Grid

Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered
Dimensions: $8 \times 8 \times 8$ cm

The light rays from the lamp are partly blocked by the shrinking design on the sphere; the resulting shadow is a regular tiling of the plane by squares. This illustrates how *stereographic projection* transforms the sphere, minus the north pole, into the plane. Note how shapes are slightly distorted near the south pole, and dramatically distorted near the north pole.

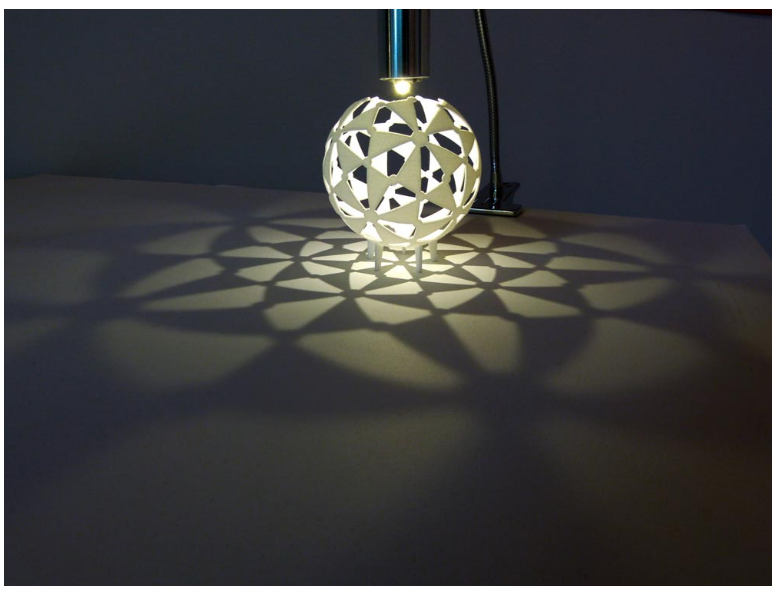
Challenge: What would happen to the pattern on the sphere if we extended the square tiling to the entire plane?

Dodecahedral Symmetries

Saul Schleimer and Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered
Dimensions: $8 \times 8 \times 9$ cm

In the plane, the three angles of a triangle must add up to π: 180 degrees. However on the sphere there is a triangle with angles $(\pi/2, \pi/3, \pi/5)$. As shown in the sculpture, the sphere is tiled by 120 copies of this triangle. The LED is positioned at the north pole of the sphere. The resulting shadows are the stereographic projection of the triangles to the plane. Note how the angles of the tiling are faithfully reproduced.

Challenge: How does the triangle with angles $(\pi/2, \pi/2, \pi/2)$ tile the sphere? How about the triangle with angles $(\pi/2, \pi/3, \pi/4)$?



24–cell

Saul Schleimer and Henry Segerman
Material: Colour 3D printed zp®150 powder
Dimensions: $18 \times 18 \times 18$ cm
On loan from the Museum of Mathematics

The 24–cell is one of the six regular four-dimensional polytopes. After radial projection, its spherical version gives a tiling of the three-sphere by regular octahedra. This is stereographically projected into three-space, giving the sculpture.

All of the regular four-dimensional polytopes, other than the 5–cell, can be obtained from symmetries in dimension three. One starts with a subgroup of SO(3), lifts to the binary group in the three-sphere, and then takes Voronoi domains. Starting with the binary tetrahedral group, $\mathbf{T}^*$, the result is the spherical 24–cell. Since the 24–cell is self-dual, its one-skeleton also gives a Cayley graph for $\mathbf{T}^*$. Each edge corresponds to a one-third rotation of the tetrahedron about a face or about a vertex.

Since every edge is an arc of a great circle, there is a unique quaternion that rotates the edge along itself, sending one endpoint to the other. Normalizing, this quaternion gives us a unit tangent vector at the identity in the three-sphere: that is, a point of the two-sphere. We colour this two-sphere in an antipodally invariant way, and pull back. The resulting coloring of the one-skeleton is consistent with the labelling, by generators of $\mathbf{T}^*$, of the Cayley graph.

Challenge: Are all of the vertices the same in S^3?

One-half 48–cell

Saul Schleimer and Henry Segerman
Material: Colour 3D printed zp®150 powder
Dimensions: $19 \times 19 \times 19$ cm
On loan from the Museum of Mathematics

The 48–cell, after radial projection, gives a tiling of the three-sphere by truncated cubes. The tiling is stereographically projected into three-space and then cut by the unit sphere. We retain the inner half, giving the sculpture.

If we start with a subgroup of SO(3), lift it to get a binary group in the three-sphere, and take Voronoi domains then we obtain a spherical polytope. For example, the cube group has 24 elements and so the binary cube group $\mathbf{C}^*$ has 48 elements. Voronoi domains about these give non-regular, truncated cubes. The result, the 48–cell, is *not* a regular polytope. The edges of the 48–cell are colored via the method for the 24–cell. Here there seems to be no interpretation in terms of Cayley graphs.

Challenge: Look for the truncated cube in the center. Find the $14 = 8 + 6$ cells that share a face with that, and $18 = 12 + 6$ half-cells that touch those.

