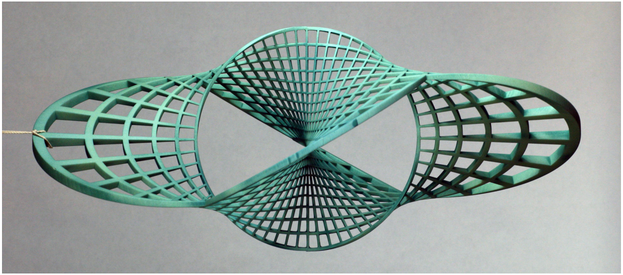
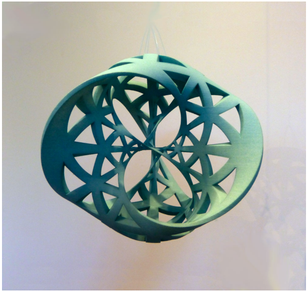
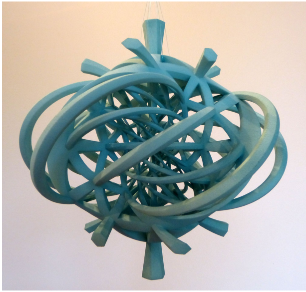


Saul Schleimer
University of Warwick

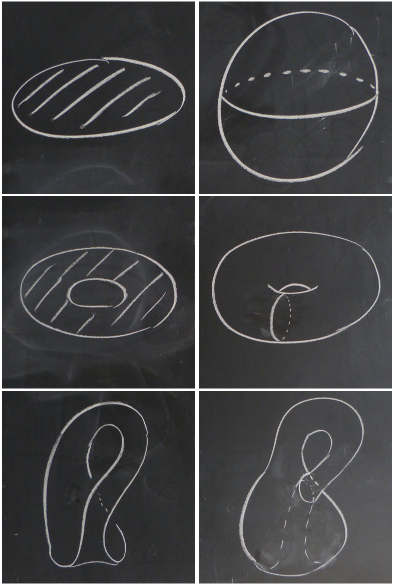
Minimal and Seifert Surfaces





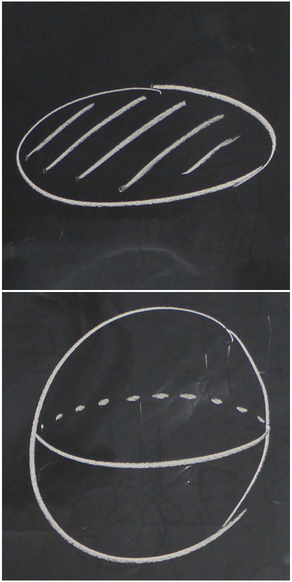
Surfaces

Rogues gallery



Dimension is a way to measure degrees of freedom. The acrobat on the high wire has one degree of freedom; the dancer on the stage has two; the astronut in space has three. A *surface* is a two-dimensional space – at every point an inhabitant has two degrees of freedom.

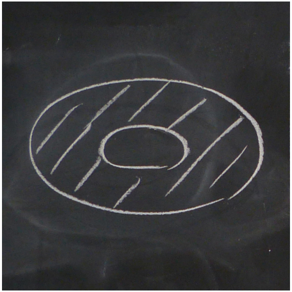
Disk and sphere

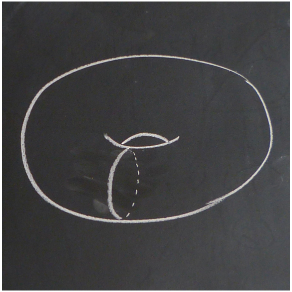


The disk is the prototypical surface – note the boundary is a circle.

We may take two copies of the disk and glue them to obtain the sphere. This is called *doubling* a surface. The two disks become the upper and lower hemispheres.

Annulus and torus

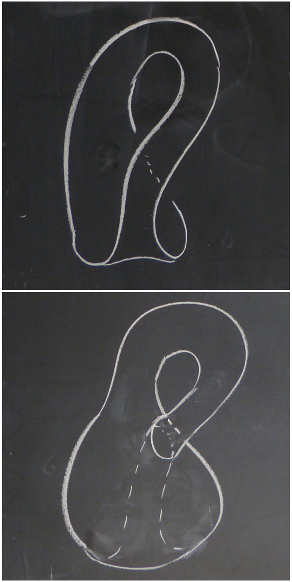




To make an annulus we can either cut a hole out of a disk, or we can glue the short edges of a 1×2 rectangle.

If we double the annulus across its boundary we obtain the torus – the boundary of a donut.

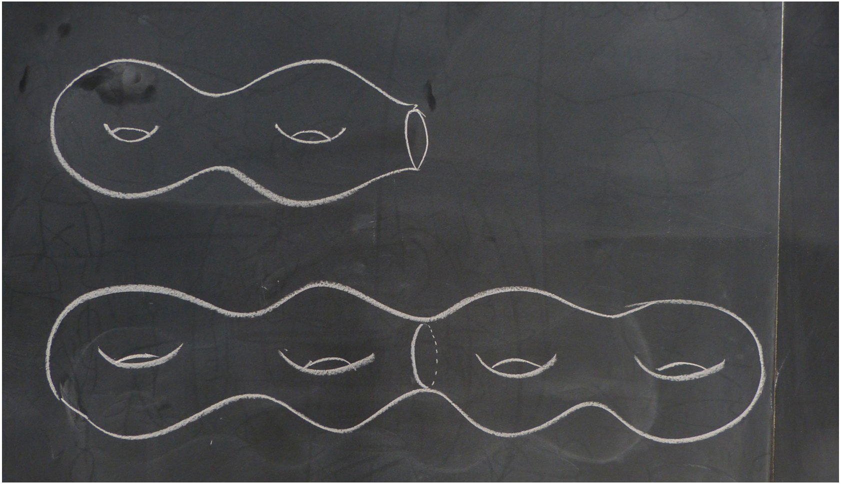
Möbius band and Klein bottle



To make a Möbius band we take a 1×2 rectangle, give it a half twist, and glue the short edges. Famously, the Möbius band is the first of the *non-orientable* surfaces.

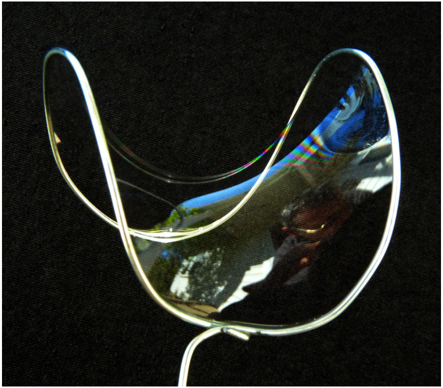
The Klein bottle is the double of the Möbius band.

