
- 12 dodecahedra at distance $2\pi/5$

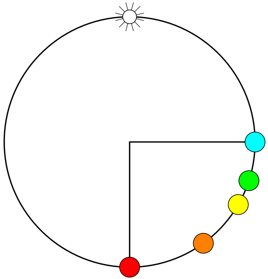




Spherical layers in the 120–cell

- 1 central dodecahedron
- 12 dodecahedra at distance $\pi/5$
- 20 dodecahedra at distance $\pi/3$
- 12 dodecahedra at distance $2\pi/5$
- 30 dodecahedra at distance $\pi/2$





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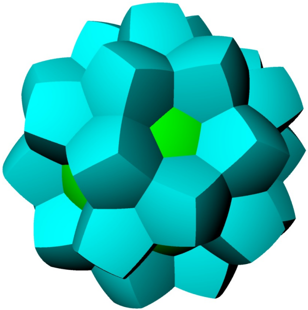
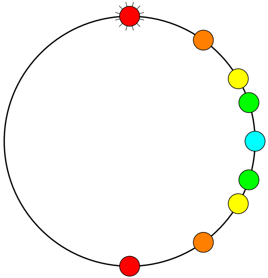
- 30 dodecahedra at distance $\pi/2$

The pattern is mirrored in the last four layers.

$1+12+20+12+30+12+20+12+1 = 120$





Hopf fibers in the 120–cell

A combinatorial version of the Hopf fibration.

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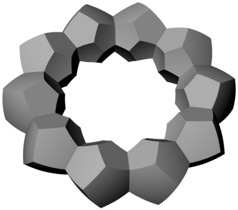
Each fiber is a “ring” of 10 dodecahedra.



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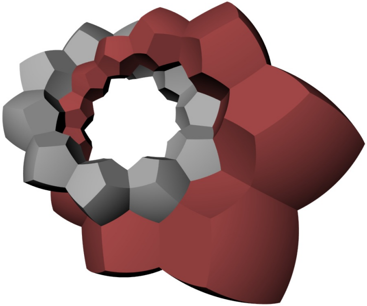
Each fiber is a “ring” of 10 dodecahedra.

The rings wrap around each other.



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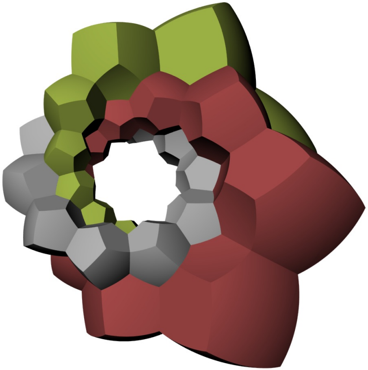

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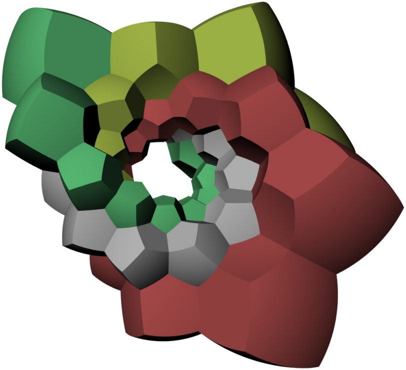

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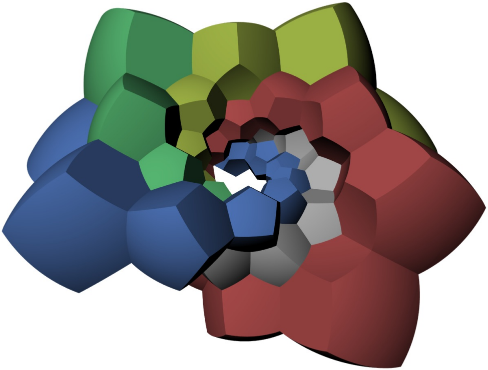
Each fiber is a “ring” of 10 dodecahedra.

The rings wrap around each other.

Each ring is surrounded by five others.