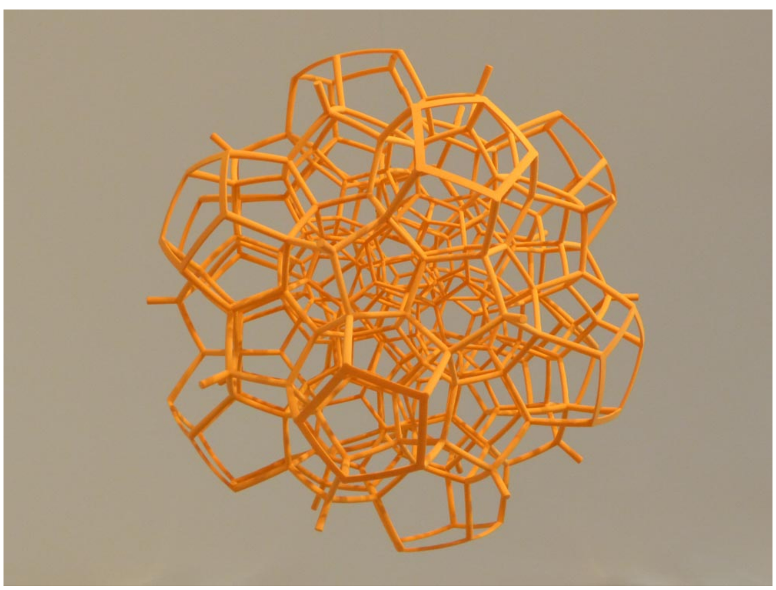
One-half 120–cell

Saul Schleimer and Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered; Rit Dye
Dimensions: $33 \times 33 \times 33$ cm

The 120–cell is one of the six regular four-dimensional polytopes. After radial projection, its spherical version gives a tiling of the three-sphere by regular dodecahedra. This is stereographically projected into three-space and then cut by the unit sphere. We retain the inner half, giving the sculpture.

The other half of the 120–cell is the spherical inversion of this half across the equatorial two-sphere. Cutting allows us to see the internal structure more clearly. The entire 120–cell, after stereographic projection, is almost 6 times larger than the sculpture shown here.

Challenge: Look for axes of symmetry. What orders do they have?



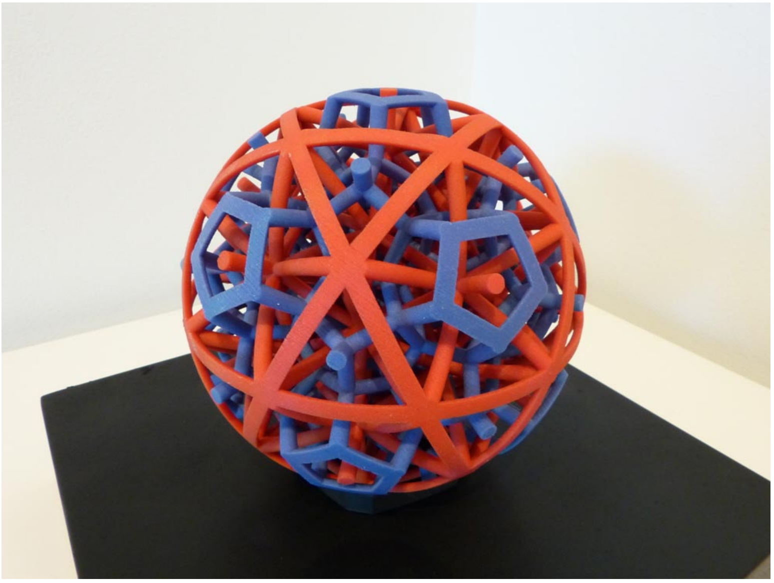
Dual half 120– and 600–cells

Saul Schleimer and Henry Segerman
Material: Colour 3D printed zp®150 powder
Dimensions: $15 \times 15 \times 15$ cm

The 120–cell (blue) and 600–cell (red) are two of the six regular four-dimensional polytopes. After radial projection, their spherical versions give tilings of the three-sphere by regular dodecahedra or tetrahedra, respectively. These are stereographically projected into three-space and then cut by the unit sphere. We retain the interlinking inner halves, giving the sculpture.

The 120–cell is obtained by taking Voronoi domains about the elements of the binary dodecahedral group $\mathbf{D}^*$. Since the 600–cell is dual to the 120, its edges give a Cayley graph for $\mathbf{D}^*$. In particular, any given edge of the 600–cell connects the centers of adjacent dodecahedral cells of the 120–cell.

Challenge: Count the number of edges of the 120–cell by counting the number of faces of the 600-cell.



Dodecahedron Chains

Saul Schleimer and Henry Segerman
Material: Colour 3D printed zp®150 powder
Dimensions: $27 \times 16 \times 15$ cm
On loan from the Museum of Mathematics

The 120–cell contains twelve *rings* of ten spherical dodecahedra each. To make a ring, start at a pentagonal face of a dodecahedral cell and move across the cell to the opposite face. Passing through this face exits one cell and enters its neighbor. Travelling straight through dodecahedra in this manner, after visiting ten cells we return to the start.