Higher genus



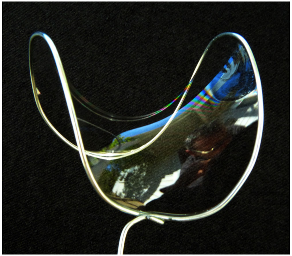
Minimal surfaces

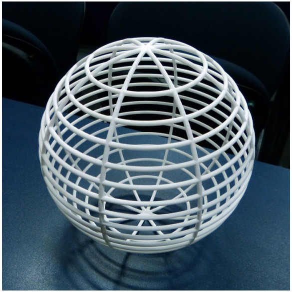
Area



The minimality of minimal surfaces refers to *area*.

Plateau and isoperimetric problems

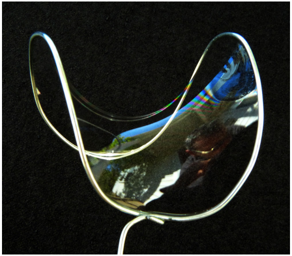


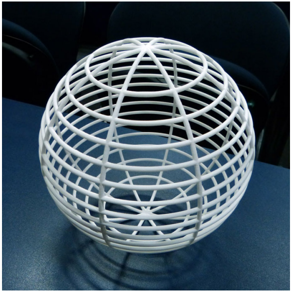


A soap film on a wire frame solves *Plateau's problem*: what is the least area surface among all surfaces with a given boundary?

A soap bubble, on the other hand, solves the *isoperimetric problem*: what is the least area surface amongst all closed surfaces containing a given volume?

Plateau and isoperimetric problems

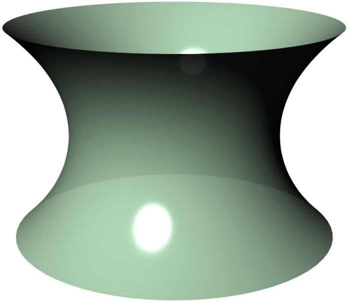




These two problems regarding area lead to surfaces with very different *geometries*. The first is saddle shaped while the second is very round.

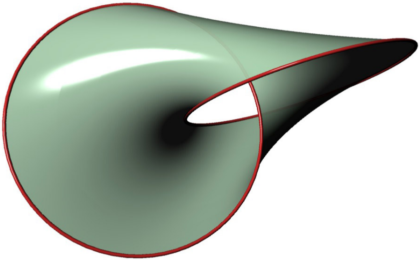
We are interested in the first type of geometry. We enshrine this in a definition: a *minimal surface* is one that locally solves Plateau's problem.

Catenoid



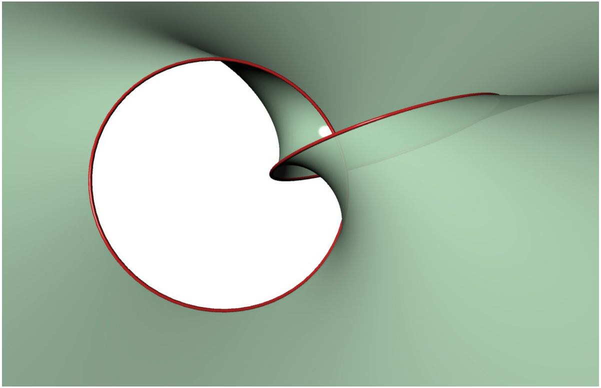
If we dip two parallel, round circles in soap solution, then we'll get either a pair of disks or the *catenoid*: an annulus of minimal area.

Hopf band and Clifford torus



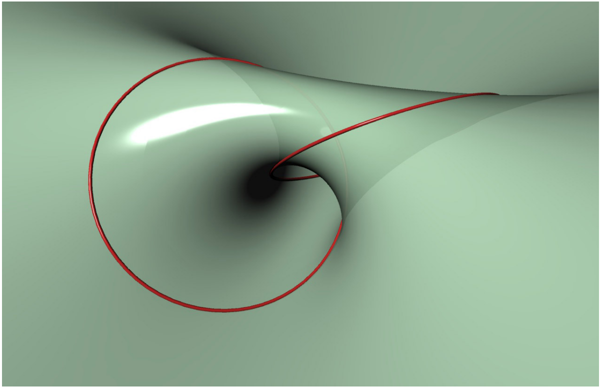
$$(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi)$$

Hopf band and Clifford torus



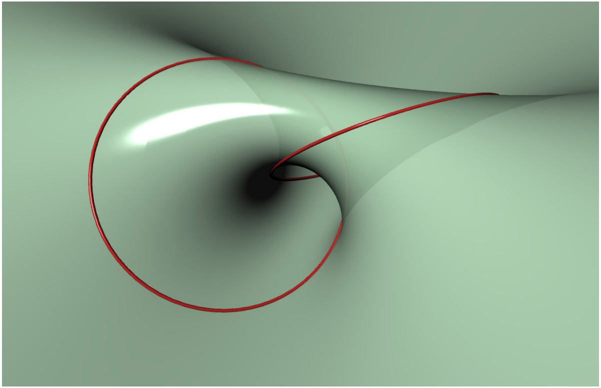
$$(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi)$$

Hopf band and Clifford torus



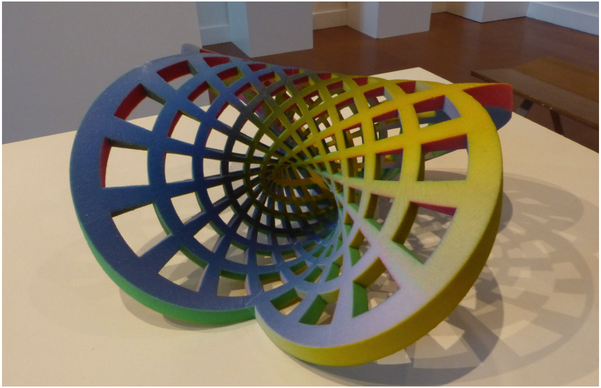
$$(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi)$$

Clifford torus



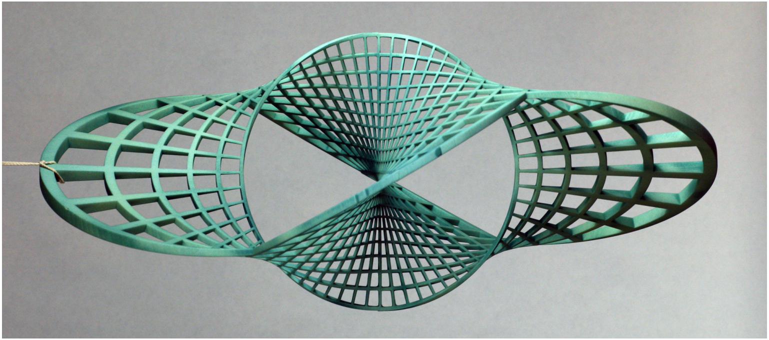
The two orthogonal families of great circles are called *Villarceau circles*.

Clifford torus



The two orthogonal families of great circles are called *Villarceau circles*.

