Math 151: Sections 1-3, 74.

Workshop 8: Linear approximation and Newton's method.

Problem 1. Consider the function $s(x) = \sqrt{x}$.

(1) Compute the derivative of s. Use this to find a formula for $T_1(x)$, the tangent line to $s(x)$ at the point $(1, 1)$.
(2) Compute $s(1.1)$, $s(1.01)$, and $s(1.001)$. How close to 1 are they?
(3) Compute $T_1(1.1)$, $T_1(1.01)$, and $T_1(1.001)$. Notice that these are fairly close to the numbers computed above.
(4) Compute directly, or use your calculator to find, a number δ so that $1 \leq x \leq 1 + \delta$ implies that $T_1(x) - s(x) < 1/100$. That is, for x in that range, the tangent line approximation gives the square root correct to about two decimal places.

Problem 2. In class e was defined as the number which satisfies the equation

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

This definition of e is somewhat unsatisfying. In particular, how get we get a sense of how big e is?

Here is the definition of e given on page 248 of the book:

$$e = \lim_{x \to 0} (1 + x)^{1/x} = \lim_{n \to \infty} (1 + 1/n)^n.$$

This is a much nicer definition – you can use the right-hand side with $n = 10000$ to find an approximation of e. Let's try to understand how good an approximation this is.

Define a function $f(x)$ piecewise: $f(x) = (1 + x)^{1/x}$ for positive values of x and also for negative values of x greater than -1. Take $f(0) = e$. Notice that $f(x)$ is continuous, by the definition of e!

(1) Check with your calculator that f is continuous at zero
(2) Compute the derivative of $f(x)$ and note that it is not defined at zero. What kind of singularity does f' have at zero? Graphing the derivative should give you an idea.
(3) Compute the limit

$$\lim_{x \to 0} f'(x).$$

This is not so easy – think carefully about the rules for limits you have learned. You will also need to recall the definition of e.

(4) Using the limit above, find a precise formula for $T_0(x)$, the tangent line to $f(x)$ at zero.
(5) Recall that the tangent line is supposed to be a good approximation to the function. Thus the error of $f(x)$ in computing $e = f(0)$ is about equal to the error of $T_0(x)$ in computing $e = T_0(0)$. That is,

$$e - f(x) \approx e - T_0(x)$$

Compute the error $e - T_0(x)$. Use this, and the fact that $e \approx 2.7$, to approximate the error $e - f(1/10000)$. Explain your steps.

(6) Can you use all of this to compute e correct to 5 decimal places?

Problem 3. Recall that Newton's method may sometimes be used to find the roots of equations. This problem finds all roots of the function $f(x) = x^3 - 5x + 1$ and explores the many possible initial guesses. You will need to use a calculator for this problem.

(1) Recall that Newton's method requires that you start with an initial guess x_0 and iteratively find $x_1, x_2, x_3, \ldots$ using the equation $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Starting with $x_0 = 4$ compute x_n for $n = 1, 2, \ldots$ until it settles down. How many digits of the root does each iteration yield?

(2) Now start with initial guess $x_0 = -6$ compute x_n for $n = 1, 2, \ldots$ until it settles down. How many digits of the root does each iteration yield?

(3) Now start with initial guess $x_0 = 0$. Compute x_n for $n = 1, 2, \ldots$ until it settles down. How many digits of the root does each iteration yield?

(4) How do initial guesses chosen near 4, -6, or 0 behave? (Say, within 1/10 of the original guess.)

(5) How do the initial guesses $x_0 = \sqrt{5/3} \pm 0.1$ differ? How about the initial guesses $x_0 = -\sqrt{5/3} \pm 0.1$?

(6) How does the initial guess $x_0 = \sqrt{5/3}$ behave? $x_0 = -\sqrt{5/3}$? Explain both with words and with a graph.

(7) As it turns out, all initial guesses greater than $\sqrt{5/3}$ converge to the largest root while all guesses less than $-\sqrt{5/3}$ converge to the smallest root. Explain why. (Graphing the function $g(x) = x - f(x)/f'(x)$ may help.)

(As an extra question – add the three roots you found above. What number do you get? Is this just a big coincidence?)

Problem 4. Again suppose that $f(x) = x^3 - 5x + 1$. Start with the initial guess $x_0 = 1.0700573501135...$ Compute, using Newton's method, x_1 and x_2 to as many digits of precision as you can. What do you notice? What is going on? Explain both in words and using a graph of f, decorated with the appropriate tangent lines.

Problem 5. Let $g(x) = x^{1/3}$.

(1) Graph g. Find any roots and explain why you have found all of the roots.

(2) Try to use Newton's method to find roots of g. Look at an initial guess of $x_0 = 1$ as well as the initial guess of $x_0 = -1$. What happens in each case?

(3) What happens for other positive initial guesses? For negative initial guesses?

(4) Explain both in words and graphically what is happening. (You will need to draw a bunch of tangent lines on the graph of g!)