These rings of dodecahedra twist around each other, giving the *combinatorial Hopf fibration.* In more detail: any one ring is adjacent to five other rings, just as any face of the dodecahedron is adjacent to five other faces. The five rings meeting a given ring together tile a solid torus making up half of the three-sphere.

This sculpture consists of a trio of three pairwise adjacent rings. It is coloured via the Gauss map, in similar fashion to the Round Klein Bottle.

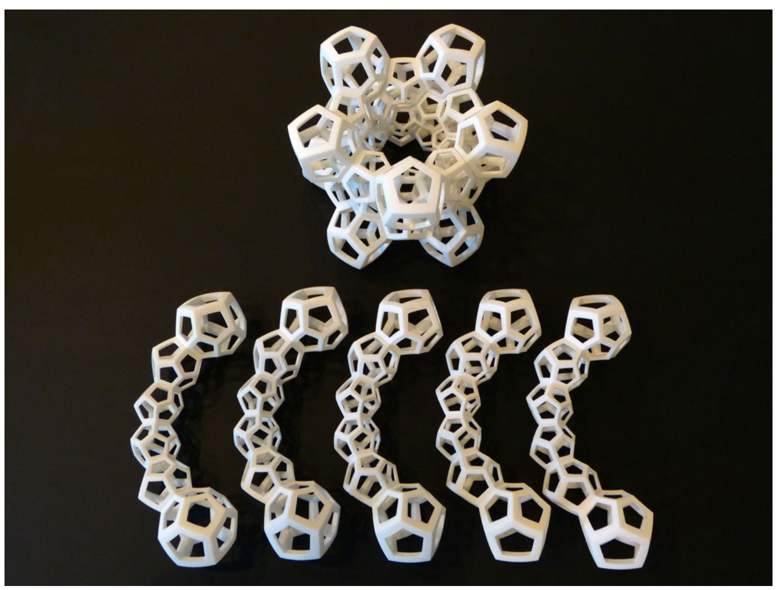
Challenge: How many pentagonal faces are shared by a pair of adjacent rings?

Dc30 Ring, assembled and unassembled

Saul Schleimer and Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered
Dimensions: $16 \times 16 \times 12$ cm assembled

In the combinatorial Hopf fibration, any one ring meets five others. These six rings cannot be disentangled. To overcome this we remove the four dodecahedra from each ring other than the "equatorial" ring. The result is the *outer 6 rib* – five of these are shown, separated. It is a bit of a puzzle to assemble these around the (missing) equator; the result is the Dc30 Ring.

Challenge: Six copies of the outer 6 rib can be assembled into a very different configuration.

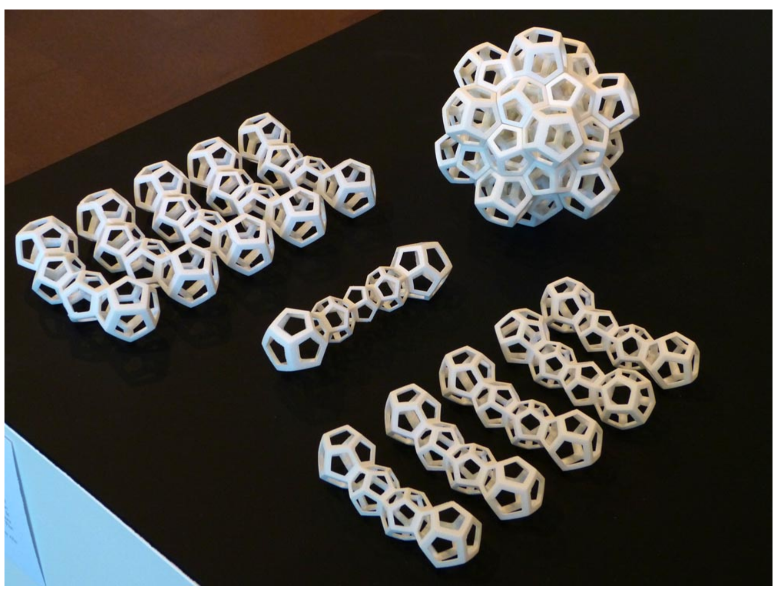


Dc45 Meteor, assembled and unassembled

Saul Schleimer and Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered
Dimensions: $13 \times 13 \times 13$ cm (assembled)

The 12 rings of the combinatorial Hopf fibration of the 120–cell each have ten dodecahedral cells. We delete the 75 cells closest to the north pole, leaving 45. This deletes the equatorial ring entirely. It leaves five cells of the "spinal" ring and four cells in each of the inner and outer ribs (meeting the spine and the equator, respectively). The ribs are shown unassembled and assembled.

Challenge: There are six different ways to assemble the given ribs to get the Dc45 Meteor.