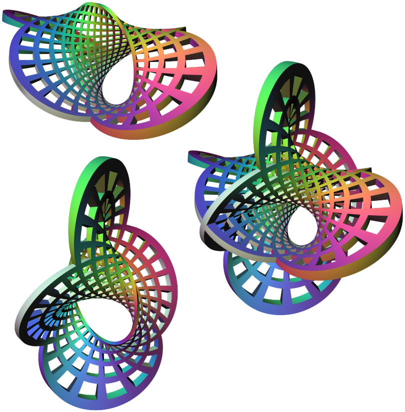
Möbius band



$$(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta\cos 2\phi,\ \sin\theta\sin 2\phi)$$

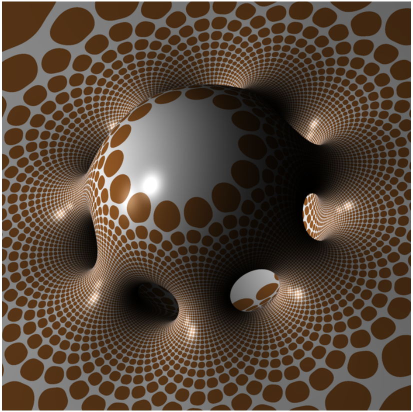
Photo: George Hart (2014)

Klein bottle



The Klein bottle is the double of the Möbius band. These realizations of the annulus, torus, Möbius band, and Klein bottle as minimal surfaces are special cases of a construction of Lawson (1970).

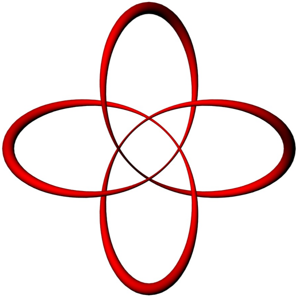
Lawson surface



Lawson gives minimal surfaces of all genera in the three-sphere. To the left we have the surface $\xi_{6,1}$, rendered by Nicholas Schmitt (Tuebingen).

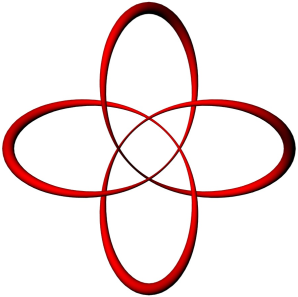
Seifert surfaces

Knots



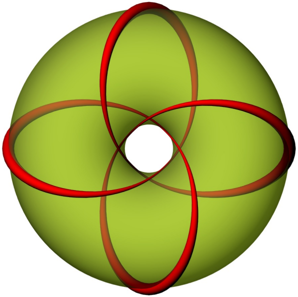
A knot K is a loop in space. Since K is a topological object, it does not come to us with a canonical position.

Torus knots and links



We say K is a *torus knot* if it can be moved to lie in the standard torus. Since the torus is "flat" we can draw the knot as a straight line in the square.

Torus knots and links



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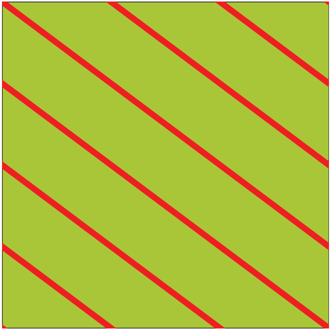
Torus knots and links



We say K is a *torus knot* if it can be moved to lie in the standard torus. Since the torus is "flat" we can draw the knot as a straight line in the square.

Seifert surfaces

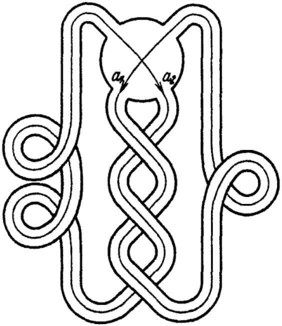


Fig. 9.

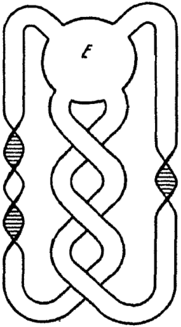


Fig. 10.

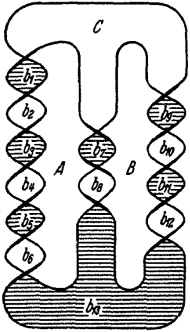


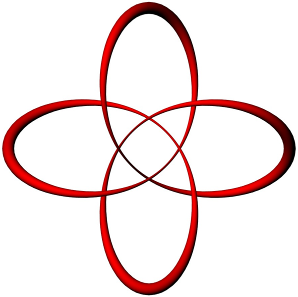
Fig 11.

Knots are one-dimensional objects living in three-dimensional space.
A *Seifert surface* is a two-dimensional bridge between them.

[1]

[1]Figure: Herbert Seifert (1934)

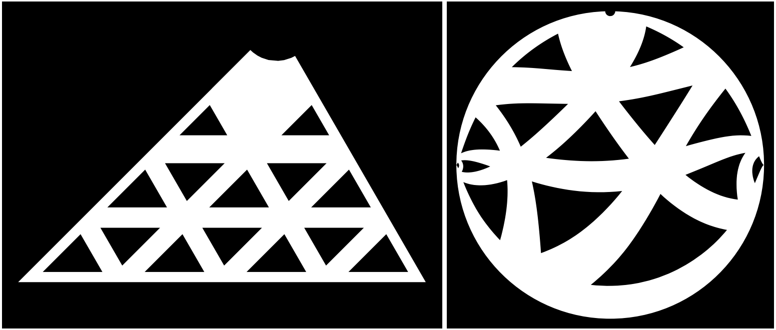
Milnor fibers



The torus knot for the rational number p/q is cut out of $S^3 \subset \mathbb{C}^2$ by the complex curve $z^p + w^q = 0$. The *Milnor fiber* at angle θ is the surface in S^3 given by the equation

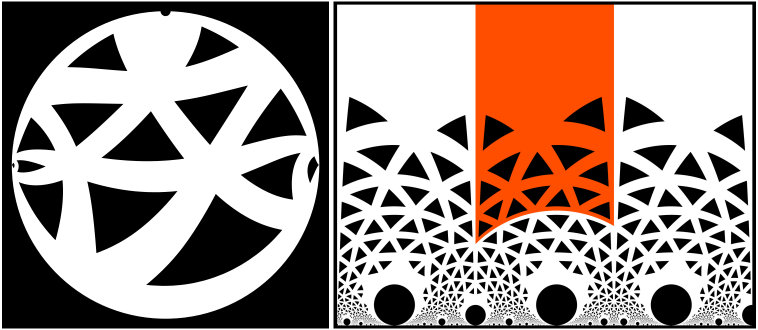
$$\arg(z^p + w^q) = \theta.$$

Euclidean triangle to Poincare disk



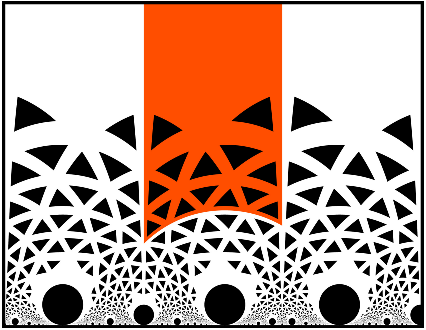
Inverting the incomplete Beta function.