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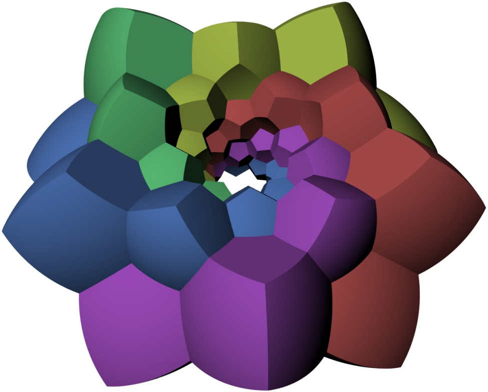

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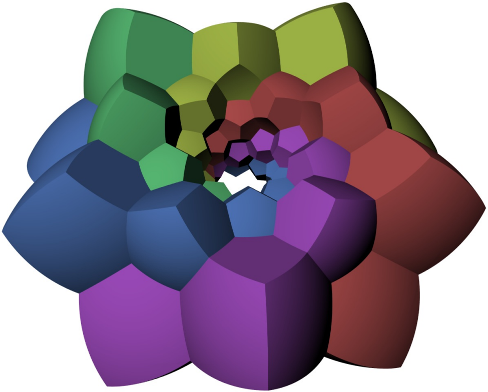

Six rings tile a solid Clifford torus (half of the 120–cell).

$$1 + 5 + 5 + 1 = 12 = 120/10$$

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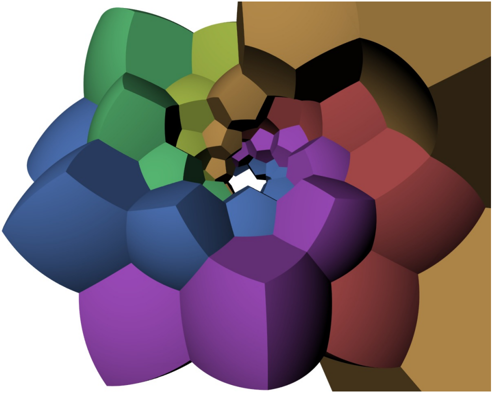

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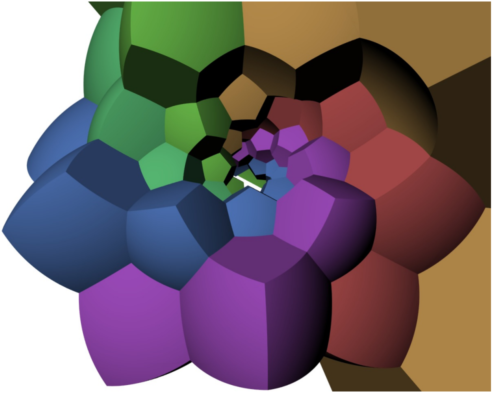

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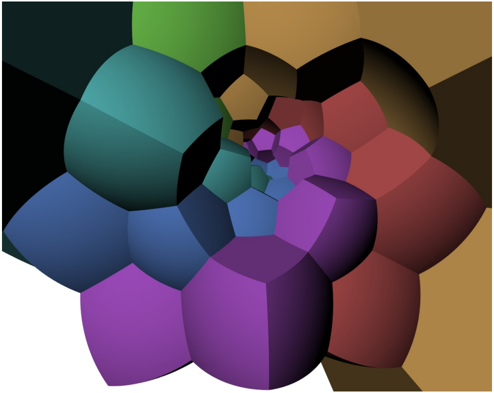

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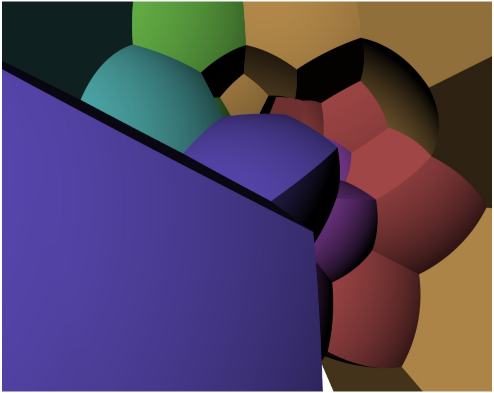