Round Möbius Band

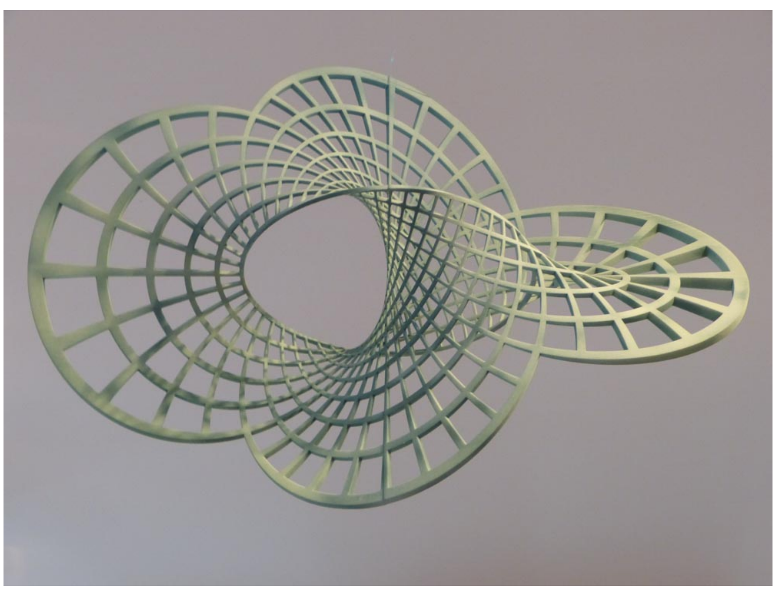
Saul Schleimer and Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered; Rit Dye
Dimensions: $60 \times 25 \times 43$ cm

The child's Möbius band is made by making a half twist in a strip of paper then joining the ends. Its boundary is an unknotted loop in space, so it can be deformed into a circle. Doing this in the most symmetric way carries the surface to the Round Möbius Band.

The boundary of the band is now a circle in the center of the sculpture, and the surface "goes through infinity": in the ideal version the surface extends outwards forever, becoming more and more flat. To avoid having an infinitely large sculpture, we have removed a square. The design is parameterised in S^3, then stereographically projected to $\mathbb{R}^3$. The parameterisation is as follows.

$$(\cos\theta\cos\phi,\, \cos\theta\sin\phi,\, \sin\theta\cos 2\phi,\, \sin\theta\sin 2\phi)$$

Challenge: Find a path in the surface that shows the surface has only one side.



Round Klein Bottle

Saul Schleimer and Henry Segerman
Material: Colour 3D printed zp®150 powder
Dimensions: $24 \times 24 \times 17$ cm

Two copies of the Möbius band, glued together along their boundaries, makes a Klein bottle. Here we start with two copies of the Round Möbius Band – the result is the Round Klein Bottle.

Let B be the boundary of the thickened Klein bottle in S^3 – the thickening is required for the stereographic projection to be printable in three-space. Away from the corners of B there is exactly one outward pointing normal. Using the quaternionic structure on S^3, the normal vector is moved to the identity, giving the *Gauss map* from B to the unit sphere (again, away from the corners). We identify the unit sphere with the sphere that is inscribed in the red/green/blue colour cube; this gives the colouring of the sculpture.

Since the Round Möbius Band goes through infinity, the Round Klein Bottle necessarily goes through infinity twice. Thus the Round Klein Bottle self intersects. The self-intersections form a great circle in the three-sphere and thus a line in three-space. The parametrisation is exactly the same as for the Round Möbius strip, but over a domain of twice the size.

Challenge: Look along the line of self intersection. What is the order of symmetry?

Clifford Torus

Saul Schleimer and Henry Segerman
Material: Colour 3D printed zp®150 powder
Dimensions: $23 \times 23 \times 10$ cm

Like the Round Klein Bottle, the Clifford Torus is a minimal surface in the three-sphere. This surface has the following parametrisation.

$$(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi)$$

The grid lines on the surface correspond to constant values of θ and ϕ. The surface is stereographically projected from the three-sphere to three-space to allow 3D printing – as usual we have placed the projection point on the surface. To avoid having an infinitely large sculpture, we have removed a square from the surface. The piece is coloured via the Gauss map, in similar fashion to the Round Klein Bottle.

Challenge: The Clifford Torus, extended to infinity, separates three-space into two identical halves. Can the two sides be swapped by a rigid motion of three-space?

Knotted Cog

Saul Schleimer and Henry Segerman
Material: Colour 3D printed zp®150 powder
Dimensions: $22 \times 19 \times 8$ cm
On loan from the Museum of Mathematics

This sculpture is based on the following parametrisation of the trefoil knot in S^3, where θ is fixed and ϕ varies.

$$\left(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta\cos\frac{3}{2}\phi,\ \sin\theta\sin\frac{3}{2}\phi\right)$$

Here, in a playful reference to steampunk semiotics, we have added intermeshing cog teeth. All of the teeth are identical in S^3, the three-sphere. The apparent differences in size are due to the distortion coming from stereographic projection. The sculpture is coloured using the colour wheel, parametrized by arclength around the the knot – thus complementary colours are adjacent.

Challenge: How many cog teeth are there? Is any number of teeth possible?