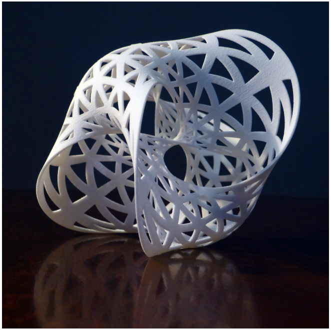
Poincare disk to hyperbolic triangle



A variant of the Schwarz-Christoffel map.

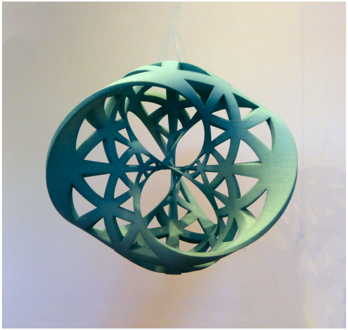
Hyperbolic triangle to S^3

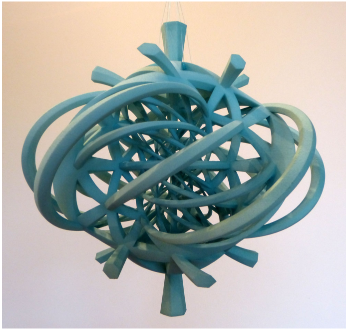




Following Milnor and Tsanov.

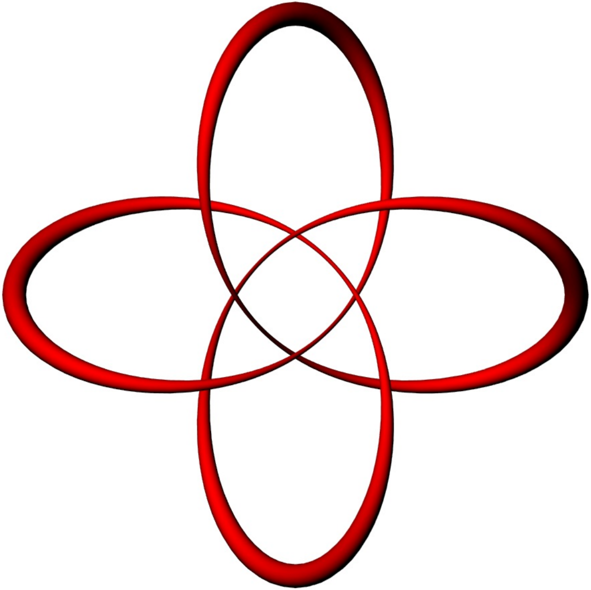
Milnor fibers and Seifert fibers



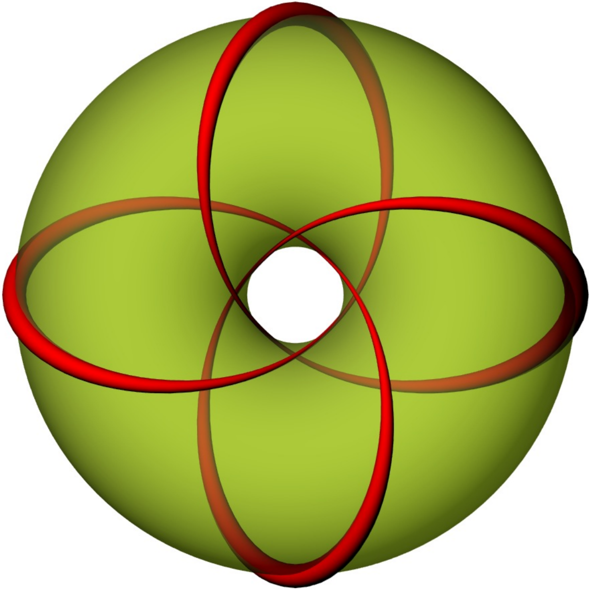


The Milnor fiber for the (3,3) torus link with and without Seifert fibers.