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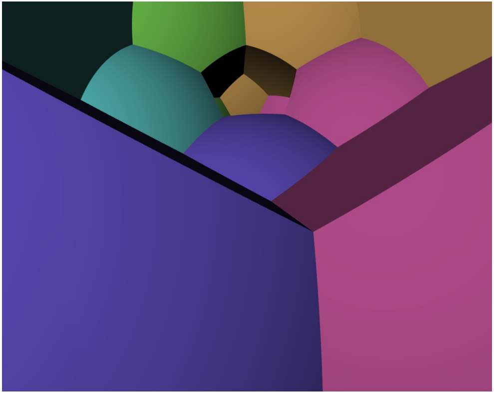
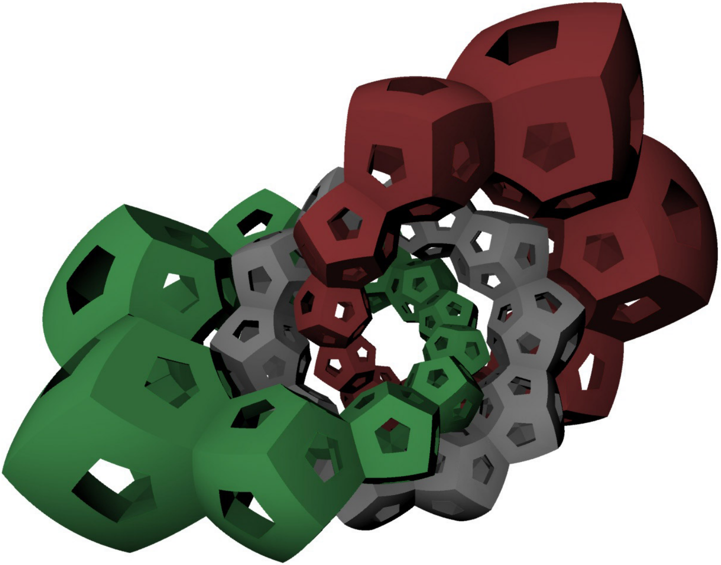
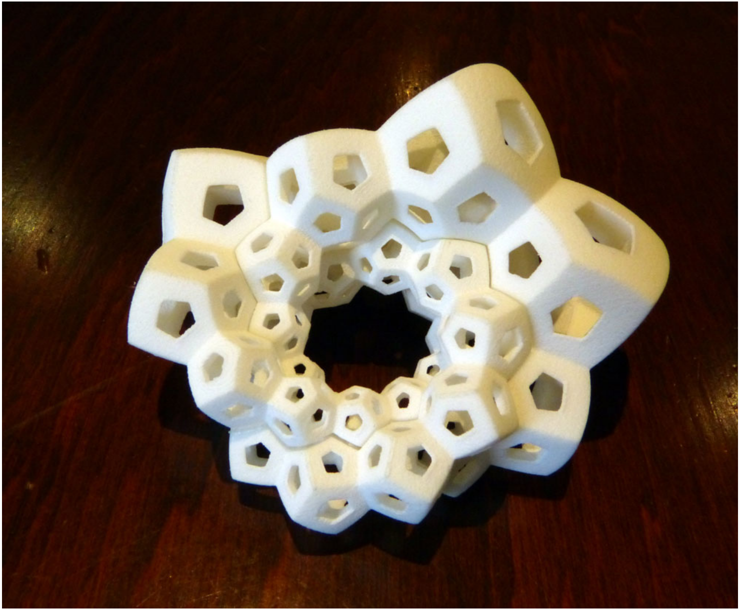


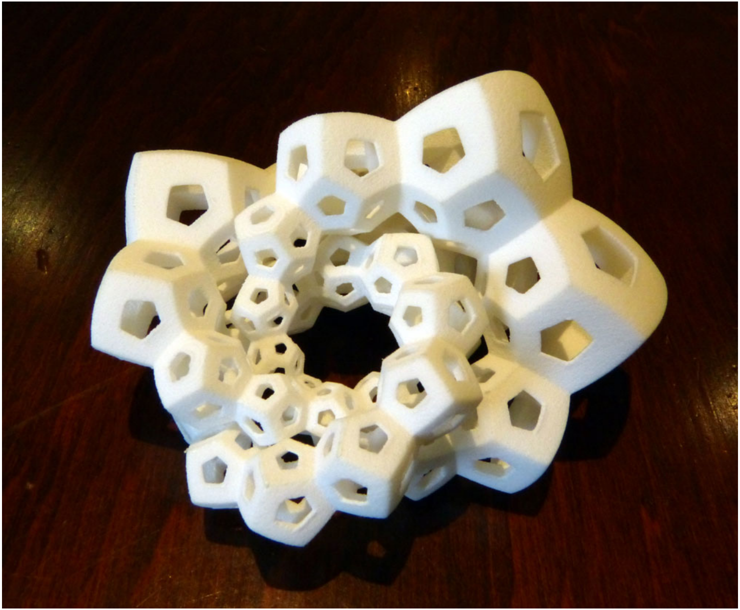
Six rings tile a solid Clifford torus (half of the 120–cell).

$$1 + 5 + 5 + 1 = 12 = 120/10$$

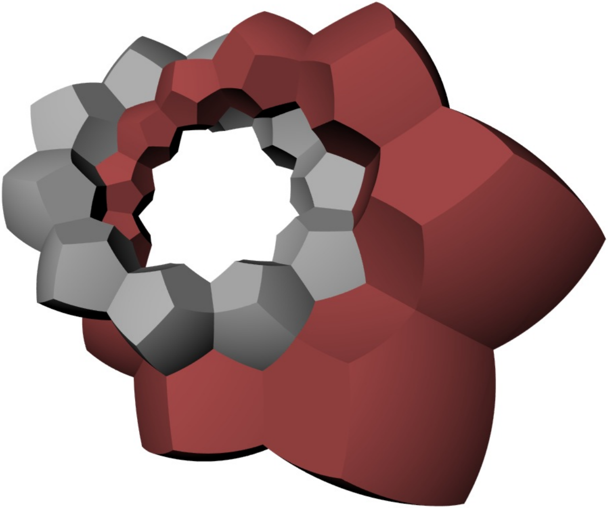
We wanted to 3D print all six of the inner rings together; it seems this cannot be done without them touching each other. (Parts intended to move must not touch during the printing process.)