$(5, 3)$ **Seifert Surface**

Saul Schleimer and Henry Segerman
Material: Colour 3D printed zp®150 powder
Dimensions: $20 \times 20 \times 16$ cm
On loan from the Museum of Mathematics

A *knot* is a loop, without beginning or end, in three-space. A *torus* is the mathematician's term for the surface of a donut. A knot K that lives on the torus is called a *torus knot*. If we count the number of times, p, that K goes through the hole of the torus, and the number of times, q, that K goes around the torus we get a fraction p/q that specifies the torus knot (up to a choice of handedness).

A *Seifert surface* for a knot L is a surface in three-space, with boundary equal to K. Any torus knot has a canonical choice of Seifert surface, namely the *Milnor fiber*: the points of the three-sphere solving

$$\arg(z^p + w^q) = 0,$$

where z and w are complex coordinates in $\mathbb{C}^2$. We parametrise the Milnor fiber, following the work of Tsanov, via fractional automorphic forms. These give a map from $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$, the canonical geometry of the torus knot complement, to the three-sphere. The pattern of triangles in each fundamental domain come from two applications of the Schwarz-Christoffel theory in complex analysis, turning a Euclidean triangle into a hyperbolic one.

Challenge: The Seifert surface can be realized topologically as a family of p disks, a family of q disks, and a collection of $p \cdot q$ copies of a very small quarter-twisted band. Find the families of five and three disks.

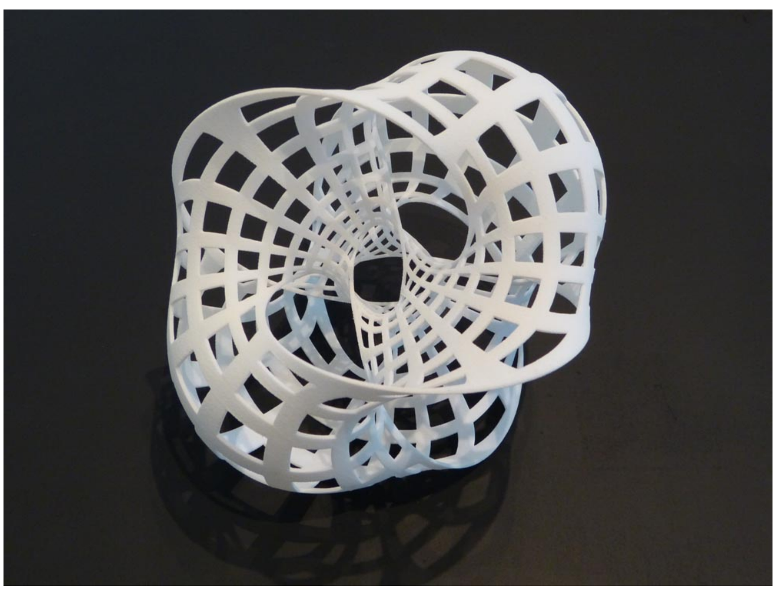
$(4, 4)$ **Seifert Surface**

Saul Schleimer and Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered
Dimensions: $14 \times 14 \times 15$ cm

A *link* is a collection of loops in three-space, each without beginning or end, that do not intersect each other. As with torus knots, a *torus link* is a link that lives on the torus. As for torus knots, Milnor fibers provide canonical Seifert surfaces for torus links.

Here, since $p = q = 4$, a highly symmetric pattern is possible for the fundamental domain of the tiling of the surface.

Challenge: Find any fundamental domain for the tiling of the Seifert surface. How many copies are there in the Seifert surface?



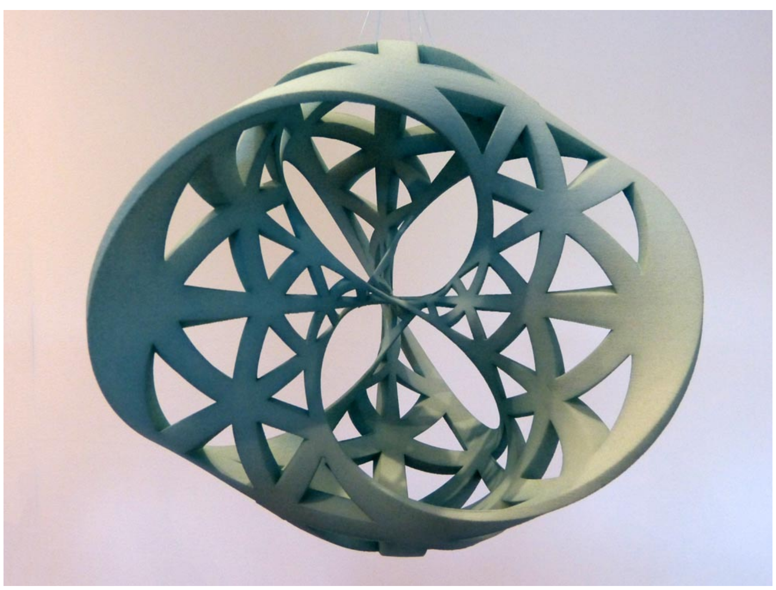
(3, 3) **Seifert Surface**

Saul Schleimer and Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered; Rit Dye
Dimensions: $16 \times 14 \times 15$ cm

The basic construction technique here is the same as for the other Seifert surfaces: we use fractional automorphic forms to give a parametrisation of the Milnor fiber. However, the surface is thickened using the so-called *elliptic flow*. This flow comes from the action of SO(2) subgroup on $\mathrm{PSL}(2, \mathbb{R})$, pushed down to S^3. The flow is transverse to the Seifert surface away from the boundary. As a point x approaches the boundary the flow line through x topples; finally the boundary is a union of flow lines.