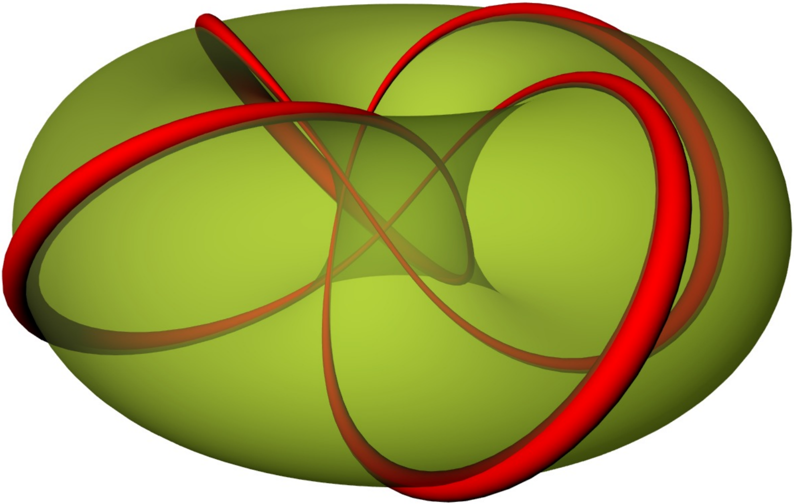
Building the Milnor fiber



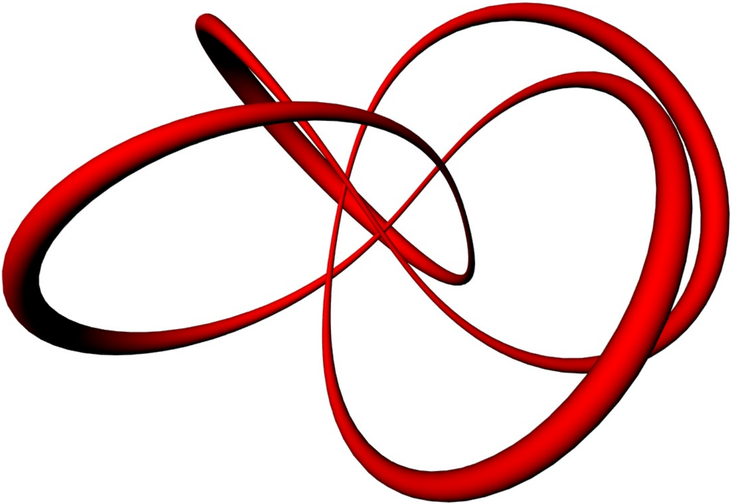
Building the Milnor fiber



Building the Milnor fiber



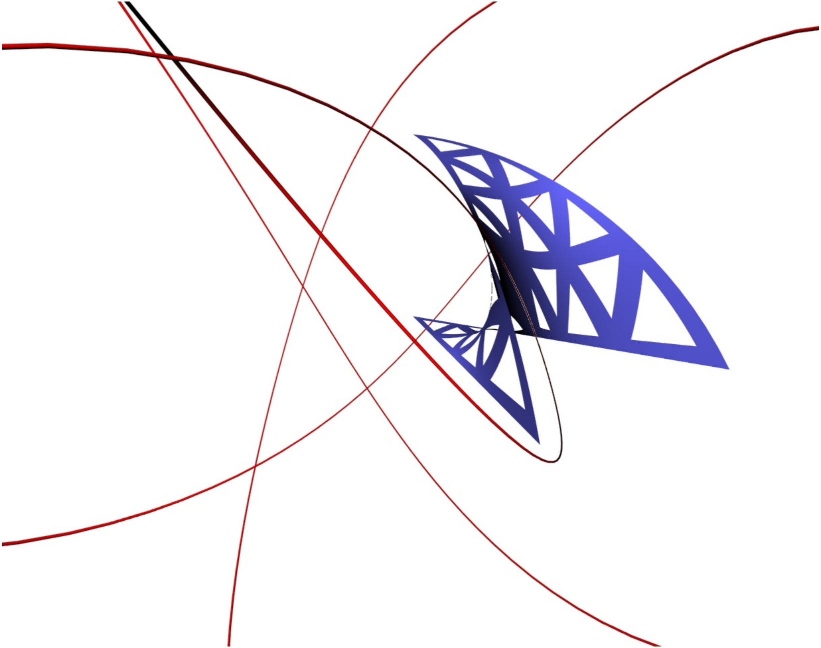
Building the Milnor fiber



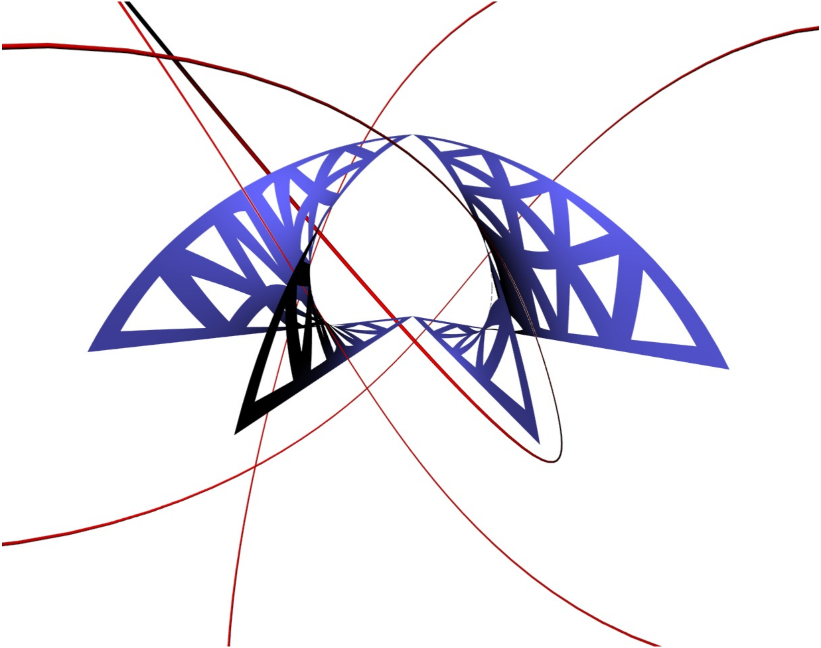
Building the Milnor fiber

Building the Milnor fiber

Building the Milnor fiber



Building the Milnor fiber



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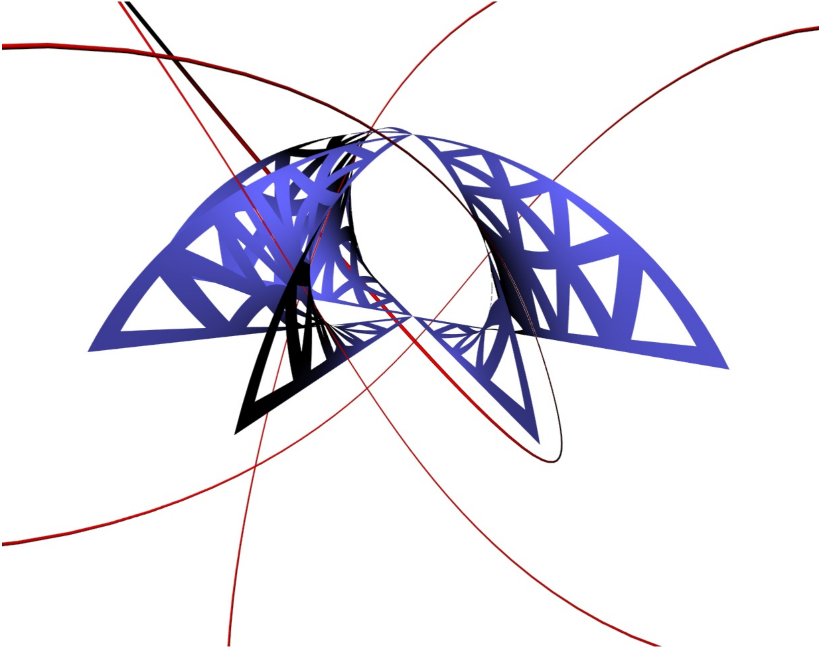

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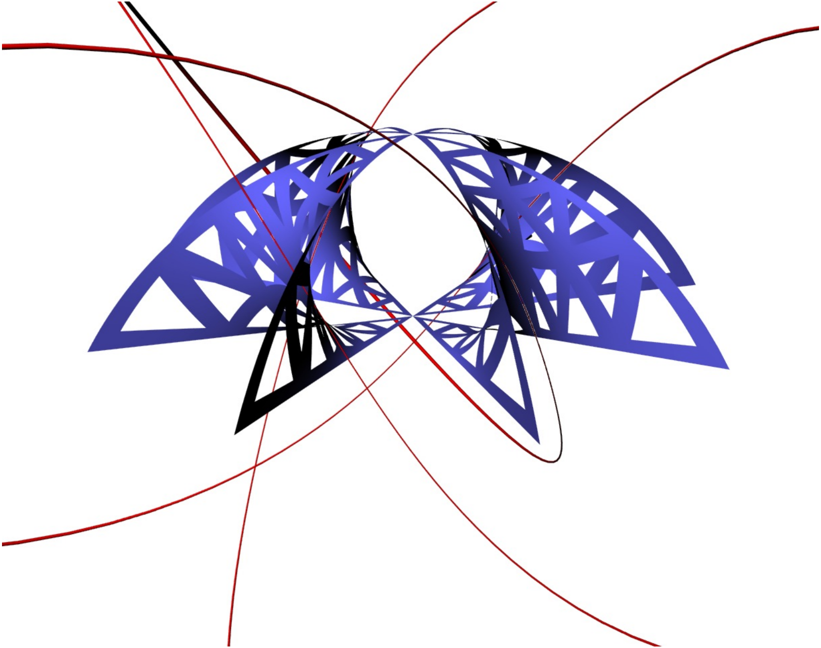