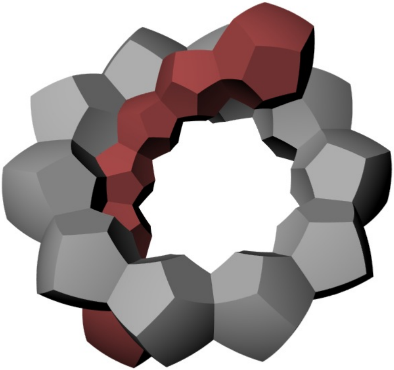




To print all five we use a trick...



To print all five we use a trick... don't print the whole ring. We call part of a ring a rib.



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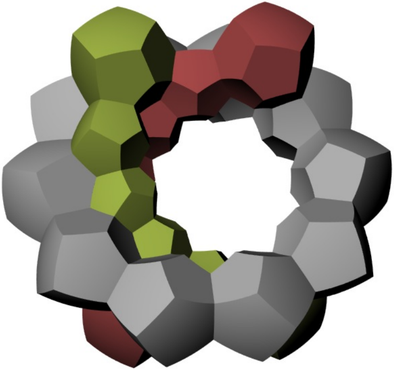

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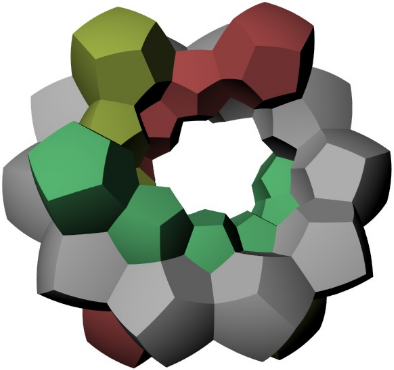
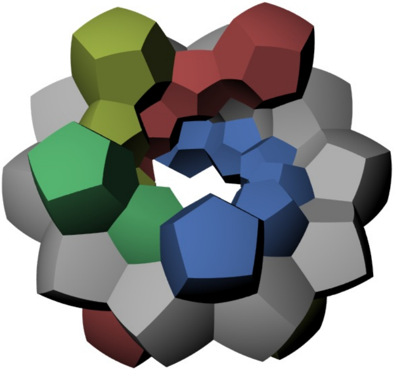
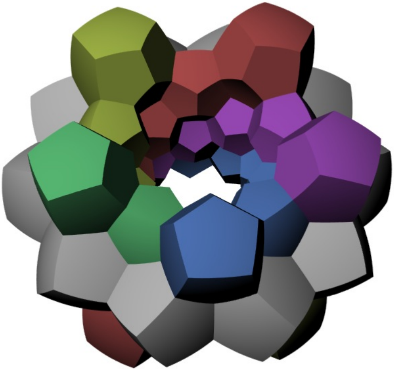


To print all five we use a trick... don't print the whole ring. We call part of a ring a rib.



To print all five we use a trick... don't print the whole ring. We call part of a ring a rib.



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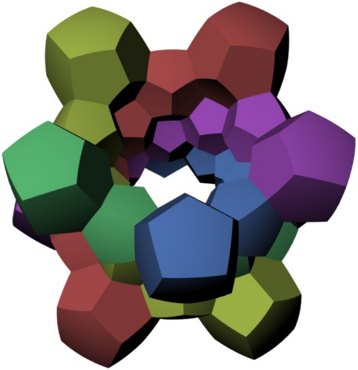
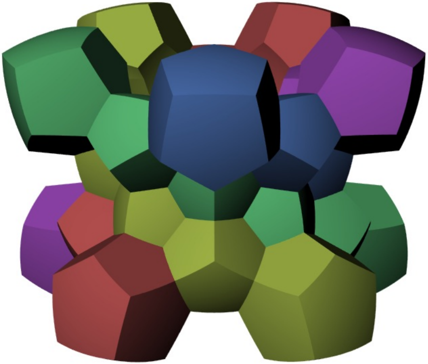
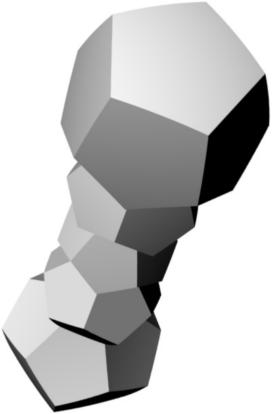


To print all five we use a trick... don't print the whole ring. We call part of a ring a rib.