Since $p = q = 3$, a highly symmetric pattern is possible for the fundamental domain of the tiling of the surface.

Challenge: Look through the windows in the thickened surface. Find the places where the toppling is as small as possible. What is special about those points?

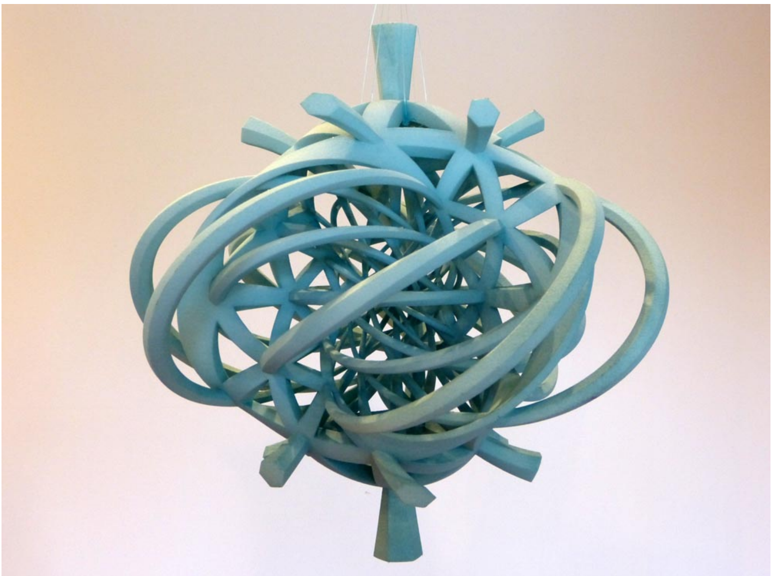


$(3, 3)$ **Seifert Surface with Fibers**

Saul Schleimer and Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered; Rit Dye
Dimensions: $21 \times 19 \times 21$ cm

When $p = q$, the flow lines of the elliptic flow are all fibers of the *Hopf fibration*: the fibration of S^3 coming from intersecting S^3 with all complex lines in $\mathbb{C}^2$. In this piece we add to the $(3, 3)$ Seifert Surface all flow lines through the small hexagons of the tiling of the Seifert surface. Flow lines going too close to infinity are cut off.

Challenge: Follow the fibers to see that they are round circles.



Twisted Earth

Saul Schleimer and Henry Segerman
Material: Colour 3D printed zp®150 powder
Dimensions: $25 \times 23 \times 20$ cm

Here we revisit the Seifert surface for the $(3,3)$ torus link, this time using it as a canvas for a distorted map of the Earth. Before projecting to three-space, the Seifert surface sits in S^3. The Hopf fibration maps from S^3 to S^2, the ordinary sphere, which we colour using the continents and oceans of the Earth. We then colour each point of S^3, including the Seifert surface, the same as the corresponding point on the Earth.

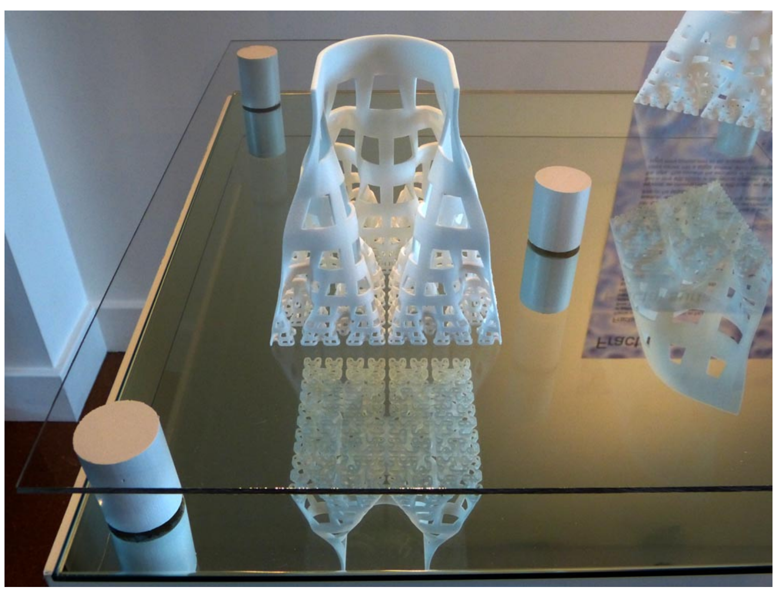
Challenge: Find all copies of Stony Brook.

Developing Hilbert Curve

Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered
Dimensions: $10 \times 10 \times 10$ cm

This sculpture shows the first few steps in the construction of the Hilbert curve, a space-filling curve in $\mathbb{R}^2$. The curve is constructed recursively: we start with very simple curve (at the top) and replace each of its arcs with a copy of the topmost curve. Each curve is given two bands below the previous – that is, the sculpture replaces motion through time with motion through space. Every other horizontal band follows one of the curves.