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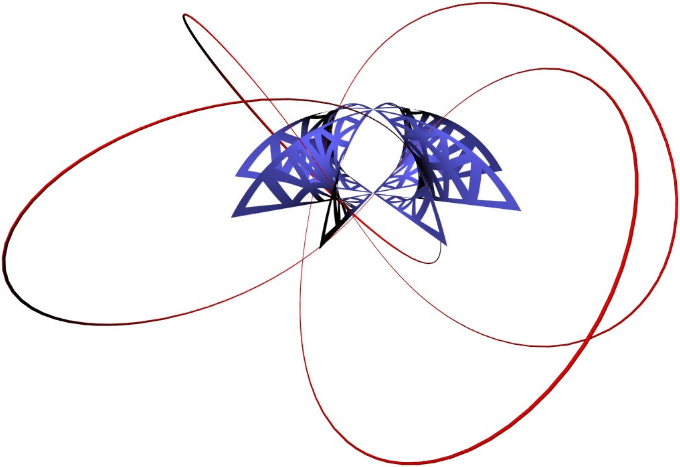

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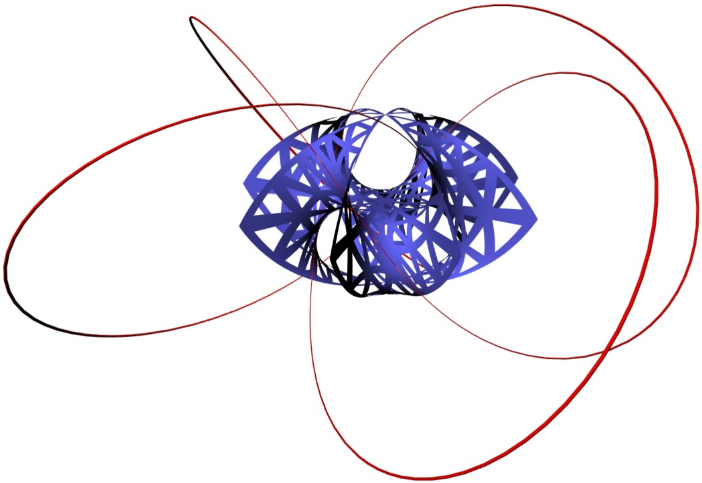
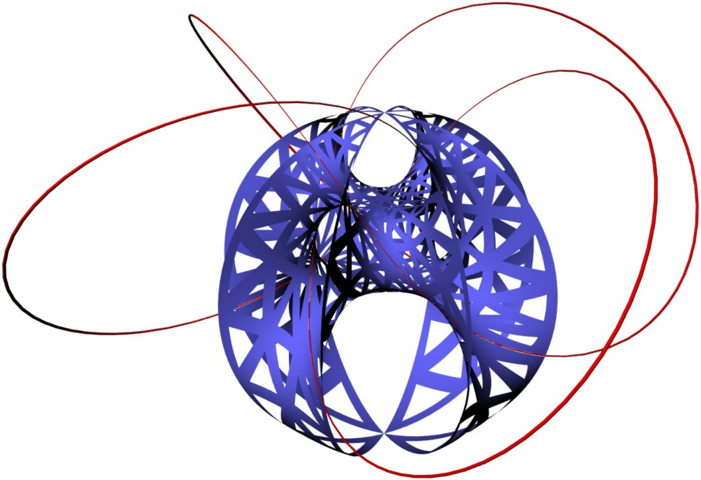
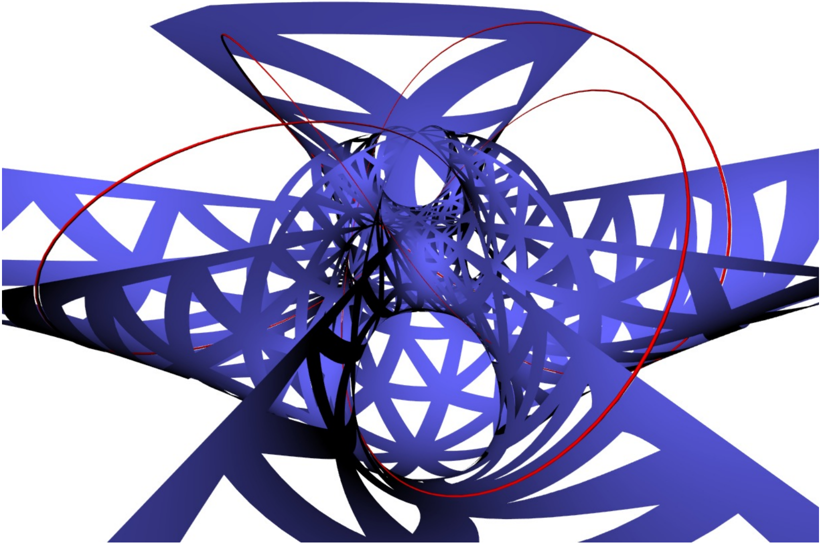


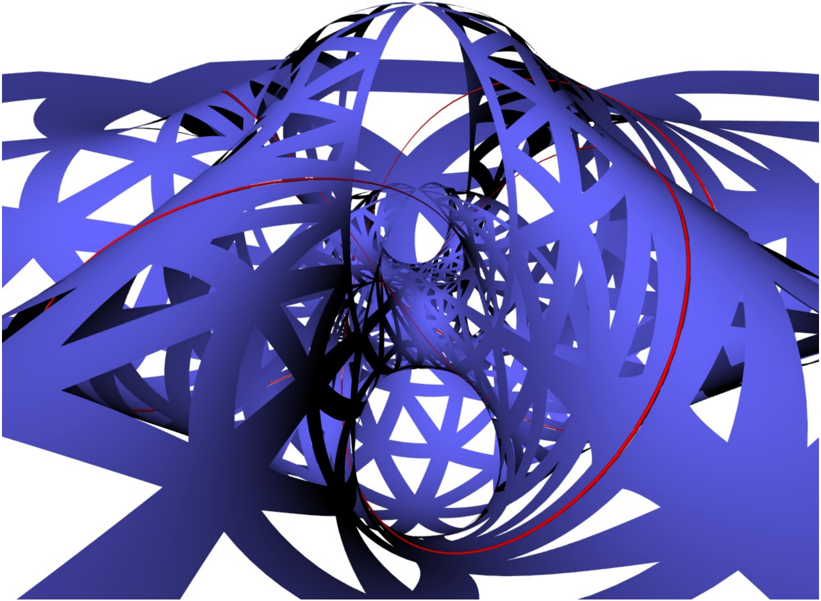
Building the Milnor fiber



Building the Milnor fiber



Building the Milnor fiber



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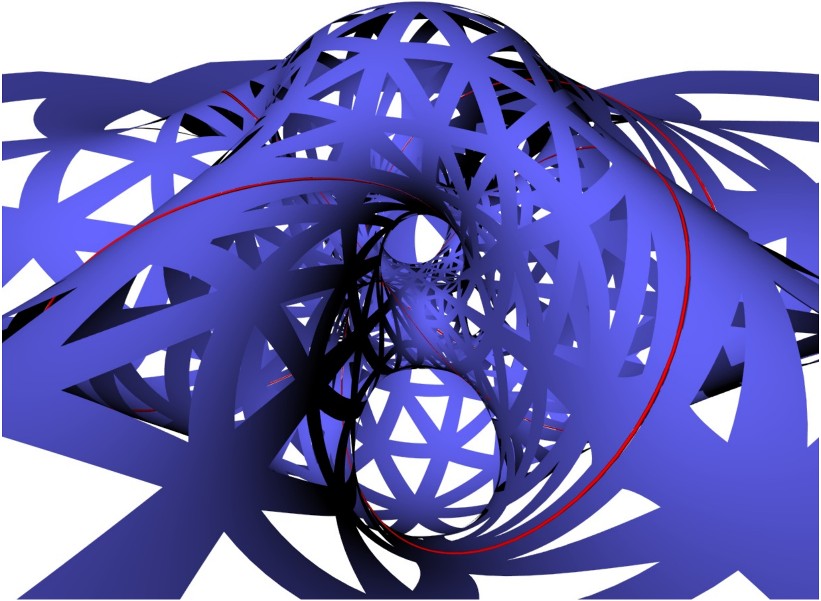