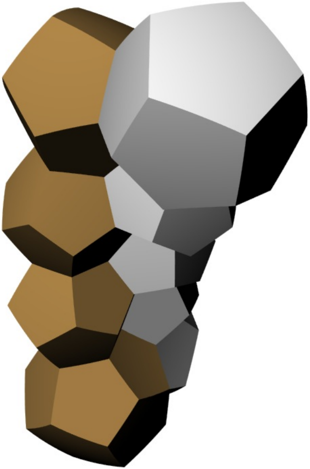


Dc30 Ring puzzle

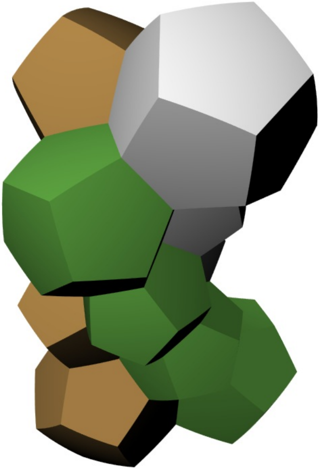
Another decomposition, with even shorter ribs.



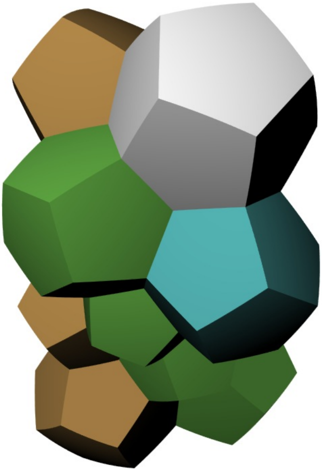
Another decomposition, with even shorter ribs.



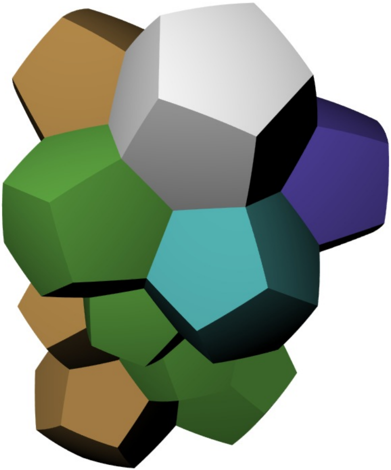
Another decomposition, with even shorter ribs.



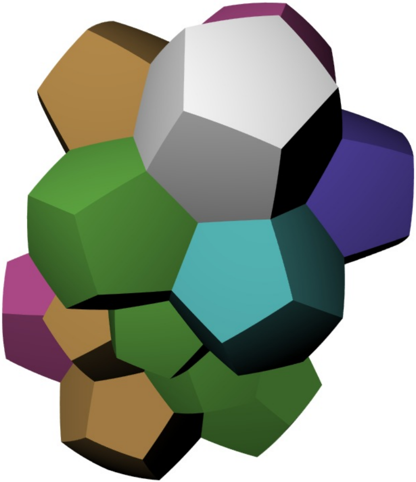
Another decomposition, with even shorter ribs.



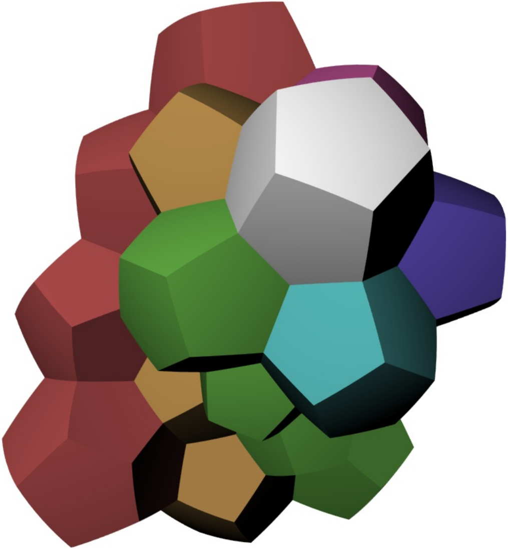
Another decomposition, with even shorter ribs.