Challenge: How many steps in the construction of the Hilbert curve are shown here?

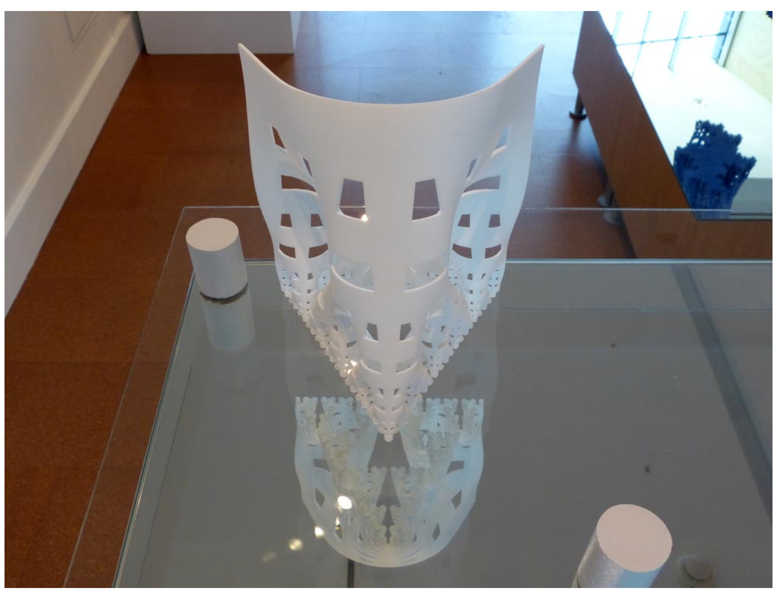


Developing Sierpińksi Arrowhead Curve

Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered
Dimensions: $13 \times 11 \times 14$ cm

This sculpture shows the first few steps in the construction of the Sierpińksi arrowhead curve, a fractal curve in $\mathbb{R}^2$ that "fills" the Sierpińksi triangle, discovered by Mandelbrot. The curve is constructed by a recursive process that starts with a simple curve and replaces it with successively more complicated curves. These steps are shown here using the third dimension – every other horizontal band follows one of the curves.

Challenge: The first curve in the construction of the Sierpińksi arrowhead curve consists of three line segments that make up half of a hexagon. How many line segments are in the second curve? What is the general formula?

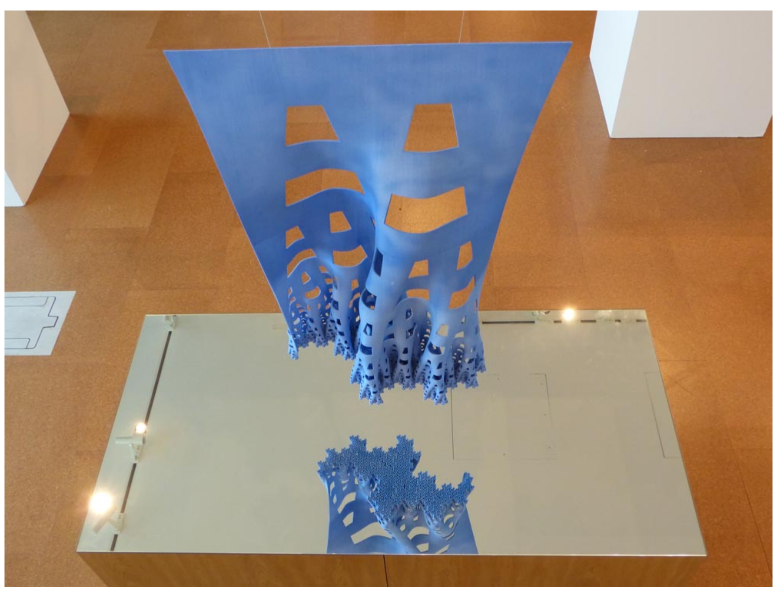


Developing Terdragon Curve

Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered; Rit Dye
Dimensions: $65 \times 52 \times 22$ cm

This sculpture shows the first few steps in the construction of the terdragon curve, a space-filling curve with fractal boundary in $\mathbb{R}^2$ discovered by Davis and Knuth. The curve is constructed by a recursive process that starts with a simple curve and replaces it with successively more complicated curves. These steps are shown here using the third dimension.

Challenge: A square can be subdivided into four squares, each half the size of the original. Similarly, the bottom of the sculpture can be divided into three identical pieces; what is the ratio of sizes?



Umbilic Rolling Link

Helaman Ferguson
Material: Bronze
Dimensions: $16 \times 16 \times 10$ cm
On loan from Tony Phillips

"This is a dynamic piece: The deltoid umbilic torus rolls through the dual cardioid umbilic torus and vice-versa, each piece free to turn. The umbilic refers to the deltoid as a hypocycloid of three cusps; the cardioid as an epicycloid of one cusp.

Each torus is the dual of the other, an outside-in instantiation: The compactification of the exterior of one is topologically the interior of the other.