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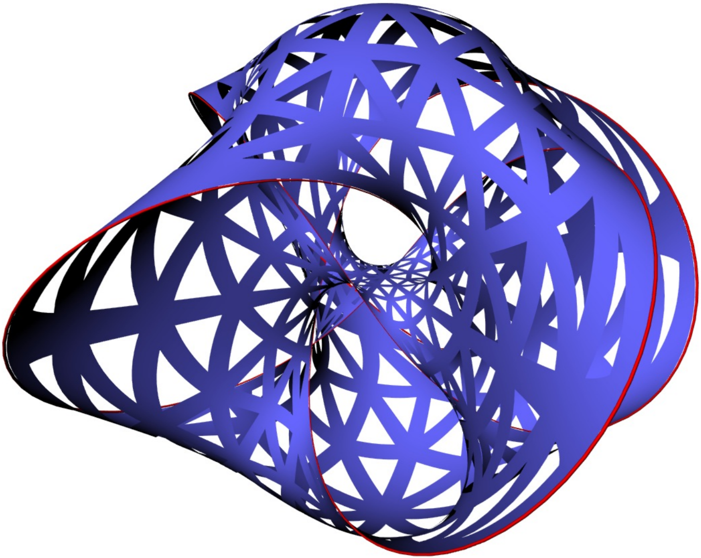

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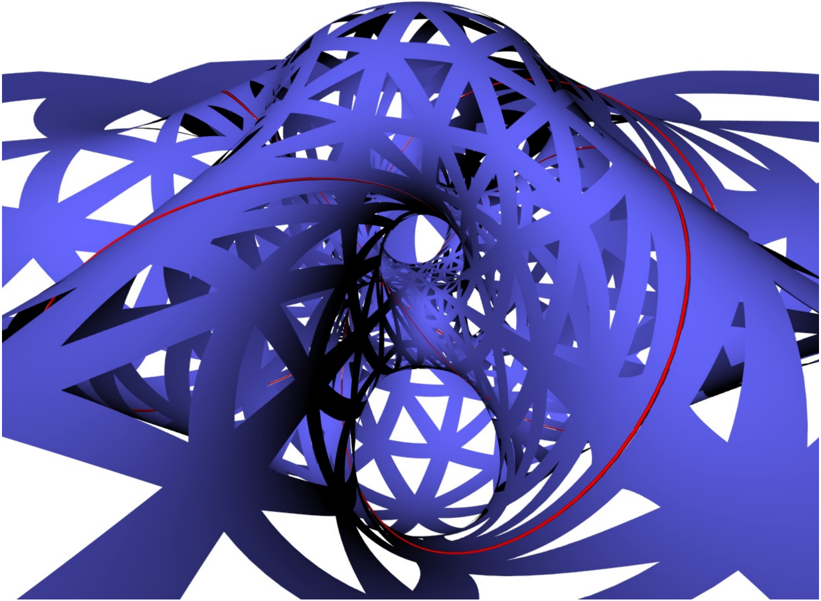

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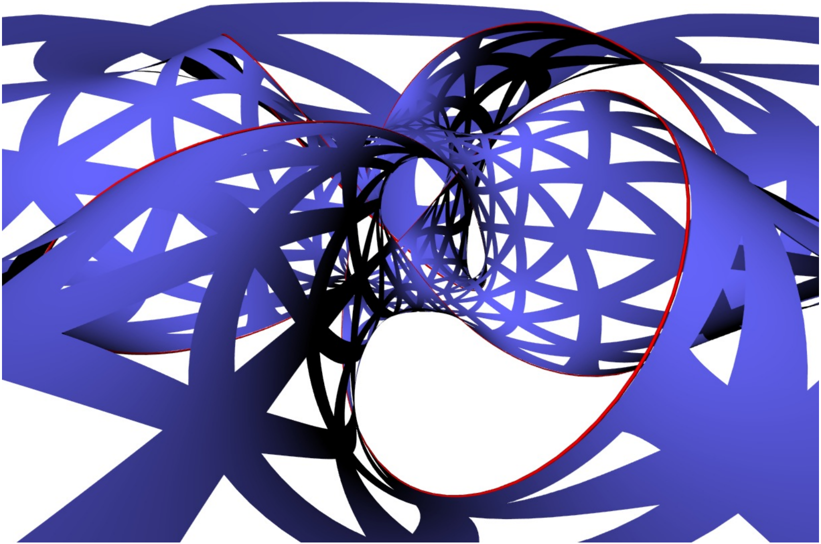

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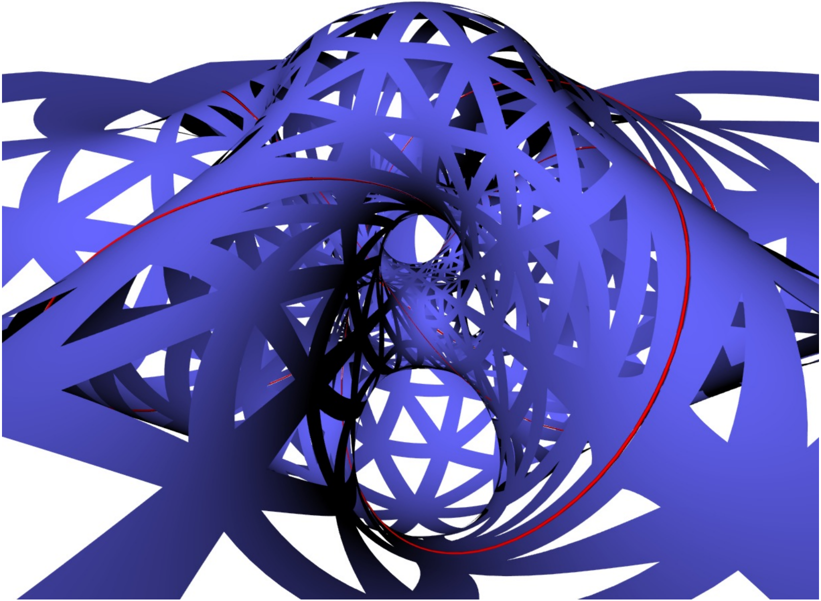
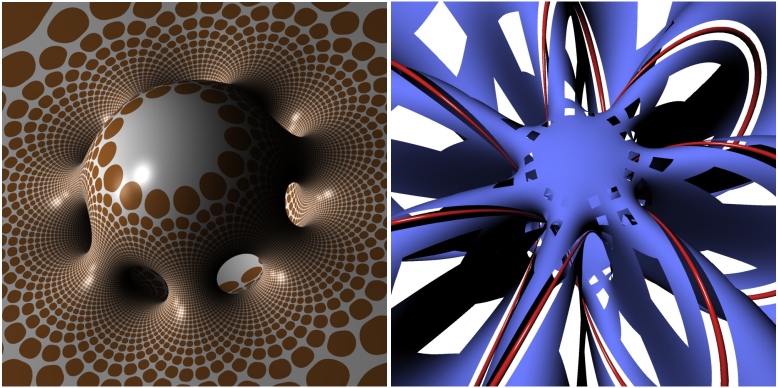


So, Milnor fibers *might* be minimal...