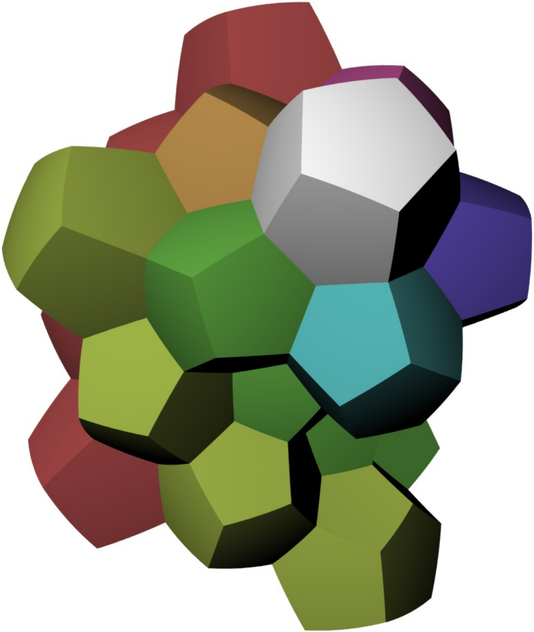
Another decomposition, with even shorter ribs.



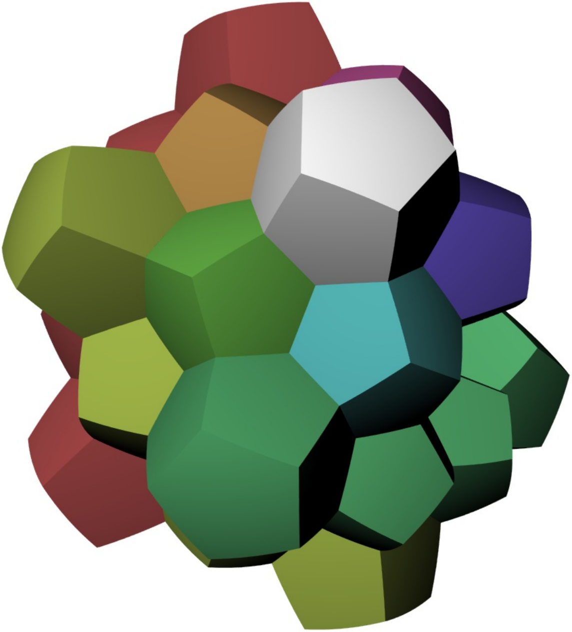
Another decomposition, with even shorter ribs.



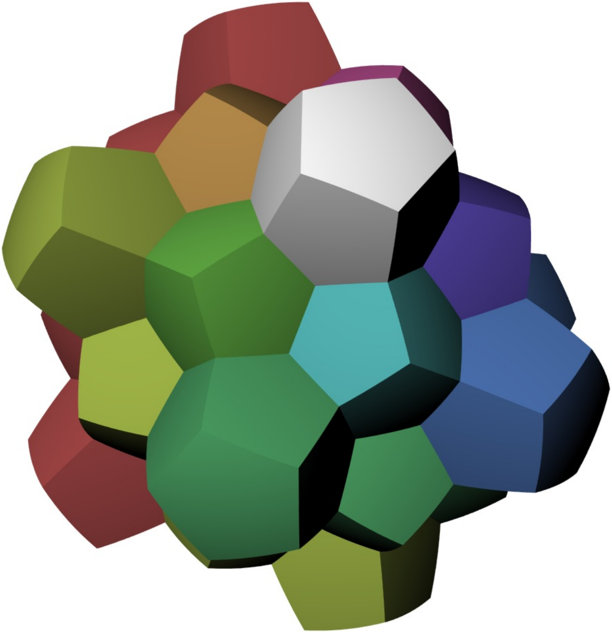
Another decomposition, with even shorter ribs.



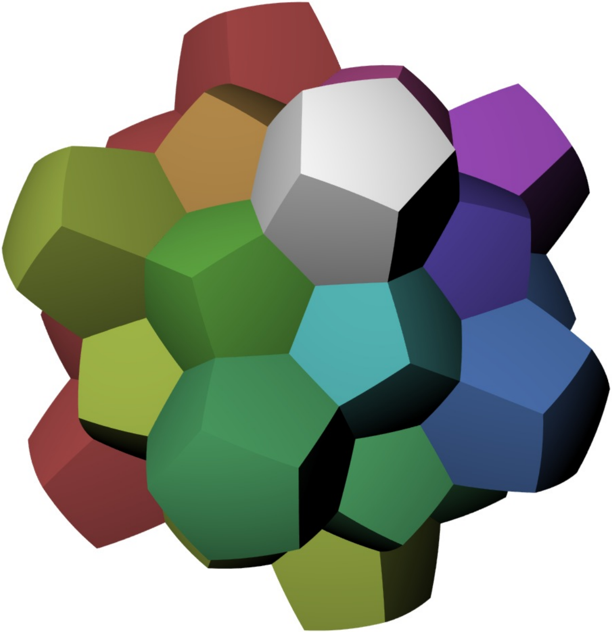
Another decomposition, with even shorter ribs.



