

Math 151: Sections 1-3, 74.

Workshop 6: Total insanity.

Problem 1. Discuss the midterm.

Problem 2. We have seen one definition of the number e . It is the magic number which “solves” the equation:

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1.$$

This came out of our efforts to find the derivative of the function e^x . Let’s think about this.

Define a function $f(a) = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$. That is, the function f , on input a , returns the corresponding limit if the limit exists. If the limit doesn’t exist then f is undefined.

- (1) Using your calculator, make a table of values of $f(a)$, for $a = 1, 2, 4, 8, 16$, accurate to two decimal places. (It should suffice to take $h = 10^{-5}$ or so.) What do you notice?
- (2) Using your calculator, make a table of values of $f(a)$, for $a = 1, 3, 9, 27, 81$. (It should suffice to take $h = 10^{-5}$ or so.) What do you notice?
- (3) Is $f(a)$ defined at $a = 0$? For negative values of a ? Why or why not?
- (4) How does $f(a)$ behave as a gets very big? (Say $a = 1, 10, 10^2, 10^3, 10^4, \dots$) Note that you will need to take fairly small values of h to get two decimal places of accuracy.
- (5) How does $f(a)$ behave as a gets very small? (Say $a = 1/10, 10^{-2}, 10^{-3}, 10^{-4}, \dots$) Again, h will need to be small.
- (6) Plot all of this data on a large-scale graph with a on the horizontal axis. Be as accurate as you can. (Perhaps a graphing package on a computer would be of use – for example `gnuplot` or *Mathematica* or a CAD program.)
- (7) Guess what $f(a)$ actually is. (Remember the laws for logs!) If your guess is correct, then what is $f(e)$?
- (8) Ed Oberhaus Here is some solid evidence for your guess. Let’s compute $f(ab)$:

$$f(ab) = \lim_{h \rightarrow 0} \frac{(ab)^h - 1}{h} = \lim_{h \rightarrow 0} \left\{ \frac{(ab)^h - b^h}{h} + \frac{b^h - 1}{h} \right\}.$$

Now break into two parts and use the fact that $\lim_{h \rightarrow 0} b^h = 1$ when b is positive. (Check this!) What formula do you find for $f(ab)$?

- (9) Finally, compute the derivative of a^x directly from the definition and using your guess for the value of $f(a)$. (You will need the laws for exponents.)
- (10) Ask the TA about anything you don’t understand.