Each is the inside out of the other/ Each is the outside in of the other/ Together they form a yin-yang pair;/ yang is the deltoid torus, and/ yin is cardioid torus.

As solid tori they can be viewed as an instantiation of a three-sphere."

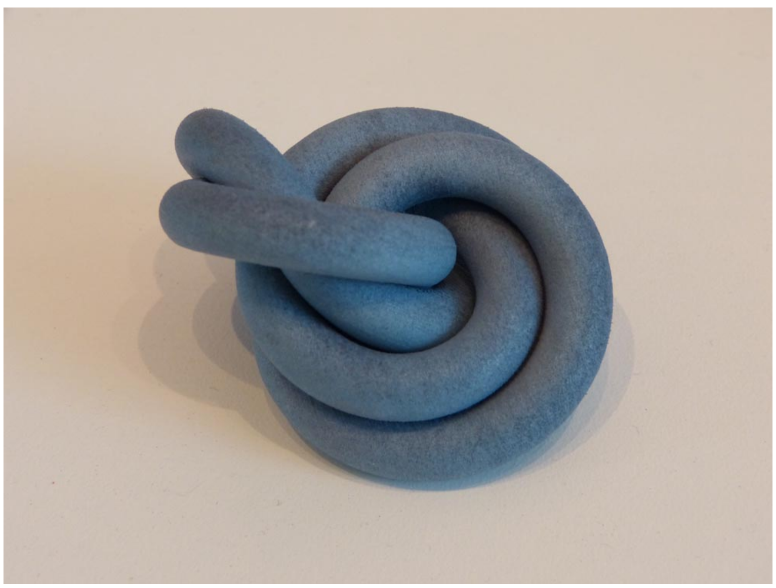
— Helaman Ferguson

Knotted Gear

Oskar van Deventer
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered; Rit Dye
Dimensions: $6 \times 4 \times 4$ cm

"These are two knots linked together. The two knots turn through each other in a gearing motion in a $2:3$ ratio."

— Oskar van Deventer



Triple Gear

Saul Schleimer and Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered;
motorized base.
Dimensions: $7.0 \times 7.4 \times 4.5$ cm without base

Three ordinary gears can be arranged into a triangle on the plane, so that each meshes with the other two. However, such an arrangement is frozen in place because meshing gears must rotate in opposite directions. This sculpture gives a non-planar, and non-frozen, arrangement of three linked gears. We were inspired by existing arrangements with two linked gears, due independently to Helaman Ferguson and Oskar van Deventer.

The gears are powered by a motor in the base of the sculpture, which rotates a central axle. We thank Adrian Goldwaser and Stuart Young for prototyping and designing the motorised base.

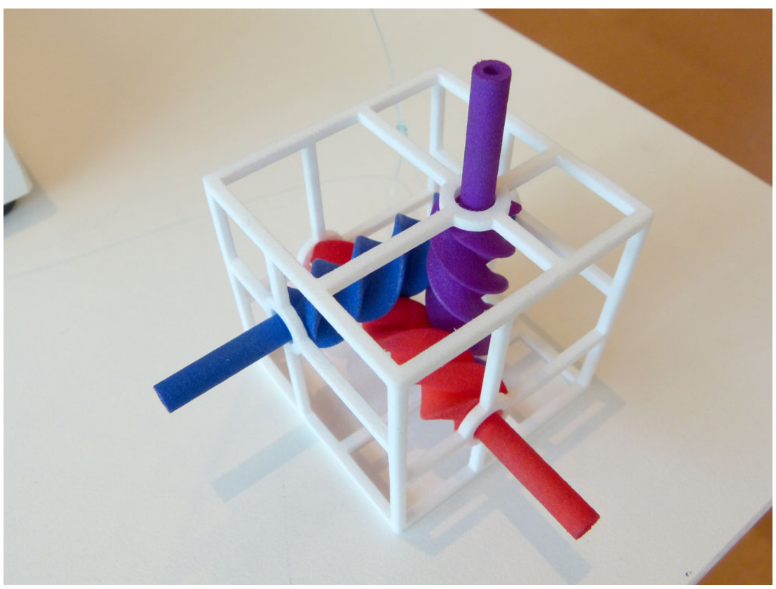
Challenge: Relate this piece to the $(3,3)$ Seifert Surface sculpture.

Triple Helix

Saul Schleimer and Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered; Dye
Dimensions: $6.5 \times 6.5 \times 6.5$ cm

This sculpture, consisting of a gearbox and three perpendicular axles, gives another, much simpler, solution to the problem of finding three pairwise meshing gears that are not frozen.

Challenge: Find a mechanism in which three gears are pairwise meshing, not frozen in place, with all three axes of rotation parallel.



Borromean Racks

Saul Schleimer and Henry Segerman
Material: PA 2200 Nylon Plastic, Selective-Laser-Sintered; Dye
Dimensions: $10 \times 10 \times 10$ cm

This sculpture consists of three identical pieces: each a loop with six racks on its inner, upper, and lower faces respectively. The racks of each piece mesh with one or two of the racks of the others. The loops interlink in the fashion of the Borromean rings.

This gives another example of a triple of gears that intermesh, but still move.