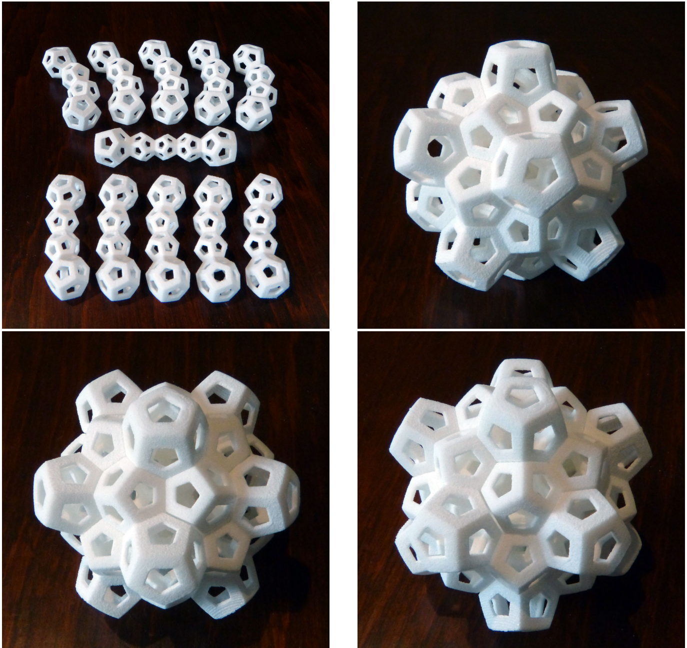
Another decomposition, with even shorter ribs.



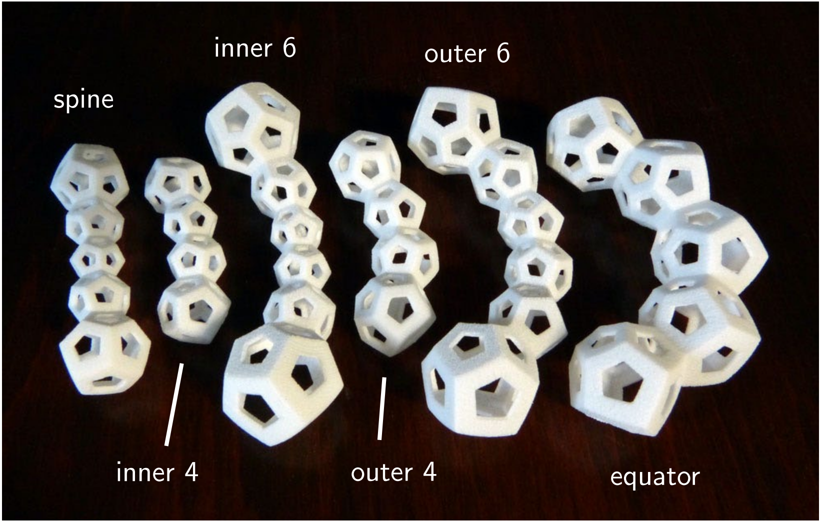
Another decomposition, with even shorter ribs.



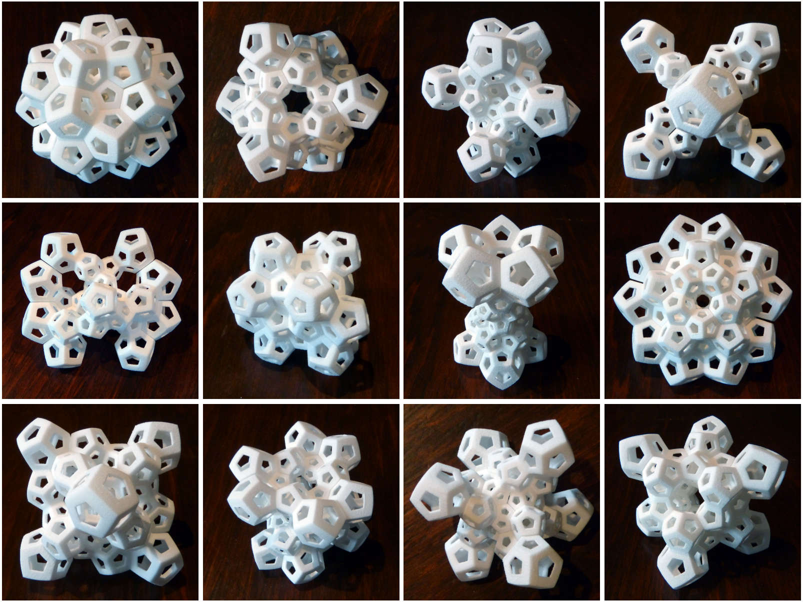
Dc45 Meteor puzzle



Six kinds of ribs



These make many puzzles, which we collectively call Quintessence.