Lawson surface $\xi_{6,1}$ and doubled Milnor fiber for the $(7, 2)$ torus knot.

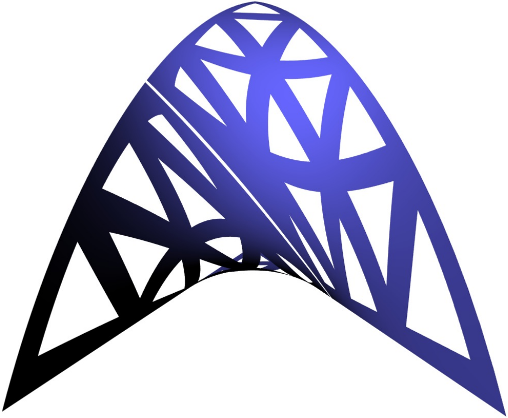
...but they are not.

Any homogeneous polynomial $f = f(x, y, u, v)$ gives a surface in $F \subset S^3 \subset \mathbb{R}^4$. When $p = q$, and for any θ, we can rewrite the equation $\arg(z^p + w^p) = \theta$ as such a polynomial. Now apply a criterion (due to Hsiang): F is minimal in S^3 if and only if

$$\Delta f \cdot |\nabla f|^2 - \nabla f \, H_f (\nabla f)^T$$

is a multiple of f. Using this we can check that Milnor fibers (for $p = q$) are not minimal.

Question: reflection?



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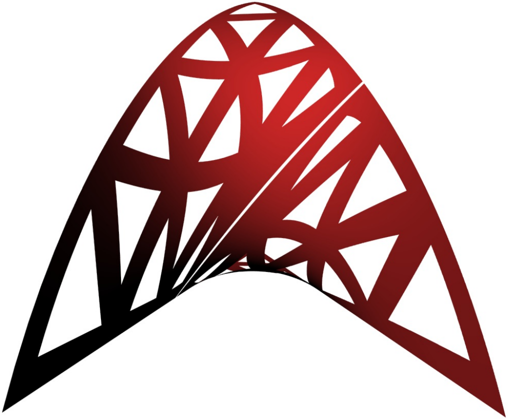

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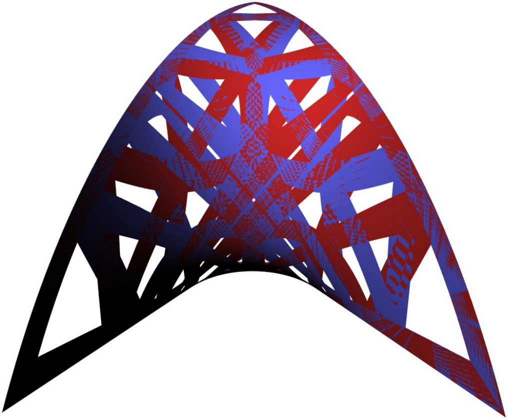
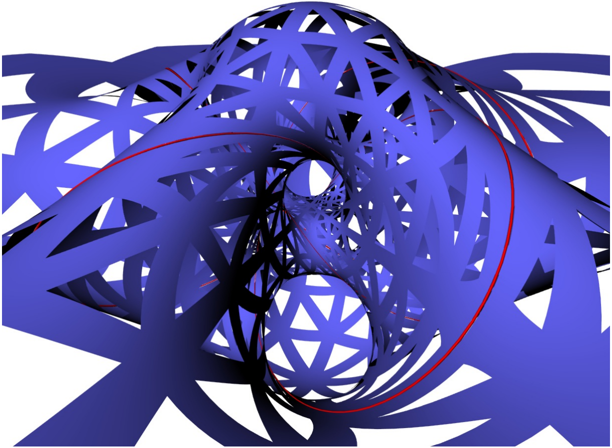
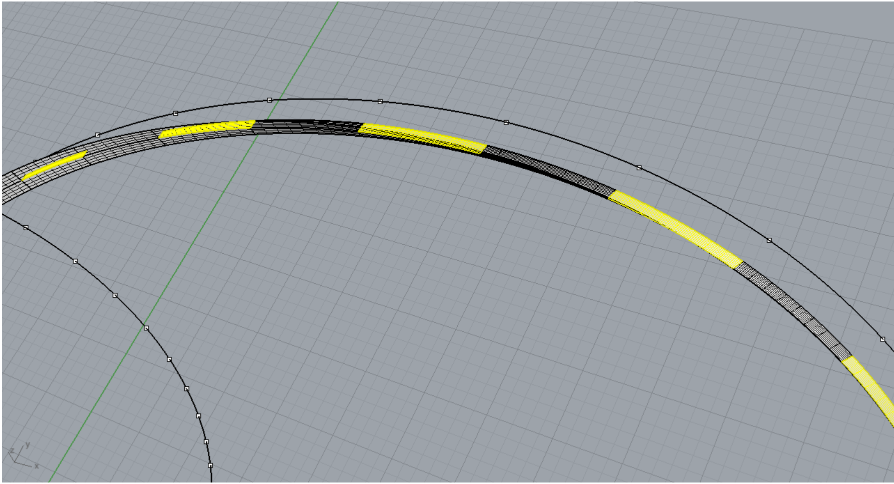


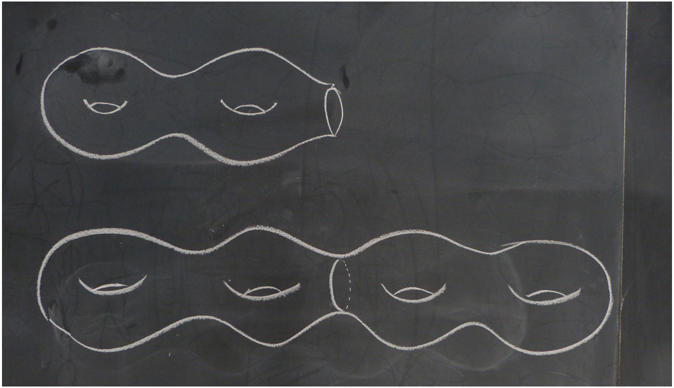
Question: reflection?



Question: arclength?



Thank you!



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