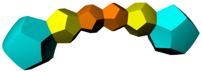
Theorem

- *At most six inner ribs are used in any puzzle.*
- *At most six outer ribs are used in any puzzle.*
- *At most ten inner and outer ribs are used in any puzzle.*

Theorem

- *At most six inner ribs are used in any puzzle.*
- *At most six outer ribs are used in any puzzle.*
- *At most ten inner and outer ribs are used in any puzzle.*

Proof.

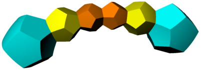


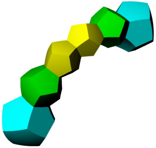
□

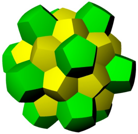
Theorem

- *At most six inner ribs are used in any puzzle.*
- *At most six outer ribs are used in any puzzle.*
- *At most ten inner and outer ribs are used in any puzzle.*

Proof.





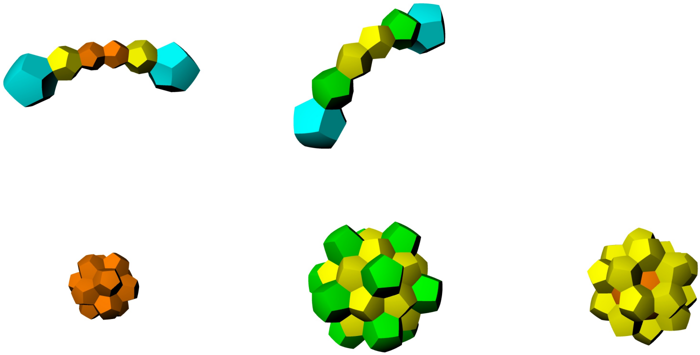


□

Theorem

- *At most six inner ribs are used in any puzzle.*
- *At most six outer ribs are used in any puzzle.*
- *At most ten inner and outer ribs are used in any puzzle.*

Proof.



□

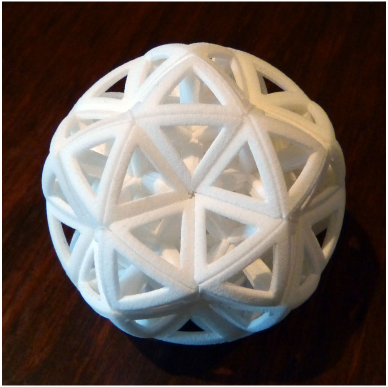
Further possibilities: vertex centered projection

Dv30 Asteroid puzzle

Further possibilities: other polytopes

The 600–cell works, although the ribs now have handedness.

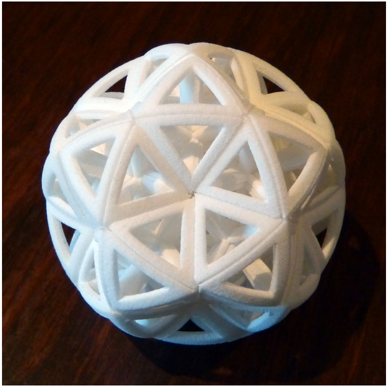
Tv270 Meteor puzzle



Further possibilities: other polytopes

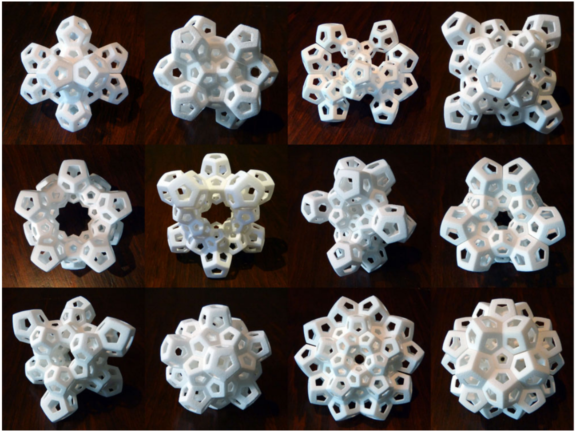
The 600–cell works, although the ribs now have handedness.

Tv270 Meteor puzzle



The other regular polytopes seem to have too few cells to make interesting puzzles.

Thanks!



http://homepages.warwick.ac.uk/~masgar/ (Schleimer)
http://segerman.org (Segerman)
http://youtube.com/user/henryseg
http://www.shapeways.com/shops/henryseg?section=Quintessence